On the generalized antiaverage problem

Bing YAO [a,1] Ming YAO [b,2] Hui CHENG [a]

a. College of Mathematics and Information Science, Northwest Normal University, Lanzhou, 730070, P.R.China

b. Department of Information Process and Control Engineering, Lanzhou Petrochemical College of Vocational Technology, Lanzhou, 730060, P.R.China

Abstract

For integers $k, \theta \geq 3$ and $\beta \geq 1$, an integer k-set S with the smallest element 0 is a $(k^0; \beta, \theta)$-free set if it does not contain distinct elements a_{i_j} $(1 \leq j \leq \theta)$ such that $\sum_{j=1}^{\theta-1} a_{i_j} = \beta a_{i_\theta}$. The largest integer of S is denoted by $\max(S)$. The generalized antiaverage number $\lambda(k; \beta, \theta)$ is equal to $\min\{\max(S) : S \text{ is a } (k^0; \beta, \theta)\text{-free set}\}$. We obtain (1) If $\beta \notin \{\theta-2, \theta-1, \theta\}$, then $\lambda(m; \beta, \theta) \leq (\theta-1)(m-2)+1$; (2) If $\beta \geq \theta-1$, then $\lambda(k; \beta, \theta) \leq \min_{k=m+n} \{\lambda(m; \beta, \theta) + \beta\lambda(n; \beta, \theta)+1\}$, where $k = m + n$ with $n > m \geq 3$; and $\lambda(2n; \beta, \theta) \leq \lambda(n; \beta, \theta)(\beta + 1) + \varepsilon$, where $\varepsilon = 1$ for $\theta = 3$, and $\varepsilon = 0$ otherwise.

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1 Introduction and concepts

The *sum-free problem* has been investigated by Erdös in 1965 (cf. [1]). A subset S of an Abelian group G is *sum-free* if $(S + S) \cap S = \emptyset$, i.e., if there are no $a, b, c \in S$ such that $a + b = c$.

Theorem 1. (Paul Erdös, 1965) *Every set of k non-zero integers contains a sum-free set of size not less than $k/3$.*

Alon and Kleitman proved in 1990 that the constant 1/3 in Theorem 1 cannot be replaced by 12/29 (or any bigger constant). The best possible constant is not known up to now.

In this paper any element of a set under consideration is a non-negative integer, unless it is explicitly stated. The shorthand symbol $[m, n]$ stands for a set $\{m, m+1, \ldots, n\}$, where m and n are non-negative integers with $m \leq n$. A set S is called a *k-set* if it contains k elements, also k is the cardinality $|S|$. The largest integer and the least integer in S are denoted

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[2] The corresponding author. E-mail: yybb918@163.com

by max(S) and min(S), respectively. The *dual set* S^* of a k-set S is defined as $S^* = \{\max(S) + \min(S) - x : x \in S\}$. A set S is *self-dual* if $S^* = S$.

For integers $k, \theta \geq 3$ and $\beta \geq 1$, a k-set S is $(k;\beta,\theta)$*-free* if it does not contain distinct elements a_{i_j} $(1 \leq j \leq \theta)$ such that

$$a_{i_1} + a_{i_2} + \cdots + a_{i_{\theta-1}} = \beta a_{i_\theta}. \tag{1}$$

A $(k^0;\beta,\theta)$*-free set* is a $(k;\beta,\theta)$-free set S whose smallest element $\min(S) = 0$. The notation $\mathcal{S}(k;\beta,\theta)$ denotes the class of $(k^0;\beta,\theta)$-free sets. The *generalized antiaverage number*, denoted by $\lambda(k;\beta,\theta)$, is equal to $\min\{\max(S) : S \in \mathcal{S}(k;\beta,\theta)\}$. A set $S \in \mathcal{S}(k;\beta,\theta)$ is *optimal* if $\max(S) = \lambda(k;\beta,\theta)$. Considering the equation (1), we define $\lambda(k;\beta,\theta) = k$ if $k < \theta$. Some $(k^0;\beta,\theta)$-free sets are shown in Table 1.

Table 1

k	(β,θ)	$(k^0;\beta,\theta)$-free sets
3	(1,3)	$\{0,1,2\}$
4		$\{0,1,2,4\}$
≥ 5		$\{0\} \cup \{2i-1 : 1 \leq i \leq k-1\}$
3	(2,3)	$\{0,1,3\}$
4		$\{0,1,3,4\}$, self-dual
5		$\{0,1,3,7,8\}$
6		$\{0,1,3,7,8,10\}$
7		$\{0,1,3,8,9,11,12\}$
8		$\{0,1,3,4,9,10,12,13\}$, self-dual
9		$\{0,2,5,6,11,13,14,18,19\}$
10		$\{0,1,4,6,10,15,17,18,22,23\}$
4	(2,4)	$\{0,1,3,4\}$
5		$\{0,1,3,4,6\}$
6		$\{0,1,2,4,8,9\}$
7		$\{0,1,3,4,9,10,12\}$
3	(3,3)	$\{0,1,2\}$
4		$\{0,1,2,4\}$
5		$\{0,1,4,5,6\}$
6		$\{0,1,4,5,6,8\}$
7		$\{0,1,2,4,7,8,9\}$

Some generalized antiaverage numbers $\lambda(k;2,3)$ for $k \in [3,12]$ are shown in Table 2.

Table 2

k	3	4	5	6	7	8	9	10	11	12
$\lambda(k;2,3)$	3	4	8	10	12	13	19	23	25	29

In researching graph labellings (cf. [2], [4]), we have a *generalized antiaverage problem* that is similar with the sum-free problem in the following.

The generalized antiaverage problem For integers $k, \theta \geq 3$ and $\beta \geq 1$, determine $\lambda(k;\beta,\theta)$ and optimal $(k^0;\beta,\theta)$-free sets.

Example 1. For a complete graph K_n, there is a one-one mapping $\pi : V(K_n) \leftrightarrow S$, where S is a $(n^0; 2, 3)$-free set such that each edge uv of K_n is labeled by $|\pi(u) - \pi(v)|$. Then, a graph obtained by deleting duplicated edges with the same labels from the labeled K_n is graceful (see [3] for the definition of a graceful graph).

2 Main results

Lemma 2. *For integers $k, \theta \geq 3$ and $\beta \geq 1$, we have*

(i) $\varepsilon \leq \lambda(k;\beta,\theta) < \lambda(k+1;\beta,\theta)$, where $\varepsilon = 2$ if $k = 3$, and $\varepsilon = k$ otherwise.

(ii) Any r-subset of a $(k;\beta,\theta)$-free set is $(r;\beta,\theta)$-free, where $r \geq 3$.

Lemma 3. *Let S be a $(k;\beta,\beta+1)$-free set with $k \geq 3$ and $\beta \geq 1$. Then*

(i) The set $\{sa + t : a \in S\}$ is also $(k;\beta,\beta+1)$-free, where s, t are integers.

(ii) The dual set of S is $(k;\beta,\beta+1)$-free.

Proof. (*i*) Let us assume that there are distinct elements $c_{i_j} = sa_{i_j} + t \in \{as + t : a \in S\}$ for $j \in [1, \beta+1]$ such that $c_{i_1} + c_{i_2} + \cdots + c_{i_\beta} = \beta c_{i_{\beta+1}}$. Then

$$\beta t + s(a_{i_1} + a_{i_2} + \cdots + a_{i_\beta}) = \beta t + \beta s a_{i_\beta+1},$$

and moreover $a_{i_1} + a_{i_2} + \cdots + a_{i_\beta} = \beta a_{i_\beta+1}$, which contradicts with the choice of S.

(*ii*) This assertion is a consequence of the assertion (*i*) by taking $s = -1$ and $t = \max(S) + \min(S)$. □

Theorem 4. *Let integers $m \geq 3$ and $\beta \geq 1$. Then*

$$\lambda(m;\beta,\theta) \leq (\theta - \varepsilon)(m-2) + 1 \tag{2}$$

for $\varepsilon = 1$ if there exists one of the following conditions

(a1) $\theta \geq 3$, $\beta \notin \{\theta-2, \theta-1, \theta\}$; and

(a2) $\theta \geq 3$, $\beta \notin \{\theta-2, \theta-1\}$ and $m > 4 + \frac{1}{2}\theta(\theta-3)$.

The inequality (2) holds for $\varepsilon = 2$ if there exists one of the following conditions

(b1) $\theta \geq 5$ and $\beta \notin \{\theta-2, \theta-1, \theta\}$; and

(b2) $\theta \geq 5$, $\beta \notin \{\theta-2, \theta-1\}$ and $m < 1 + \frac{1}{4}\theta(\theta-1)$.

Proof. We, first, prove the inequality (2) for $\varepsilon = 1$ under the condition $(a1)$. Let $C = [(\theta-2)k+1, (\theta-1)k+1]$ for $k \geq 1$. First, consider case $1 \leq \beta \leq \theta - 3$. Assume that $\sum_{i=1}^{\theta-2} c_i = \beta c$ for distinct $c_i, c \in C$ $(1 \leq i \leq \theta - 2)$, thus,

$$\sum_{i=1}^{\theta-2}[(\theta-2)k+i] \leq \sum_{i=1}^{\theta-2} c_i = \beta c \leq \beta[(\theta-1)k+1],$$

and moreover

$$(\theta-2)^2 k + \frac{(\theta-1)(\theta-2)}{2} \leq (\theta-3)[(\theta-1)k+1], \tag{3}$$

because $\beta \leq \theta - 3$. From (3) we have that $k + \frac{1}{2}(\theta-2)(\theta-3) \leq -1$, which is absurd since $\theta \geq 3$. By the analogous method, we have that there is no $\sum_{i=1}^{\theta-1} c_i = \beta c$ for distinct $c_i, c \in C$ $(1 \leq i \leq \theta - 1)$.

Now, we consider case $\beta \geq \theta + 1$. Suppose that there are distinct $c_i, c \in C$ such that $\sum_{i=1}^{m} c_i = \beta c$. In this case, we have

$$\sum_{i=1}^{m}((\theta-1)k+2-i) \geq \sum_{i=1}^{m} c_i = \beta c \geq \beta((\theta-2)k+1),$$

and furthermore

$$m((\theta-1)k+2) + \frac{1}{2}m(m+1) \geq \beta((\theta-2)k+1). \tag{4}$$

Taking $m = \theta - 2$ and $\beta = \theta + 1$ in (4) gives us that

$$2(\theta-2) - \frac{1}{2}(\theta-1)(\theta-2) \geq 2(\theta-2)k + \theta + 1 \geq 6(\theta-2),$$

a contradiction. Next, letting $m = \theta - 1$ and $\beta = \theta + 1$ in (4) products $3k + \theta - \frac{1}{2}\theta(\theta-1) \geq \theta k + 3 \geq 3k + 3$, and moreover $\theta(3-\theta) \geq 6$, which conflicts to$\theta \geq 3$.

Therefore, the set C is $(k+1;\beta,\theta)$-free for $\beta \notin \{\theta-2, \theta-1, \theta\}$. Immediately, the set $\{0\} \cup C$ is $((k+2)^0;\beta,\theta)$-free, which shows that $\lambda(k+2;\beta,\theta) \leq (\theta-1)k+1$, that is the inequality (2) when $m = k+2$.

While using the hypothesis $(a2)$, we will obtain contrary forms if we take $m = \theta - 2$ and $\beta = \theta$, or $m = \theta - 1$ and $\beta = \theta$ in (4), respectively. In other words, the set C is $(k+1;\beta,\theta)$-free under the hypothesis $(a2)$, as a result, it implies the inequality (2) when $\varepsilon = 1$.

To show the inequality (2) for $\varepsilon = 2$, we take an integer set $B = [(\theta-3)k+1, (\theta-2)k+1]$ with $k \geq 3$. It is easy to see that $\lambda(k+2;\beta,\theta) \leq (\theta-2)k+1$ if the set $\{0\} \cup B$ is $((k+2)^0;\beta,\theta)$-free. The rest of proof is very similar with that of proving the inequality (2) with $\varepsilon = 1$, so we omit it. □

Lemma 5. *Let $k, \theta \geq 3$ and $\beta \geq 1$.*

(*i*) *Let $k = s_1 + s_2$ such that $s_1 \geq s_2 \geq 3$. Then*

$$\max_{k=s_1+s_2} \{\lambda(s_1; \beta, \beta+1) + \lambda(s_2; \beta, \beta+1)\} < \lambda(k; \beta, \beta+1). \tag{5}$$

(*ii*) *Let $k = m + n$ with $n > m \geq 3$. If $\beta \geq \theta - 1$, then*

$$\lambda(k; \beta, \theta) \leq \min_{k=m+n} \{\lambda(m; \beta, \theta) + \beta\lambda(n; \beta, \theta) + 1\}. \tag{6}$$

(*iii*) *Given $\lambda(n_0; \beta, \theta) = \alpha$ for an integer $n_0 \geq 3$. If $\beta \geq \theta - 1$, then*

$$\lambda(2n_0; \beta, \theta) \leq \alpha(\beta+1) + \varepsilon, \tag{7}$$

where $\varepsilon = 1$ for $\theta = 3$, and $\varepsilon = 0$ otherwise.

Proof. (*i*) To show the inequality (5), we take a $(k^0; \beta, \beta+1)$-free set $S = \{0, a_1, \ldots, a_{k-1}\} \in \mathcal{S}(k; \beta, \beta+1)$, where $k = s_1 + s_2$ and $s_1 \geq s_2 \geq 3$. Clearly, the proper subset $S_1 = \{0, a_1, a_2, \ldots, a_{s_1-1}\} \subset S$ is $(s_1^0; \beta, \beta+1)$-free by Lemma 2, so that $\lambda(s_1; \beta, \beta+1) \leq a_{s_1-1}$. If there are distinct $a_{s_1+j_i} - a_{s_1} \in S_2 = \{a_{s_1+j} - a_{s_1} : j \in [0, s_2-1]\}$, $1 \leq i \leq \beta+1$, such that

$$\sum_{i=1}^{\beta}(a_{s_1+j_i} - a_{s_1}) = \beta(a_{s_1+j_{\beta+1}} - a_{s_1}),$$

however, the above form conflicts to the choice of S since $S_2 \subset S$. Notice that S_2 is $(s_2^0; \beta, \beta+1)$-free, thus, we have $\lambda(s_2; \beta, \beta+1) \leq a_{s_1+s_2-1} - a_{s_1}$. It is not hard to see $\lambda(s_1; \beta, \beta+1) + \lambda(s_2; \beta, \beta+1) \leq a_{s_1-1} + a_{s_1+s_2-1} - a_{s_1} < a_{s_1+s_2-1} = \lambda(s_1+s_2; \beta, \beta+1)$, thus, it implies the inequality (5).

(*ii*) We take two optimal sets $S_1 = \{a_0, a_1, \ldots, a_{m-1}\} \in \mathcal{S}(m; \beta, \theta)$ with $\max(S_1) = a_{m-1} = \lambda(m; \beta, \theta)$ and $S_2 = \{b_0, b_1, \ldots, b_{n-1}\} \in \mathcal{S}(n; \beta, \theta)$ with $\max(S_2) = b_{n-1} = \lambda(n; \beta, \theta)$. So $a_{m-1} < b_{n-1}$ from $n > m$, by the assertion (i) of Lemma 2.

We, now, define a new set $S = S_2 \cup \{c_i = a_i + \beta b_{n-1} + 1 : a_i \in S_1\}$. Clearly, $\max(S) = a_{m-1} + \beta b_{n-1} + 1$. Our goal is to show that S is $((m+n)^0; \beta, \theta)$-free in the following. Let $x + y = \theta - 1 \geq 2$, where integers $x, y \geq 0$.

Suppose first that there exist distinct $c_j, b_{i_l}, c_{i_k} \in S$, $1 \leq l \leq x$ and $1 \leq k \leq y$, such that $\beta c_j = \sum_{l=1}^{x} b_{i_l} + \sum_{k=1}^{y} c_{i_k}$, equivalently,

$$\beta^2 b_{n-1} + \beta(a_j + 1) = y\beta b_{n-1} + \sum_{l=1}^{x} b_{i_l} + \sum_{k=1}^{y}(a_{i_k} + 1). \tag{8}$$

Immediately, from (8), we have

$$\beta^2 b_{n-1} + \beta(a_j + 1) \leq (x + y\beta + y)b_{n-1} \tag{9}$$

since $\max(S_1) < \max(S_2) = b_{n-1}$. By the hypothesis $\beta \geq \theta - 1$ $(= x + y)$, we have $x\beta \geq \beta \geq x+y$ if $x \geq 1$ and $y\beta \geq \beta \geq x+y$ if $y \geq 1$. Furthermore, $\beta^2 \geq (x+y)\beta \geq (x+y)+y\beta$. Then, the inequality (9) products $\beta(a_j+1) \leq 0$, a contradiction because $a_j \geq \min(S) = 0$ whenever $a_j \in S$.

We consider the following case

$$\beta^2 b_{n-1} + \beta(a_j + 1) = \beta c_j = \sum_{k=1}^{\theta-1} c_{i_k} = (\theta - 1)\beta b_{n-1} + \sum_{k=1}^{\theta-1}(a_{i_k} + 1). \quad (10)$$

Notice that $\beta \geq \theta - 1$. If $\beta = \theta - 1$, thus, the form (10) leads to a contradiction with the choice of S_1. If $\beta - 1 \geq \theta - 1 = 0 + y$, then $\beta^2 = (\beta - 1 + 1)\beta \geq (\theta - 1)\beta + (\theta - 1)$. From (10) we have

$$\beta^2 b_{n-1} + \beta(a_j + 1) \leq (\theta - 1)\beta b_{n-1} + (\theta - 1)b_{n-1},$$

furthermore $\beta(a_j + 1) \leq 0$, which is absurd.

The case $\beta^2 b_{n-1} + \beta(a_j + 1) = \beta c_j = \sum_{l=1}^{\theta-1} b_{i_l}$ will product $\beta^2 b_{n-1} + \beta(a_j + 1) \leq (\theta - 1)b_{n-1}$. By the hypothesis $\beta \geq \theta - 1$, we still get this ridiculous inequality $\beta(a_j + 1) \leq 0$.

Now, suppose that there are distinct $b_j, b_{i_l}, c_{i_k} \in S$ for $1 \leq l \leq x$ and $1 \leq k \leq y$ such that $\beta b_j = \sum_{l=1}^{x} b_{i_l} + \sum_{k=1}^{y} c_{i_k}$. Then $y \geq 1$ since S_2 is $(n^0; \beta, \theta)$-free. We have

$$\sum_{l=1}^{x} b_{i_l} + \sum_{k=1}^{y}(a_{i_k} + 1) = \beta b_j - y\beta b_{n-1} \leq \beta(b_j - b_{n-1}) \leq 0,$$

which is false. Therefore, S is $((m + n)^0; \beta, \theta)$-free. The inequality (6) follows since $\lambda(m + n; \beta, \theta) \leq \max(S)$.

(*iii*) Let $S_1 = \{a_0, a_1, \ldots, a_{n_0-1}\}$ be an optimal $(n_0^0; \beta, \theta)$-free set, that is, $\max(S_1) = \alpha$, where $\alpha \geq 2$ by Lemma 2. Let $M = \alpha(\beta + 1) + \varepsilon$, where $\varepsilon = 1$ for $\theta = 3$, and $\varepsilon = 0$ otherwise. We construct a set $S = S_1 \cup S_2$, where $S_2 = \{M - a_i : a_i \in S_1\}$. Let integers $x \geq 0$ and $y \geq 0$ such that $x + y = \theta - 1$.

Case 1. Suppose that there is an element $a_j \in S_1$ such that

$$\beta a_j = \sum_{s=1}^{x} a_{i_s} + \sum_{t=1}^{y}(M - a_{j_t}), \quad (11)$$

where distinct $a_{i_s} \in S \setminus \{a_j\}$ for $1 \leq s \leq x$ and distinct $(M - a_{j_t}) \in S$ for $1 \leq t \leq y$.

There is an obvious mistake when $y = 0$ in (11). Taking $x = 0$ in (11), immediately, we have

$$\alpha\beta + (\theta - 1)\alpha \geq \beta a_j + \sum_{t=1}^{\theta-1} a_{j_t} = (\theta - 1)M \geq (\theta - 1)\alpha(\beta + 1),$$

and furthermore $\theta - 2 \leq 0$, which contradicts with $\theta \geq 3$. So, the following discussion will be restricted to $x \geq 1$ and $y \geq 1$.

If some $a_{i_s} = \alpha$, no $a_{j_t} = \alpha$ in (11), we then have

$$\alpha\beta \geq \beta a_j = \sum_{s=1}^{x} a_{i_s} + \sum_{t=1}^{y}(M - a_{j_t}) \geq \alpha + M - a_{j_1} \geq \alpha(\beta + 1) + \varepsilon,$$

an obvious mistake since $\alpha \geq 2$.

Consider no $a_{i_s} = \alpha$, no $a_{j_t} = \alpha$ in (11). Hence,

$$\alpha\beta \geq \beta a_j = \sum_{s=1}^{x} a_{i_s} + \sum_{t=1}^{y}(M - a_{j_t}) \geq a_{i_1} + M - (\alpha - 1) = a_{i_1} + \alpha\beta + 1 + \varepsilon,$$

which is impossible.

If some $a_{j_t} = \alpha$ in (11). For $\theta = 3$ we have that $x = 1$ and $y = 1$, so that

$$\alpha\beta \geq \beta a_j = a_{i_1} + (M - \alpha) = a_{i_1} + \alpha\beta + \varepsilon = a_{i_1} + \alpha\beta + 1,$$

a wrong form. We consider case $\theta \geq 4$, then $\varepsilon = 0$ in this case. If $x = 1$ and $y \geq 2$, we have

$$\alpha\beta \geq \beta a_j = \sum_{s=1}^{x} a_{i_s} + \sum_{t=1}^{y}(M - a_{j_t}) \geq a_{i_1} + (M - \alpha) + (M - a_{j_2}) \geq 1 + 2\alpha\beta,$$

evidently, a contradiction.

For $x \geq 2$ and $y = 1$. From (11) we have a contrary form as follows:

$$\alpha\beta \geq \beta a_j \geq a_{i_1} + a_{i_2} + (M - \alpha) \geq 1 + \alpha\beta,$$

since $\max\{a_{i_1}, a_{i_2}\} \geq 1$.

If $\min\{x, y\} > 1$, thus,

$$\alpha\beta \geq \beta a_j = \sum_{s=1}^{x} a_{i_s} + \sum_{t=1}^{y}(M - a_{j_t}) \geq a_{i_1} + a_{i_2} + (M - \alpha) \geq 1 + \alpha\beta,$$

which means that the form (11) is wrong.

Case 2. Let $x \geq 1$ and $y \geq 1$. Suppose that $\beta(M - a_j) = \sum_{s=1}^{x} a_{i_s} + \sum_{t=1}^{y}(M - a_{j_t})$ for distinct $a_{i_s} \in S \setminus \{M - a_j\}$ $(1 \leq s \leq x)$ and distinct $(M - a_{j_t}) \in S \setminus \{M - a_j\}$ $(1 \leq t \leq y)$. Then we have

$$xM + \sum_{t=1}^{y} a_{j_t} = [(\theta - 1) - y]M + \sum_{t=1}^{y} a_{j_t} \leq (\beta - y)M + \sum_{t=1}^{y} a_{j_t} = \beta a_j + \sum_{s=1}^{x} a_{i_s}$$

and moreover

$$\alpha\beta + x\alpha + 1 \leq xM + y \leq xM + \sum_{t=1}^{y} a_{j_t} = \beta a_j + \sum_{s=1}^{x} a_{i_s} \leq \alpha\beta + x\alpha,$$

a contradiction.

Case 3. Suppose that $\beta(M-a_j) = \sum_{t=1}^{\theta-1}(M-a_{i_t})$ for distinct $M-a_{i_t} \in S \setminus \{M-a_j\}$, $1 \leq t \leq \theta - 1$.

Notice the hypothesis $\beta \geq \theta - 1$. This case never occurs if $\beta = \theta - 1$ because S_1 is $(n_0^0; \beta, \theta)$-free. For $\beta > \theta - 1$, we have

$$\alpha(\beta+1) = M < [\beta - (\theta - 1)]M = \beta a_j - \sum_{t=1}^{\theta-1} a_{i_t} \leq \beta a_j \leq \alpha\beta,$$

which causes a contradiction from $\alpha \geq 2$.

The above three cases show that S is $(2n_0^0; \beta, \theta)$-free. It follows $\max(S) = M$ that the inequality (7) holds. □

Theorem 6. *Let integers $k, \theta \geq 3$ and $\beta \geq 1$.*

(i) For integers $n_0 \geq 3$ and $t \geq 1$, if $\beta \geq \theta - 1 \geq 3$, we have

$$\lambda(tn_0; \beta, \theta) \leq \lambda(n_0; \beta, \theta)\frac{\beta^t - 1}{\beta - 1}. \tag{12}$$

(ii) For integers $m \geq 3$ and $n \geq 1$, if $\beta \notin \{\theta-2, \theta-1, \theta\}$, then

$$\lambda(nm; \beta, \theta) \leq (\theta - 1)\left(2(m-1)\beta^{n-2} + (m-2)\sum_{l=0}^{n-3}\beta^l\right) + \sum_{l=0}^{n-2}\beta^l. \tag{13}$$

Proof. (i) Let $M = \lambda(n_0; \beta, \theta)$. Using the induction on parameter m. By Lemma 5 and the condition $\beta \geq \theta - 1 \geq 3$ we have

$$\lambda(3n_0; \beta, \theta) \leq M + \beta\lambda(2n_0; \beta, \theta) \leq M + \beta(\beta+1)M = M\frac{\beta^3 - 1}{\beta - 1}.$$

According to the inductive hypothesis,

$$\lambda(tn_0; \beta, \theta) \leq M + \beta\lambda((t-1)n_0; \beta, \theta) \leq M + \beta M\frac{\beta^{t-1} - 1}{\beta - 1} = M\frac{\beta^t - 1}{\beta - 1},$$

now the assertion (i) is proofed.

(ii) Let $\lambda_1 = \lambda(m; \beta, \theta)$ and $\lambda_2 = \lambda(2m; \beta, \theta)$ with $m \geq 3$. By Theorem 4 and Lemma 5, we have

$$\lambda(3m; \beta, \theta) \leq \lambda_1 + \beta\lambda_2 = \lambda_1\sum_{l=0}^{3-3}\beta^l + \lambda_2\beta^{3-2},$$

and furthermore

$$\begin{aligned}\lambda(nm;\beta,\theta) &\le \lambda(m;\beta,\theta) + \beta\lambda((n-1)m;\beta,\theta)\\ &\le \lambda_1 + \beta\left(\lambda_1 \sum_{l=0}^{n-4} \beta^l + \lambda_2\beta^{n-3}\right)\\ &= \lambda_1 \sum_{l=0}^{n-3} \beta^l + \lambda_2\beta^{n-2},\end{aligned}$$

by the inductive hypothesis. The inequality (13) follows since $\lambda(m;\beta,\theta) \le (\theta-1)(m-2)+1$ and $\lambda(2m;\beta,\theta) \le 2(\theta-1)(m-1)+1$. □

Corollary 7. (1) $\lambda(2^m;2,3) \le \frac{1}{2}(3^m-1)$ *for* $m \ge 4$.
(2) $\lambda(mn_0;\beta,3) \le \beta^{m-2} + \lambda(n_0;\beta,3)\sum_{i=1}^{m}\beta^{m-i}$ *for* $m \ge 2$.

Lemma 8. *Let* $S = \{a_1, a_2, \ldots, a_k\}$ *be a* $(k;\beta,\theta)$*-free set such that* $\min(S) = 0$ *and* $\max(S) = N$. *If* $\beta \ge \theta-1 \ge 2$, *and* $2 \le a_i \le N-2$ *for* $a_i \in S\backslash\{0,N\}$, *we have* $\lambda(2k;\beta,\theta) \le (\beta+1)N-1$, *and* $\lambda(2(k-1);\beta,\theta) \le (\beta+1)N-4$.

Proof. We make a set $U = S \cup T$, where $T = \{\beta N + a_i - 1 : a_i \in S\}$. Clearly, $\max(U) = (\beta+1)N-1$, and $2 \le x \le (\beta+1)N-3$ for $x \in U \setminus \{0,N\}$. Our goal is to show that U is $((2k)^0;\beta,\theta)$-free. Clearly, the second inequality follows from the structure of U. Let integers $x, y \ge 0$ hold $x+y=\theta-1$.

Case 1. Suppose that there exist distinct $a_j, a_{i_s}, (\beta N + a_{j_t} - 1) \in U$ such that

$$\beta a_j = \sum_{s=1}^{x} a_{i_s} + \sum_{t=1}^{y}(\beta N + a_{j_t} - 1) = \sum_{s=1}^{x} a_{i_s} + \sum_{t=1}^{y} a_{j_t} + y(\beta N - 1). \quad (14)$$

If $x=1$ and $y=1$ in (14), we may meet $\beta a_j = \beta N - 1$, but no $a_j \in U$ can keep this equality.

In (14) if $x=0$ and $y \ge 2$, or $x+y \ge 3$, we have $2 \le \sum_{s=1}^{x} a_{i_s} + \sum_{t=1}^{y} a_{j_t} = \beta a_j - y(\beta N-1) \le 1$ from (14) since $2 \le a_{i_s}, a_{j_t}$ for $i_s \ne 0$ and $j_t \ne 0$, a contradiction. And $x \ge 2$ and $y=0$, which contradicts with the choice of S.

Case 2. Suppose that there exist distinct $(\beta N + a_j - 1), a_{i_s}, (\beta N + a_{j_t} - 1) \in U$ such that $\beta(\beta N + a_j - 1) = \sum_{s=1}^{x} a_{i_s} + \sum_{t=1}^{y}(\beta N + a_{j_t} - 1)$, or

$$\beta(\beta N + a_j - 1) = \sum_{s=1}^{x} a_{i_s} + \sum_{t=1}^{y} a_{j_t} + y(\beta N - 1). \quad (15)$$

Case A1. If $x=0$ in (15), thus, $y=\theta-1$. For $\beta=\theta-1$, we are done since the form (15) contradicts with the choice of S. So, we consider case $\beta > \theta-1$, that is, $\beta \ge \theta$. From (15) we obtain

$$\beta^2 N + \beta(a_j - 1) \le N + (\theta-2)(N-2) + (\theta-1)(\beta N - 1) \le ((\beta+1)\theta - \beta)N. \quad (16)$$

Notice that $\beta^2 = (\beta-1)\beta+\beta \geq (\theta-1)\beta+\theta$ from $\beta \geq \theta$. Hence, by inequalities (16), we have $\beta(a_j-1) \leq 0$, and moreover $\theta N + \beta(N-1) \leq (\beta+\theta)N + \beta(a_j-1) \leq 0$, which is impossible.

Case A2. If $x=1$ and $a_j=0$ in (15), so $a_{j_t} \neq 0$ for $t \in [1, \theta-2]$, and we have

$$(\beta-\theta+2)(\beta N-1) = a_{i_1} + \sum_{t=1}^{\theta-2} a_{j_t}. \tag{17}$$

Subcase A2.1. If $\beta = \theta-1$, thus, the equation (17) gives us

$$\begin{aligned}\beta N &= (a_{i_1}+1) + \sum_{t=1}^{\beta-1} a_{j_t} \\ &\leq N-1+2+N+(N-2)+(N-3)+\cdots+(N-\beta-1) \\ &= 2+\beta N - \frac{\beta(\beta+1)}{2}\end{aligned}$$

that is, $\beta(\beta+1) \leq 4$, a contradiction because $\beta \geq 3$.

Subcase A2.2. If $\beta \geq \theta$ in (17), thus,

$$2(\beta N-1) \leq (\beta-\theta+2)(\beta N-1) = a_{i_1} + \sum_{t=1}^{\theta-2} a_{j_t} \leq (\theta-1)N \leq (\beta-1)N,$$

and furthermore $\beta N \leq 0$, which is impossible since $\beta \geq \theta \geq 3$ and $N \geq 2$.

Case A3. If $x=1$ and $a_j \neq 0$ in (15), we have

$$\begin{aligned}\beta^2 N + \beta(a_j-1) &= a_{i_1} + \sum_{t=1}^{y} a_{j_t} + y(\beta N-1) \\ &\leq 2N + (\theta-3)(N-2) + (\theta-2)(\beta N-1) \\ &\leq ((\theta-2)\beta+\theta-1)N.\end{aligned} \tag{18}$$

Since $\beta \geq \theta-1$, so $\beta^2 = (\beta-1)\beta+\beta \geq (\theta-2)\beta+\theta$, using (18), we obtain a wrong inequality $N+\beta(a_j-1) \leq 0$, according to $a_j \geq 2$ and $\beta \geq \theta-1 \geq 2$.

Case A4. Consider $x \geq 2$ in (15). If $a_j = 0$, from (15) we have

$$2(\beta N-1) \leq x(\beta N-1) \leq (\beta-y)(\beta N-1) = \sum_{s=1}^{x} a_{i_s} + \sum_{t=1}^{y} a_{j_t} \leq (\theta-1)N \leq \beta N,$$

that is, $\beta N \leq 2$, an absurd inequality because $\beta \geq \theta-1 \geq 2$ and $N > 2$.

If $a_j \geq 2$, we have

$$\begin{aligned} \beta^2 N + \beta(a_j - 1) &= \sum_{s=1}^{x} a_{i_s} + \sum_{t=1}^{y} a_{j_t} + y(\beta N - 1) \\ &\leq 2N + (\theta - 3)(N - 2) + y(\beta N - 1) \\ &\leq (\theta + y\beta - 1)N. \end{aligned} \tag{19}$$

Notice that $\beta^2 \geq (\theta - 1)\beta = (x - 1 + y)\beta \geq \beta + y\beta \geq \theta - 1 + y\beta$ since $x \geq 2$ and $\beta \geq \theta - 1$. Therefore, the form (19) leads to $\beta(a_j - 1) \leq 0$, a contradiction.

The discussion through all above cases is the proof of this therorem. □

Example 2. $S = \{0, 2, 3, 7, 8, 10\}$ is a $(6^0; 2, 3)$-free set. We have $T = \{20 + a - 1 : a \in S\} = \{19, 21, 22, 26, 27, 29\}$ such that $S \cup T$ is $((12)^0; 2, 3)$-free, which means that $\lambda(12; 2, 3) \leq 29$ and $\lambda(10; 2, 3) \leq 26$.

Corollary 9. *Let $S = \{a_1, a_2, \ldots, a_k\}$ be a $(k; \beta, \theta)$-free set such that $0 = a_1 < 2 \leq a_i \leq a_k - 2$ for $i \in [2, k-1]$. If $\beta \geq \theta - 1 \geq 2$, then*

$$\lambda(2^m k; \beta, \theta) \leq a_k(\beta + 1)^m - \sum_{l=0}^{m-1} (\beta + 1)^l.$$

3 Problems

Clearly, the result in Theorem 4 is not the best one. To improve it may be very interesting. Thereby, we propose the following problem.

Problem 1. Find bounds of $\lambda(k; \beta, \theta)$ for integers $\beta \geq 1$ and $k, \theta \geq 3$.

Notice that $n > 4 + \frac{1}{2}\theta(\theta - 3)$ as $n \to \infty$ and $\theta \geq 3$. From Lemma 2 and Theorem 4 we have $0 \leq \lambda(n+1; \beta, \theta) - \lambda(n; \beta, \theta) \leq \theta - 1$, immediately,

$$\lim_{n \to \infty} \frac{\lambda(n + 1; \beta, \theta) - \lambda(n; \beta, \theta)}{n} = 0,$$

for $\beta \notin \{\theta - 2, \theta - 1\}$. Naturally, we ask

Problem 2. For $\beta \in \{\theta - 2, \theta - 1\}$, does $\lim_{n \to \infty} \frac{1}{n}(\lambda(n+1; \beta, \theta) - \lambda(n; \beta, \theta))$ converge?

We are working on finding $\lambda(k; \beta, \theta)$ by computer for some particular values of k, β and θ. Some $((4m)^0; 2, 3)$-free sets support the following conjecture:

Conjecture 3. $S(k;\beta,\theta)$ denotes the class of $(k^0;\beta,\theta)$-free sets. Then $|S(4m;2,3)| = 1$ for $m \geq 1$.

We will define a particular class of sets before proposing the last problem. For non-negative real numbers $r_1, r_2, \ldots, r_{\theta-1}$ ($\theta \geq 3$), an integer k-set $S = \{a_i : 1 \leq i \leq k\}$ with $0 = a_1 < a_2 < \cdots < a_k$ and $k \geq \theta$ is called a $f(k; r_1, \ldots, r_{\theta-1})$*-free set* if it holds

$$r_1 a_{i_1} + r_2 a_{i_2} + \cdots + r_{\theta-1} a_{i_{\theta-1}} \neq a_{i_\theta} \tag{20}$$

for distinct $a_{i_j} \in S$, $1 \leq j \leq \theta$. The $f(k; r_1, \ldots, r_{\theta-1})$*-number*, denoted by $\lambda(k; r_1, \ldots, r_{\theta-1})$, is equal to $\min_S\{a_k\}$ over all $f(k; r_1, \ldots, r_{\theta-1})$-free k-sets S. A $f(k; r_1, \ldots, r_{\theta-1})$-free k-set S is *optimal* if $\max(S) = \lambda(k; r_1, \ldots, r_{\theta-1})$.

Problem 4. Let integers $k \geq \theta \geq 3$, and let $r_1, r_2, \ldots, r_{\theta-1}$ be non-negative real numbers. Determine the $f(k; r_1, \ldots, r_{\theta-1})$-numbers $\lambda(k; r_1, \ldots, r_{\theta-1})$, and optimal $f(k; r_1, \ldots, r_{\theta-1})$-free sets.

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References

[1] Noga Alon, Joel H. Spencer and Paul Erdös. The Probabilistic Methods. *A Wiley-Interscience Publication*, John Wiley & Sons, Inc., 1991.

[2] Hui Cheng, Bing Yao, Xiang-en Chen and Zhong-fu Zhang. On Graceful Generalized Spiders and Caterpillars. *Ars Combinatoria* 87 (2008), 181-191.

[3] Joseph A. Gallian. A Dynamic Survey of Graph Labeling. *The Electronic Journal of Combinatorics*, 14 (2007), $\#DS6$.

[4] Bing Yao, Hui Cheng, Ming Yao and Meimei Zhao. A Note on Strongly Graceful Trees. *Ars Combinatoria* 92 (2009), 155-169.