The Laplacian spectral radius of graphs with given matching number *

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Abstract

In this paper, we obtain the largest Laplacian spectral radius for bipartite graphs with given matching number and use them to characterize the extremal general graphs.

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1 Introduction

Let G be a simple connected graph. The matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of the graph G, where $D(G) = diag(d_u, u \in V)$ is the diagonal matrix of vertex degrees of G and $A(G)$ is the adjacency matrix of G. It is known that $L(G)$ is a positive semi-definite and singular matrix. The largest eigenvalue, $\lambda_1(G)$ of $L(G)$, is called the *Laplacian spectral radius* of G and is denoted by $\lambda = \lambda_1(G)$. Suppose $Q(G) = D(G) + A(G)$, we call this matrix the *Q–matrix* and its largest eigenvalue is denoted by $\mu(G)$ or μ for simplicity. It is well known that $Q(G)$ is irreducible nonnegative for a connected graph G, so from the Perron-Frobenius theorem, there is a unique positive unit eigenvector corresponding to $\mu(G)$. We call such an eigenvector as Perron vector. For the background on the Laplacian eigenvalues of a graph, the reader is referred to [12] and the references therein.

We introduce some graph notation next. Two distinct edges in a graph G are *independent* if they are not incident with a common vertex in G. A set

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of pairwise independent edges in G is called a *matching* in G. A matching of maximum cardinality is a maximum matching in G. The *matching number* $\beta(G)$ (or just β, for short) of G is the cardinality of a maximum matching of G. It is well known that $\beta(G) \leq \frac{n}{2}$ with equality if and only if G has a perfect matching. Given a vertex subset S of G, the subgraph induced by S is denoted by $G[S]$. For a vertex v in G, $N_G(v)$ (or just $N(v)$ for short) denotes the set of neighbors of v and $d_G(v)$ (or just $d(v)$ for short) denotes the degree of v. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The *union* $G_1 \cup G_2$ is defined to be $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The *join* $G_1 \bigvee G_2$ of G_1 and G_2 is obtained from $G_1 \cup G_2$ by joining each vertex of G_1 to each vertex of G_2 by an edge. We denoted by $\overline{K_t}$ the graph on t vertices with no edges. For other notation in graph theory, we follow [2].

In [3], Brualdi and Solheid proposed the following problem concerning spectral radius: *Given a set of graphs $\mathcal{G}$ find an upper bound for the spectral radius of graphs in $\mathcal{G}$ and characterize the graphs in which the maximal spectral radius is attained.* This problem is well studied, see for example [1, 5, 6, 11, 13]. For the Laplacian spectral radius of this problem, see [14, 15].

Let $\mathcal{G}_{n,\beta}$ be the set of graphs on n vertices with matching number β. Now, we consider the following problems: *What is the structure of the graphs with maximal Laplacian spectral radius in $\mathcal{G}_{n,\beta}$?*

In this paper, we solve the above problem, we obtain the largest Laplacian spectral radius for bipartite graphs with given matching number. Then by using these results, we characterize the extremal general graphs with given matching number.

2 Bipartite graphs

Let $\mathcal{G}^+_{n,\beta}$ be the set of bipartite graphs on n vertices with matching number β. If $\beta = 1$, the only bipartite graph is $K_{1,n-1}$, the star on n vertices and $\lambda(K_{1,n-1}) = \mu(K_{1,n-1}) = n$. So in the following, we always assume $2 \leq \beta \leq \frac{n}{2}$.

Lemma 2.1 *[8] For a connected graph G, we have $\lambda(G) \leq \mu(G)$, the equality holds if and only if G is bipartite.*

Lemma 2.2 *[4] For a connected graph G, we have $\lambda(G) \leq \max\{d(u) + d(v) - |N(u) \cap N(v)| : uv \in E(G)\}$. This upper bound for $\lambda(G)$ does not exceed n.*

Lemma 2.3 *[10] Let G be a connected graph and u, v be two vertices of G. Suppose $v_1, v_2, \cdots, v_s \in N(v) \setminus (N(u) \cup \{u\})$ $(1 \leq s \leq d(v))$, and G^* is the graph obtained from G by deleting the edges vv_i and adding the edges uv_i*

$(1 \le i \le s)$. Let $X = (x_1, x_2, \cdots, x_n)^t$ be the Perron vector of $Q(G)$, where x_i corresponds to v_i $(1 \le i \le n)$. If $x_u \ge x_v$, then $\mu(G) < \mu(G^*)$.

We generalize Lemma 2.3 next.

Lemma 2.4 *[10] Let A be a nonnegative symmetric matrix with spectral radius $\rho(A)$. Let X be a unit vector in $\mathcal{R}^n$. If $\rho(A) = X^tAX$, then $AX = \rho(A)X$.*

Lemma 2.5 *Let G be a connected graph of order n and S, T be its two disjoint nonempty vertex subset. Suppose $S = \{v_1, v_2, \cdots, v_s\}$ and the neighbors of v_i in T are $v_{i1}, v_{i2}, \cdots, v_{il_i}$ $(l_i \ge 1,\ i = 1, 2, \cdots, s)$. Let $X = (x_{v_1}, x_{v_2}, \cdots, x_{v_n})^t$ be the Perron vector of $Q(G)$, where x_{v_k} corresponds to the vertex v_k $(1 \le k \le n)$. Suppose $x_{v_1} = \max\{x_{v_i} : i = 1, 2, \cdots, s\}$. Let H be the graph obtained from G by deleting edges v_iv_{ij} and adding the edges v_1v_{ij} $(i = 2, 3, \cdots, s; j = 1, 2, \cdots, l_i)$. Then we have $\mu(G) < \mu(H)$.*

Proof. The proof is similar to Theorem 2.1 in [10], we present it here for completeness. Obviously,

$$\begin{aligned} X^t(Q(H) - Q(G))X &= X^t(D(H) + A(H) - D(G) - A(G))X \\ &= \sum_{i=2}^{s}\sum_{j=1}^{l_i}\Big((x_{v_1} + x_{v_{ij}})^2 - (x_{v_i} + x_{v_{ij}})^2\Big) \\ &= \sum_{i=2}^{s}\sum_{j=1}^{l_i}\Big((x_{v_1}^2 - x_{v_i}^2) + 2x_{v_{ij}}(x_{v_1} - x_{v_i})\Big) \\ &\ge 0. \end{aligned}$$

Thus,

$$\mu(H) = \max_{||Y||=1} Y^tQ(H)Y \ge X^tQ(H)X \ge X^tQ(G)X = \mu(G).$$

If $\mu(H) = \mu(G)$, then the inequalities above should be equalities. So

$$\mu(H) = X^tQ(H)X = X^tQ(G)X = \mu(G).$$

By Lemma 2.4, we have $\mu(H)X = Q(H)X$ and $Q(G)X = \mu(G)X$. Thus,

$$\mu(H)x_{v_1} = d_H(v_1) + \sum_{w \in N_H(v_1)} x_w.$$

$$\mu(G)x_{v_1} = d_G(v_1) + \sum_{w \in N_G(v_1)} x_w.$$

Since $d_H(v_1) \ge d_G(v_1)$, $\sum_{w \in N_H(v_1)} x_w > \sum_{w \in N_G(v_1)} x_w$, therefore $\mu(H)x_{v_1} > \mu(G)x_{v_1}$. Since $x_{v_1} > 0$, hence $\mu(H) > \mu(G)$, a contradiction. ■

Theorem 2.6 *Let $G \in \mathcal{G}^+_{n,\beta}$ be a connected bipartite graph with n vertices and matching number β ($2 \leq \beta \leq \frac{n}{2}$). Then we have $\lambda(G) \leq n$ with equality holding if and only if $G \cong K_{\beta,n-\beta}$.*

Proof. From Lemma 2.1, it would be convenient to consider the Q−matrix. Let $G = (U, W)$ be a maximizing graph among all bipartite graphs of order n with matching number β. Since G is bipartite, every edge of G joins two vertices in different part. Let $E = \{e_1 = u_1v_1, e_2 = u_2v_2, \cdots, e_\beta = u_\beta v_\beta\}$ be a set of edge independent set with cardinality β, where $u_i \in U$, $v_i \in W$ ($i = 1, 2, \cdots, \beta$). Further, let $U' = \{u_1, u_2, \cdots, u_\beta\}$, $W' = \{v_1, v_2, \cdots, v_\beta\}$.

We claim that either $U \setminus U' = \emptyset$ or $W \setminus W' = \emptyset$ holds.

Without loss of generality, we suppose on the contrary that both $U \setminus U' \neq \emptyset$ and $W \setminus W' \neq \emptyset$. Note that the vertices in $U \setminus U'$ are adjacent to some vertices in W' and the vertices in $W \setminus W'$ are adjacent to some vertices in U'. Now we consider the eigencomponents of $\mu(G)$ corresponding to the vertices in U'. By Lemma 2.3 or 2.5, comparing the eigencomponents corresponding to the vertices in U', we get a new graph G_1 such that all the vertices in $W \setminus W'$ are adjacent to one vertex (say, u_1) in U' and $\mu(G) < \mu(G_1)$. Doing the similar thing for W' in G_1 , we get a new graph G_2 such that all the vertices in $U \setminus U'$ are adjacent to one vertex (say, v_i and perhaps $i \neq 1$) in W' and $\mu(G_1) < \mu(G_2)$. In G_2, by Lemma 2.3, comparing the eigencomponents of the vertices u_1 and v_i, we get a bipartite graph G_3 satisfying $\mu(G_2) < \mu(G_3)$ such that one partition of G_3 has β vertices and the other has $n - \beta$ vertices. Note that although the matching number of G_2 might not be β, but G_3 has matching number β. Hence, we have $\mu(G) < \mu(G_1) < \mu(G_2) < \mu(G_3)$.

This contradicts the assumption that G has the maximum Laplacian spectral radius. So the claim holds.

From the above claim, we can suppose $U = U'$, $W = V \setminus U$, then $G = (U, W)$ must be complete bipartite since adding any edge between U and W would increase the Q-spectral radius of graphs. Thus we get the result. It is well known that $\mu(K_{\beta,n-\beta}) = \lambda(K_{\beta,n-\beta}) = n$. ■

3 General graphs

Lemma 3.1 *[9] Let G be a connected graph on n vertices. Suppose that $v_1, v_2, \cdots, v_s$ ($s \geq 2$) are s vertices of G, $G[v_1, v_2, \cdots, v_s] = sK_1$, namely, the s vertices induce an empty subgraph on s vertices and $N(v_1) = N(v_2) = \cdots = N(v_s)$. Let G_t be a graph obtained from G by adding any t ($0 \leq t \leq \frac{s(s-1)}{2}$) edges among $v_1, v_2, \cdots, v_s$. Then $\lambda(G) = \lambda(G_t)$.*

Let $K_{\beta,n-\beta}$ be the complete bipartite graph with bipartition (U, W)) such that $|U| = \beta$, $|W| = n - \beta$.

If $n = 2\beta$ or $2\beta + 1$, then adding any edges in U or W, the matching number of the resulting graph G_1^+ is still β. Note that $G_1^+ \subseteq K_n$, the complete graph on n vertices.

If $n \geq 2\beta + 2$, then adding any edge in U, the matching number of the resulting graph G_2^+ is still β. But adding any edge in W, the matching number of the resulting graph would be larger that β. Hence in this case we can only add edges in U. Note that $G_2^+ \subseteq K_\beta \bigvee \overline{K_{n-\beta}}$.

From Theorem 2.6, Lemma 3.1 and the above discussion, we have the following main result of this paper.

Theorem 3.2 *Let G be a connected graph with maximal Laplacian spectral radius in $\mathcal{G}_{n,\beta}$. Then we have*
(1). If $n = 2\beta$ or $2\beta + 1$, then G is of the form like G_1^+ and

$$K_{\beta,n-\beta} \subseteq G \subseteq K_n;$$

(2). If $n \geq 2\beta + 2$, then G is of the form like G_2^+ and

$$K_{\beta,n-\beta} \subseteq G \subseteq K_\beta \bigvee \overline{K_{n-\beta}}.$$

Moreover, $\lambda(G) = n$.

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