COMPLEX FACTORIZATIONS OF THE JACOBSTHAL AND JACOBSTHAL LUCAS NUMBERS

SAADET ARSLAN AND FİKRİ KÖKEN

ABSTRACT. In this paper, we present the complex factorizations of the Jacobsthal and Jacobsthal Lucas numbers by determinants of tridiagonal matrices.

1. INTRODUCTION

Over the years, many number and polynomial sequences have been defined, characterized and evaluated. They have fascinated both amateurs and professional mathematicians. They appear, often surprisingly, as answers to intricate problems, in conventional and in recreational mathematics. In this paper, we illustrate the relationships between Chebyshev polynomials and the Jacobsthal numbers and Jacobsthal Lucas numbers by considering the determinants of a sequence of tridiagonal matrices . We then obtain the complex factorizations of the Jacobsthal and Jacobsthal Lucas numbers by using these relationships.

We present the well known following lemma that is used to obtain determinants of tridiagonal matrices.

Lemma 1. *Let $\{H(n),\ n=1,2,...\}$ be a sequence of tridiagonal matrices of the form $H=[h_{ij}]_{nxn}$ $1\le i,j\le n$. Then, the successive determinants of $H(n)$ are given by the recursive formula in the* [2] *:*

$$\begin{aligned} |H(1)| &= h_{1,1}, \\ |H(2)| &= h_{1,1}h_{2,2}-h_{1,2}h_{2,1}, \\ |H(n)| &= h_{n,n}\,|H(n-1)|-h_{n-1,n}h_{n,n-1}\,|H(n-2)|. \end{aligned}$$

2. COMPLEX FACTORIZATION OF THE JACOBSTHAL NUMBERS

In order to derive the complex factorization of the Jacobsthal numbers, we introduce the sequence of matrices $\{J(n),\ n=1,2,...\}$, where $J(n)$ is

Date: January 25, 2010.

This paper is in final form and no version of it will be submitted for publication elsewhere.

ARS COMBINATORIA 96(2010), pp. 289-294

the $n \times n$ tridiagonal matrix with entries $j_{k,k} = 1$, $1 \le k \le n$, $j_{k-1,k} = i$ and $j_{k,k-1} = 2i$, $2 \le k \le n$. According to Lemma 1, the successive determinants of $J(n)$ are given by the recursive formula:

$$\begin{aligned} |J(1)| &= 1 \text{ and } |J(2)| = 1^2 - 2i^2 = 3 \\ |J(n)| &= 1.|J(n-1)| - 2i^2.|J(n-2)| = |J(n-1)| + 2|J(n-2)|. \end{aligned}$$

Obviously, this sequence is the Jacobshal sequence, starting with J_2. Hence,

$$J_{n+1} = |J(n)|, \quad n \ge 1. \tag{2.1}$$

Theorem 1. *Let J_n be n^{th} Jacobsthal number, then*

$$J_{n+1} = \prod_{k=1}^{n} (1 - 2\sqrt{2}i \cos \tfrac{\pi k}{n+1}), \quad n \ge 1.$$

Proof. The determinant of $J(n)$ can be found by taking the product of its eigenvalues. Therefore, we will compute the spectrum of $J(n)$. We now introduce another sequence of matrices $\{G(n),\ n = 1, 2, ...\}$, where $G(n)$ is the $n \times n$ tridiagonal matrix with entries

$$g_{k,k} = 0, \quad 1 \le k \le n, \ g_{k-1,k} = 1 \text{ and } g_{k,k-1} = 2, \quad 2 \le k \le n. \tag{2.2}$$

Note that $J(n) = (I + iG(n))$. Let λ_k, $k = 1, 2, ..., n$, be the eigenvalues of $G(n)$ and x_k be the eigenvectors corresponding to the eigenvalues λ_k. Then, for each k,

$$J(n)x_k = [I + iG(n)]x_k = (x_k + i\lambda_k x_k) = (1 + i\lambda_k)x_k. \tag{2.3}$$

Therefore, $\mu_k = 1 + i\lambda_k$, $k = 1, 2, ..., n$, are the eigenvalues of $J(n)$. Hence,

$$|J(n)| = \prod_{k=1}^{n} \mu_k = \prod_{k=1}^{n} (1 + i\lambda_k) \quad n \ge 1. \tag{2.4}$$

We have known that each λ_k is a zero the characteristic polynomial $p_n(\lambda) = |G(n) - \lambda I|$, thus we have to determine roots of the characteristic polynomial $p_n(\lambda)$. Since $(G(n) - \lambda I\ )$ is a tridiagonal matrix,

$$|G(n) - \lambda I| = 0, \tag{2.5}$$

we use Lemma 1 to establish a recursive formula for the characteristic polynomials of $\{G(n),\ n = 1, 2, ...n\}$:

$$\begin{aligned} p_1(\lambda) &= -\lambda \text{ and } p_2(\lambda) = \lambda^2 - 2 \\ p_n(\lambda) &= -\lambda p_{n-1}(\lambda) - 2p_{n-2}(\lambda). \end{aligned}$$

This family of characteristic polynomials can be transformed into another family $\left\{2^{\frac{n}{2}} U_n(x),\ n \ge 1\right\}$ by the transformation $\lambda \equiv -2\sqrt{2}x$:

$$\begin{aligned} p_1(-2\sqrt{2}x) &= 2^{\frac{1}{2}} 2x = 2^{\frac{1}{2}} U_1(x), \\ p_2(-2\sqrt{2}x) &= 2^{\frac{2}{2}} \left(4x^2 - 1\right) = 2^{\frac{2}{2}} U_2(x), \\ p_n(-2\sqrt{2}x) &= 2^{\frac{n}{2}}.(2xU_{n-1}(x) - U_{n-2}(x)) = 2^{\frac{n}{2}} U_n(x) \end{aligned} \tag{2.6}$$

The family $\{U_n(x),\ n \geq 1\}$ is the set of Chebyshev polynomials of the second kind. It is a well-known fact that defining $x \equiv \cos\theta$ allows the Chebyshev polynomials of the second kind to be written as [1]:

$$U_n(x) = \frac{\sin[(n+1)\theta]}{\sin\theta}. \tag{2.7}$$

From (2.6) it follows that roots of the polynomial $p_n(\lambda)$ can be obtained from the $U_n(x)$. Therefore, we can easily derive from (2.7) that the roots of $U_n(x) = 0$ are given by $\theta_k = \frac{\pi k}{n+1}$, $k = 1, 2, ...n$ or $x_k = \cos\theta_k = \cos\frac{\pi k}{n+1}$, $k = 1, 2, ...n$ [1]. Applying the transformation $\lambda \equiv -2\sqrt{2}x$, we now have the eigenvalues of $G(n)$:

$$\lambda_k = -2\sqrt{2}\cos\tfrac{\pi k}{n+1}, \quad k = 1, 2, ..., n. \tag{2.8}$$

From (2.3), we have known that $\mu_k = 1 + i\lambda_k$, $k = 1, 2, ..., n$, are the eigenvalues of $J(n)$ .Combining (2.1), (2.4), and (2.8), we obtain the following identity:

$$J_{n+1} = |J(n)| = \prod_{k=1}^{n}(1 - 2\sqrt{2}i\cos\tfrac{\pi k}{n+1}), \quad n \geq 1.$$

This completes the proof of the complex factorization of the Jacobsthal numbers. □

Theorem 2. *Let J_n be n^{th} Jacobsthal number, then*

$$J_{n+1} = \left(\sqrt{2}i\right)^n \frac{\sin\left((n+1)\cos^{-1}(\frac{-i}{2\sqrt{2}})\right)}{\sin(\cos^{-1}(\frac{-i}{2\sqrt{2}}))}, \quad n \geq 0.$$

Proof. From (2.5), we can see that Chebyshev polynomials of the second kind are generated by determinants of successive matrices of the $n \times n$ tridiagonal matrix $A(n, x)$ with entries

$$a_{k,k} = 2\sqrt{2}x,\ 1 \leq k \leq n,\ a_{k-1,k} = 1 \text{ and } a_{k,k-1} = 2,\ \ 1 \leq k \leq n.$$

If we note that $J(n) = i.A(n, \frac{-i}{2\sqrt{2}})$, then we have:

$$|J(n)| = i^n.\left|A(n, \tfrac{-i}{2\sqrt{2}})\right| = \left(\sqrt{2}i\right)^n U_n(\tfrac{-i}{2\sqrt{2}}). \tag{2.9}$$

Combining (2.1), (2.7) and (2.9) yields

$$J_{n+1} = \left(\sqrt{2}i\right)^n \frac{\sin\left((n+1)\cos^{-1}(\frac{-i}{2\sqrt{2}})\right)}{\sin(\cos^{-1}(\frac{-i}{2\sqrt{2}}))}, \quad n \geq 1.$$

Furthermore, the complex factorization of the Jacobsthal numbers yields for $n = 0$, i.e.,

$$J_1 = 1 = \left(\sqrt{2}i\right)^0 \frac{\sin\left((0+1)\cos^{-1}(\frac{-i}{2\sqrt{2}})\right)}{\sin(\cos^{-1}(\frac{-i}{2\sqrt{2}}))}.$$

□

3. COMPLEX FACTORIZATION OF THE JACOBSTHAL LUCAS NUMBERS

The process shown in proof of Theorem 1 can be applied to obtain the complex factorization of the Jacobsthal Lucas numbers. Firstly, we consider the sequence of matrices $\{j(n),\ n = 1, 2, ...n\}$, where $j(n)$ is the $n \times n$ tridiagonal matrix with entries

$$j_{1,1} = \frac{1}{2},\ j_{k,k} = 1,\ j_{k-1,k} = i \text{ and } j_{k,k-1} = 2i, 2 \le k \le n.$$

According to Lemma 1, the successive determinants of $j(n)$ are given by recursive formula:

$$\begin{aligned} |j(1)| &= \frac{1}{2}, \text{ and } |j(2)| = \frac{1}{2} - 2i^2 = \frac{5}{2} \\ |j(n)| &= 1.\,|j(n-1)| - 2i^2.\,|j(n-2)| = |j(n-1)| + 2\,|j(n-2)|\,. \end{aligned}$$

Clearly, each number in this sequence is half of the corresponding Jacobsthal Lucas number. We have

$$j_n = 2.\,|j(n)|\,,\quad n \ge 1. \tag{3.1}$$

Theorem 3. *Let j_n be n^{th} Jacobsthal Lucas number, then,*

$$j_n = \prod_{k=1}^{n} (1 - 2\sqrt{2}i \cos \tfrac{\pi(k-\frac{1}{2})}{n}),\ n \ge 1.$$

Proof. We will consider that determinant of $j(n)$ can be written following :

$$|j(n)| = \frac{1}{2}\left|(I + e_1 e_1^T)\, j(n)\right| \tag{3.2}$$

where e_j is the j^{th} column of the identity matrix. Furthermore, we can express the right-hand side of (3.2) in the following way:

$$\frac{1}{2}\left|(I + e_1 e_1^T)\, j(n)\right| = \frac{1}{2}\left|[I + i(G(n) + e_1 e_2^T)]\right|,$$

where $G(n)$ is the matrix given in (2.2). Let $\gamma_k, k = 1, 2, ..., n$ be the eigenvalues of $[G(n) + e_1 e_2^T]$ and y_k be the eigenvectors corresponding to the eigenvalues γ_k. Then, for each k,

$$[I + i(G(n) + e_1 e_2^T)]y_k = Iy_k + i[G(n) + e_1 e_2^T]y_k = (y_k + i\gamma_k y_k) = (1 + i\gamma_k)y_k.$$

Therefore,

$$\frac{1}{2}\left|I + i(G(n) + e_1 e_2^T)\right| = \frac{1}{2}\prod_{k=1}^{n} (1 + i\gamma_k). \tag{3.3}$$

In order to determine the γ_k, we recall that each γ_k is a zero of the characteristic polynomial $q_n(\gamma) = \left|G(n) + e_1 e_2^T - \gamma I\right|$. Since $\left|I - \frac{1}{2} e_1 e_1^T\right| = \frac{1}{2}$, we can alternately represent the characteristic polynomial as

$$q_n(\gamma) = 2\left|(I - \frac{1}{2}e_1e_1^T)\left(G(n) + e_1e_2^T - \gamma I\right)\right|. \tag{3.4}$$

Since $q_n(\gamma)$ is twice the determinant of a tridiagonal matrix, i.e.,

$$\frac{1}{2}q_n(\gamma) = \left|(I - \frac{1}{2}e_1e_1^T)\left(G(n) + e_1e_2^T - \gamma I\right)\right| \tag{3.5}$$

we can use Lemma 1 to establish a recursive formula for $\frac{1}{2}q_n(\gamma)$:

$$\begin{aligned} \tfrac{1}{2}q_1(\gamma) &= -\tfrac{\gamma}{2} \text{ and } \frac{1}{2}q_2(\gamma) = \tfrac{\gamma}{2}^2 - 2 \\ \tfrac{1}{2}q_n(\gamma) &= -\gamma q_{n-1}(\gamma) - 2q_{n-2}(\gamma). \end{aligned}$$

The family $\{\frac{1}{2}q_n(\gamma), n \geq 1\}$ of polynomials can be transformed into another family $\left\{2^{\frac{n}{2}}T_n(x),\ n \geq 1\right\}$ by the transformation $\gamma \equiv -2\sqrt{2}x$:

$$\begin{aligned} \frac{1}{2}q_1(-2\sqrt{2}x) &= 2^{\frac{1}{2}}x = 2^{\frac{1}{2}}T_1(x), \\ \frac{1}{2}q_2(-2\sqrt{2}x) &= 2^{\frac{2}{2}}\left(2x^2 - 1\right) = 2^{\frac{2}{2}}T_2(x), \\ \frac{1}{2}q_n(-2\sqrt{2}x) &= 2^{\frac{n}{2}}.\left(2xT_{n-1}(x) - T_{n-2}(x)\right) = 2^{\frac{n}{2}}T_n(x) \end{aligned} \tag{3.6}$$

The family $\{T_n(x),\ n \geq 1\}$ is the set of Chebyshev polynomials of the first kind. It is a fact that defining $x \equiv \cos\theta$ allows the Chebyshev polynomials of the first kind to be written as [1]:

$$T_n(x) = \cos n\theta. \tag{3.7}$$

From (3.7), we can see that the roots of $T_n(x) = 0$ are given by

$$\theta_k = \tfrac{\pi(k-\frac{1}{2})}{n}, \text{ or } x_k = \cos\theta_k = \cos\tfrac{\pi(k-\frac{1}{2})}{n},\quad k = 1, 2, ..., n \text{ [1]}.$$

From (3.6), it follows that roots of the polynomial $q_n(\gamma)$ can be obtained from $T_n(x)$. Applying the transformation $\gamma = -2\sqrt{2}x$ and considering that the roots of (3.4) are also roots of $\left|G(n) + e_1e_2^T - \gamma I\right| = 0$, we now have the eigenvalues of $[G(n) + e_1e_2^T]$:

$$\gamma_k = -2\sqrt{2}\cos\tfrac{\pi(k-\frac{1}{2})}{n},\quad k = 1, 2, ..., n. \tag{3.8}$$

From (3.1), (3.3) and (3.8), we get

$$j_n = \prod_{k=1}^{n}(1 - 2\sqrt{2}i\cos\tfrac{\pi(k-\frac{1}{2})}{n}),\ n \geq 1,$$

which is identical to the complex factorization of the Jacobsthal Lucas numbers. □

Theorem 4. *Let j_n be n^{th} Jacobsthal Lucas number, then,*

$$j_n = 2^{\frac{n+2}{2}}.i^n \cos\left(n\cos^{-1}(\tfrac{-i}{2\sqrt{2}})\right), \quad n \geq 0.$$

Proof. From (3.5), we can think of Chebyshev polynomials of the first kind as being generated by determinants of successive matrices of the $n \times n$ tridiagonal matrix $B(n,x)$ with entries

$$b_{k,k} = 2\sqrt{2}x, \ 1 \leq k \leq n, \ b_{k-1,k} = 1 \text{ and } b_{k,k-1} = 2, \quad 2 \leq k \leq n.$$

If we note that $j(n) = iB(n, \frac{-i}{2\sqrt{2}})$, then we have

$$|j(n)| = i^n \left|B(n, \tfrac{-i}{2\sqrt{2}})\right| = i^n T_n(\tfrac{-i}{2\sqrt{2}}). \tag{3.9}$$

Using (3.1), (3.7) and (3.9), we write following identity,

$$j_n = 2^{\frac{n+2}{2}}.i^n \cos\left(n\cos^{-1}(\tfrac{-i}{2\sqrt{2}})\right), \quad n \geq 1.$$

Furthermore, the last identity yields for $n = 0$, i.e.,

$$j_0 = 2 = 2^{\frac{0+2}{2}}.i^0 \cos\left(0.\cos^{-1}(\tfrac{-i}{2\sqrt{2}})\right).$$

Thus, the proof is completed. □

REFERENCES

[1] J. C. Mason and D. C. Handscomb, Chebyshev Polynomials, Chapman and Hall, CRC, New York, 2003.

[2] N. D. Cahill, J. R. D'Ericco and J.P. Sipense, Fibonacci and Lucas numbers as Tridiagonal Matrix Determinants, The Fibonacci Quarterly 42(3) (2003), 216-221

AMS Classifications Numbers: 11B39, 11C20, 33C52

SELÇUK UNIVERSITY, FACULTY OF EDUCATION, DEPARTMENT OF MATHEMATICS, 42090 MERAM, KONYA, TURKEY

E-mail address: sarslan@selcuk.edu.tr

SELÇUK UNIVERSITY, FACULTY OF SCIENCE, DEPARTMENT OF MATHEMATICS, 42075 KAMPUS, KONYA, TURKEY

E-mail address: fkoken@selcuk.edu.tr