Number of disjoint 5-cycles in graphs

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Abstract

Let $k \geq 1$, $l \geq 3$ and $s \geq 5$ be integers. In 1990, Erdös and Faudree conjectured that if G is a graph of order $4k$ with $\delta(G) \geq 2k$, then G contains k vertex-disjoint 4-cycles. In this paper, we consider an analogous question for 5-cycles; that is to say if G is a graph of order $5k$ with $\delta(G) \geq 3k$, then G contains k vertex-disjoint 5-cycles? In support of this question, we prove that if G is a graph of order $5l$ with $\sigma_2(G) \geq 6l - 2$, then, unless $\overline{K_{l-2}} + K_{2l+1,2l+1} \subseteq G \subseteq K_{l-2} + K_{2l+1,2l+1}$, G contains $l-1$ vertex-disjoint 5-cycles and a path of order 5, which is vertex-disjoint from the $l-1$ 5-cycles. In fact, we prove a more general result that if G is a graph of order $5k+2s$ with $\sigma_2(G) \geq 6k+2s$, then, unless $\overline{K_k} + K_{2k+s,2k+s} \subseteq G \subseteq K_k + K_{2k+s,2k+s}$, G contains $k+1$ vertex-disjoint 5-cycles and a path of order $2s-5$, which is vertex-disjoint from the $k+1$ 5-cycles. As an application of this theorem, we give a short proof for determining the exact value of $ex(n,(k+1)C_5)$, and characterize the extremal graph.

1 Introduction

We consider only undirected, finite and simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, where $e(G)$ denotes $|E(G)|$. For $v \in V(G)$, the degree of v in G is denoted by $d_G(v)$ (or simply by $d(v)$). We define $\sigma_2(G)$ to be the minimum of the sum of the degrees of two non-adjacent vertices in G, i.e., $\sigma_2(G) = \min\{d(x)+d(y) \mid x, y \in V(G), x \neq y, xy \notin E(G)\}$. In the case where $G \cong K_n$, we take $\sigma_2(G) = \infty$. The minimum degree of G is denoted by $\delta(G)$. For graphs G_1 and G_2 with $V(G_1) \cap V(G_2) = \emptyset$, $G_1 + G_2$ denotes the join of G_1 and G_2, i.e., $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1+G_2) = E(G_1) \cup E(G_2) \cup \{u_1u_2 \mid u_1 \in V(G_1), u_2 \in V(G_2)\}$. Further we let $G_1 \cup G_2$ denote the union of G_1 and G_2, i.e., $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$

(whenever we use the notation $G_1 + G_2$ or $G_1 \cup G_2$, it is assumed that $V(G_1) \cap V(G_2) = \emptyset$). For a graph G and an integer $k \geq 1$, kG denotes the graph consisting of k vertex-disjoint copies of G; thus $kG = G_1 \cup \cdots \cup G_k$, where $G_i \cong G$ for each $1 \leq i \leq k$. We let $K_{n_1,n_2,\cdots,n_k}$ denote the complete k-partite graph with color classes of sizes $n_1, n_2, \cdots, n_k$, and let C_m and P_m denote the cycle of order m and the path of order m, respectively.

The following conjecture is well-known.

Conjecture A (El-Zahar [8]). Let n, l be integers with $n \geq 3l$ and $l \geq 1$, and write $n = n_1 + n_2 + \cdots + n_l$, where $n_i \geq 3$ for all $1 \leq i \leq l$. Let G be a graph of order n, and suppose that $\delta(G) \geq \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2}{2} \rceil + \cdots + \lceil \frac{n_l}{2} \rceil$. Then $G \supseteq C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_l}$.

El-Zahar [8] proved Conjecture A for $l = 2$, and Dirac's theorem [6] corresponds to the case $l = 1$. In 1998, Abbasi [1] proved that Conjecture A holds for graphs with sufficiently large order. It is a difficult and an interesting question to exclude the word "sufficiently large" from Abbasi's result. On the other hand, Wang [13] proved that if G is a graph of order $n \geq 3l \geq 6$ with $\delta(G) \geq \frac{n+l-1}{2}$, then $G \supseteq (l-1)C_3 \cup C_{n-3(l-1)}$. This in particular implies that Conjecture A holds in the case where $n_i = 3$ for all $1 \leq i \leq l$. The cases where $n_i = 4$ for all i and $n_i = 5$ for all i of Conjecture A can be stated in the following forms.

Conjecture B (Erdös and Faudree). Let $l \geq 1$ be an integer, and let G be a graph of order $4l$ with $\delta(G) \geq 2l$. Then $G \supseteq lC_4$.

Conjecture C. Let $l \geq 1$ be an integer, and let G be a graph of order $5l$ with $\delta(G) \geq 3l$. Then $G \supseteq lC_5$.

Conjecture B had already been mentioned in Erdös and Faudree [9]. In connection with Conjecture B, Johansson [10] and Randerath etc. [11] independently proved that if G is a graph of order $4l \geq 8$ with $\delta(G) \geq 2l$, then $G \supseteq (l-1)C_4 \cup P_4$. In this paper, we prove the following result.

Theorem 1.1. *Let $k \geq 1, s \geq 5$ be integers, and let G be a graph of order $5k + 2s$ with $\sigma_2(G) \geq 6k + 2s = |V(G)| + k$. Then one of the following holds:*

(i) $\overline{K_k} + K_{2k+s,2k+s} \subseteq G \subseteq K_k + K_{2k+s,2k+s}$; *or*

(ii) $G \supseteq (k+1)C_5 \cup P_{2s-5}$.

If we let $k = l-2$ and $s = 5$ in Theorem 1.1, then we obtain the following result in connection with Conjecture C.

Corollary 1.2. *Let $l \geq 3$ be an integer, and let G be a graph of order $5l$ with $\sigma_2(G) \geq 6l-2$. Then one of the following holds:*

(i) $\overline{K_{l-2}} + K_{2l+1,2l+1} \subseteq G \subseteq K_{l-2} + K_{2l+1,2l+1}$; *or*

(ii) $G \supseteq (l-1)C_5 \cup P_5$.

For a graph H and an integer n, the Turán number $ex(n, H)$ is the maximum possible number of edges in a simple graph of order n that contains no copy of H. The Turán graph $T_r(n)$ stands for the complete r-partite graph of order n whose color classes are as equal as possible. For any two integers $n \geq r \geq 1$, Turán's theorem states that $ex(n, K_{r+1}) = e(T_r(n))$, with equality only for $T_r(n)$. In 1968, Simonovits [12] extended Turán's theorem for graphs of sufficiently large order as follows. For any $r \geq 1$, $t \geq 1$ and $m \geq 2rt$ and n sufficiently large (at least as large as exponential in m), the Turán number $ex(n, T_r^t(m))$ is equal to $e(T_r(n))$ when $t = 1$ and $e(K_{t-1} + T_r(n-t+1))$ when $t \geq 2$, with equality only for $T_r(n)$ when $t = 1$ and $K_{t-1} + T_r(n-t+1)$ when $t \geq 2$, where $T_r^t(m)$ is a graph obtained from $T_r(m)$ by adding t independent edges to the same smallest color class of it. As an application of Theorem 1.1, we give a short proof of determining the exact value of $ex(n, (k+1)C_5)$ for all sufficiently large n, where $k \geq 1$. We remark that the above theorem of Simonovits is much stronger than our result for sufficiently large order graphs, since $T_2^{k+1}(6k+6) \supseteq (k+1)C_5$ and $K_k + T_2(n-k) \not\supseteq (k+1)C_5$. However, our result has an advantage that it significantly improves the minimum n for which the conclusion concerning 5-cycles holds. A precise statement is the following.

Corollary 1.3. *Let $k \geq 1$ and $n \geq 8k^2+17k+10$. Then $ex(n, (k+1)C_5) = e(K_k + T_2(n-k))$, with equality only when $K_k + T_2(n-k)$.*

Our notation is standard and taken from [4]. Possible exceptions are as follows. Let G be a graph. The neighborhood of a vertex $v \in V(G)$ is denoted by $N_G(v)$ (or simply by $N(v)$); thus $d_G(v) = |N_G(v)|$. For a subset A of $V(G)$, we define $N_G(v, A) = N_G(v) \cap A$, and set $d_G(v, A) = |N_G(v, A)|$. When there is no danger of confusion, we write $N(v, A)$ and $d(v, A)$ for $N_G(v, A)$ and $d_G(v, A)$, respectively. For $A, B \subset V(G)$, we denote by $E(A, B)$ the set of edges of G with one endvertex in A and the other in B, and let $e(A, B) = |E(A, B)|$. For a subset S of $V(G)$, $G[S]$ and $G - S$ denote the subgraph induced by S and $V(G) - S$, respectively. We denote by $e(S)$ the cardinality of $E(G[S])$. A subgraph of G is often identified with its vertex set. For example, $N_G(v, A)$ means $N_G(v, V(A))$ for a subgraph A of G and a vertex v of G, $E(A, B)$ means $E(V(A), B)$ for a subgraph A of G and a subset B of $V(G)$, etc. A cycle of order m is referred to as an m-cycle for short. Let $C = c_1c_2 \cdots c_mc_1$ be an m-cycle. We set $c_i^+ = c_{i+1}$, $c_i^{++} = c_{i+2}$, $c_i^- = c_{i-1}$ and $c_i^{--} = c_{i-2}$ (whenever we represent an m-cycle in the form $c_1c_2\cdots c_mc_1$, indices are to be read modulo m). We use similar notations for a path $P = p_1p_2\cdots p_k$ (except that indices are not to be read modulo k). For a cycle C of G and for $v \in V(G)$, we also introduce the following additional notations:

$$N'(v,C) = \{u \in V(C) \mid \{u^+, u^-\} \subseteq N(v,C)\},$$
$$N^{\pm}(v,C) = \{u^+, u^- \mid u \in N(v,C)\},$$
$$N^{\pm\pm}(v,C) = \{u^{++}, u^{--} \mid u \in N(v,C)\}.$$

Finally if V,X,Y are sets such that $V = X \cup Y$ and $X \cap Y = \emptyset$, then we write $V = X \dot{\cup} Y$.

2 Preliminaries

The main result of this section is Lemma 2.6. We start with simple claims. The first claim, Claim 2.1, follows immediately from the definition of $N'(v,C)$ (see Figure 1 for pictorial descriptions of (ii) and (iii) of Claim 2.1).

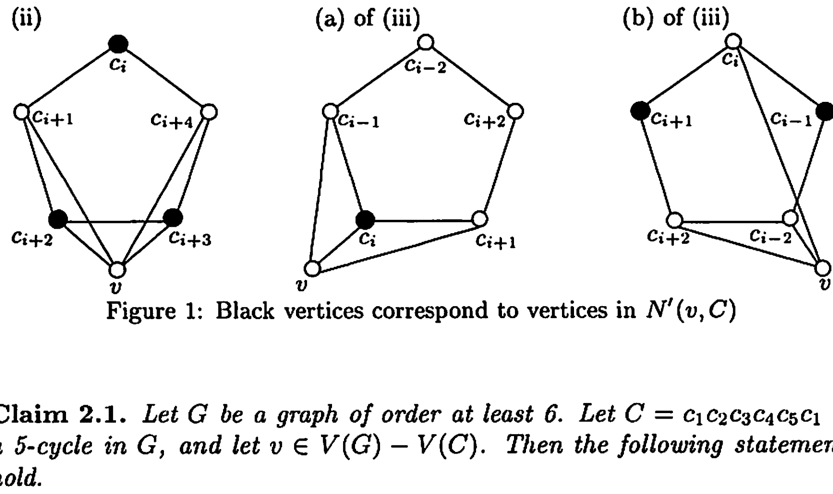

Figure 1: Black vertices correspond to vertices in $N'(v,C)$

Claim 2.1. *Let G be a graph of order at least 6. Let $C = c_1c_2c_3c_4c_5c_1$ be a 5-cycle in G, and let $v \in V(G) - V(C)$. Then the following statements hold.*

(i) *If $|N(v,C)| = 5$, then $|N'(v,C)| = 5$.*

(ii) *If $|N(v,C)| = 4$, then $|N'(v,C)| = 3$.*

(iii) *If $|N(v,C)| = 3$, then $1 \le |N'(v,C)| \le 2$, and the following hold.*

- (a) *If $|N'(v,C)| = 1$, then there exists i such that $N(v,C) = \{c_{i-1}, c_i, c_{i+1}\}$.*

(b) *If* $|N'(v,C)| = 2$, *then there exists* i *such that* $N(v,C) = \{c_{i-2}, c_i, c_{i+2}\}$.

(iv) *If* $|N(v,C)| = 2$, *then* $|N'(v,C)| \leq 1$, *and the following hold.*

(a) *If* $|N'(v,C)| = 0$, *then there exists* i *such that* $N(v,C) = \{c_i, c_{i+1}\}$.

(b) *If* $|N'(v,C)| = 1$, *then there exists* i *such that* $N(v,C) = \{c_{i-1}, c_{i+1}\}$.

□

Claim 2.2. *The following three statements hold.*

(i) *Let* A, C *be subgraphs of a graph* G *such that* $V(G) = V(A) \dot\cup V(C)$, $A \cong 2K_1$ *and* $C \cong C_5$. *Write* $V(A) = \{a_1, a_2\}$ *and* $C = c_1c_2c_3c_4c_5c_1$, *and suppose that* $e(A, C) \geq 7$. *Suppose further that* $N(a_1, C) \cap N'(a_2, C) = \emptyset$. *Then* $N'(a_1, C) \cap N(a_2, C) \neq \emptyset$.

(ii) *Let* A, C *be subgraphs of a graph* G *such that* $V(G) = V(A) \dot\cup V(C)$, $A \cong 2K_1$ *and* $C \cong C_5$, *and suppose that* $e(A, C) \geq 7$. *Then* $G \supseteq C_5 \cup K_2$.

(iii) *Let* G *be a graph of order* 4. *Write* $V(G) = A \dot\cup B$ *with* $|A| = |B| = 2$, *and suppose that* $e(A, B) \geq 3$. *Then* $G \supseteq 2K_2$.

Proof. (i) From $N(a_1,C) \cap N'(a_2,C) = \emptyset$, we get $d(a_1,C) + |N'(a_2,C)| \leq 5$. We also have $d(a_1,C) + d(a_2,C) \geq 7$ by assumption. Hence by Claim 2.1, we have either $d(a_1,C) = 5$ and $d(a_2,C) = 2$, or $d(a_1,C) = 4$ and $d(a_2,C) = 3$ (see Figure 2). Then $|N'(a_1,C) \cap N(a_2,C)| \geq \min\{5+2-5, 3+3-5\} > 0$, and hence $N'(a_1,C) \cap N(a_2,C) \neq \emptyset$.

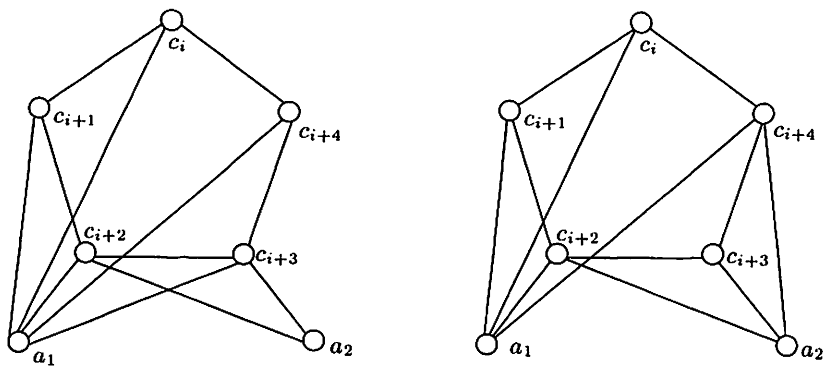

Figure 2: Two cases such that $N(a_1,C) \cap N'(a_2,C) = \emptyset$

(ii) Write $V(A) = \{a_1, a_2\}$. Then by (i), either $N(a_1,C) \cap N'(a_2,C) \neq \emptyset$ or $N'(a_1,C) \cap N(a_2,C) \neq \emptyset$ holds, which implies the desired conclusion.

(iii) From $e(A,B) \geq 3$, we get $G \supseteq P_4$, and hence $G \supseteq 2K_2$. □

Claim 2.3. *Let A, C be subgraphs of a graph G such that $V(G) = V(A) \dot{\cup} V(C)$, $A \cong K_2$ and $C \cong C_5$, and write $V(A) = \{a_1, a_2\}$ and $C = c_1c_2c_3c_4c_5c_1$.*

(i) *If $e(A,C) \geq 5$, $d(a_1,C) > 0$ and $d(a_2,C) > 0$, then there exists $i \in \{1, 2, 3, 4, 5\}$ such that $G[\{a_1, a_2, c_i, c_{i+1}, c_{i+2}\}] \supseteq C_5$.*

(ii) *If $e(A,C) \geq 7$, then there exists $i \in \{1, 2, 3, 4, 5\}$ such that $G[\{c_i, c_{i+1}, c_{i+2}, a_1, a_2\}] \supseteq C_5$ and $G[\{c_{i-2}, c_{i-1}, c_i, a_1, a_2\}] \supseteq C_5$.*

Proof. (i) We may assume that $d(a_1, C) \geq d(a_2, C)$, so $d(a_1,C) \geq 3$. It suffices to show that $N^{\pm\pm}(a_1, C) \cap N(a_2,C) \neq \emptyset$. Now if $d(a_1,C) = 3$, then $|N^{\pm\pm}(a_1,C)| \geq 4$, and hence $|N^{\pm\pm}(a_1,C) \cap N(a_2,C)| \geq (4+2) - 5 > 0$; if $d(a_1,C) \geq 4$, then $|N^{\pm\pm}(a_1,C)| = 5$, and hence $|N^{\pm\pm}(a_1,C) \cap N(a_2,C)| \geq (5+1) - 5 > 0$.

(ii) We may assume that $d(a_1, C) \geq d(a_2, C)$, so $d(a_1,C) \geq 4$. It suffices to show that there exists $x \in N(a_2,C)$ such that $\{x^{++}, x^{--}\} \subseteq N(a_1,C)$. If $d(a_1,C) = 5$, then any $x \in N(a_2,C)$ will do. Thus we may assume $d(a_1,C) = 4$. Write $V(C) - N(a_1,C) = \{v\}$. Since $d(a_2,C) \geq 3$, we have $|\{v^-, v, v^+\} \cap N(a_2,C)| \geq (3+3) - 5 > 0$. Take $x \in \{v^-, v, v^+\} \cap N(a_2,C)$. Then $\{x^{++}, x^{--}\} \subseteq V(C) - \{v\} = N(a_1,C)$, and hence x has the desired property. □

Claim 2.4. *Let $m \geq 3$, and let A, P, C be subgraphs of a graph G such that $V(G) = V(A) \dot{\cup} V(P) \dot{\cup} V(C)$, $A \cong K_2, P \cong P_{m-1}, C \cong C_5$. Write $P = p_1p_2 \cdots p_{m-1}$, and suppose that $e(V(A) \cup \{p_1, p_{m-1}\}, C) \geq 13$. Then $G \supseteq C_5 \cup P_{m+1}$.*

Proof. Write $V(A) = \{a_1, a_2\}$ and $C = c_1c_2c_3c_4c_5c_1$. We may assume that $d(a_1,C) \geq d(a_2,C)$ and $d(p_1,C) \geq d(p_{m-1},C)$. We divide the proof into three cases according to the value of $e(A,C)$.

Case 1. $7 \leq e(A,C) \leq 10$.
By (ii) of Claim 2.3, there exists i such that $G[\{c_i, c_{i+1}, c_{i+2}, a_1, a_2\}] \supseteq C_5$ and $G[\{c_{i-2}, c_{i-1}, c_i, a_1, a_2\}] \supseteq C_5$. Since $e(\{p_1, p_{m-1}\}, C) \geq 13 - e(A,C) \geq 3$, we have $d(p_1,C) \geq 2$. Hence $N(p_1,C) \cap \{c_{i+1}, c_{i+2}\} \neq \emptyset$ or $N(p_1,C) \cap \{c_{i-1}, c_{i-2}\} \neq \emptyset$. In the former case, $G[\{c_{i+1}, c_{i+2}\} \cup V(P)]$ contains a path of order $m+1$ which, together with the 5-cycle in

$G[\{c_{i-2}, c_{i-1}, c_i, a_1, a_2\}]$, forms a spanning subgraph of G having the desired properties, and we can similarly find a desired spanning subgraph in the latter case.

Case 2. $5 \le e(A, C) \le 6$.

Subcase 2.1. $d(a_1, C) = 5$ and $d(a_2, C) = 0$.
We have either $d(p_1, C) = 4$ and $d(p_{m-1}, C) = 4$, or $d(p_1, C) = 5$ and $d(p_{m-1}, C) \ge 3$. Hence by (i) and (ii) of Claim 2.1, $|N'(p_1, C) \cap N(p_{m-1}, C) \cap N(a_1, C)| \ge \min\{(3+4+5) - 2 \cdot 5, (5+3+5) - 2 \cdot 5\} > 0$. We may assume $c_1 \in N'(p_1, C) \cap N(p_{m-1}, C) \cap N(a_1, C)$. Then $p_1 c_2 c_3 c_4 c_5 p_1$ and $p_2 p_3 \cdots p_{m-1} c_1 a_1 a_2$ are subgraphs with the desired properties.

Subcase 2.2. Otherwise.
By (i) of Claim 2.3, there exists i such that $G[\{a_1, a_2, c_i, c_{i+1}, c_{i+2}\}] \supseteq C_5$. Since $d(p_1, C) \ge 4$, $N(p_1, C) \cap \{c_{i-1}, c_{i-2}\} \ne \emptyset$. Hence $G[\{c_{i-1}, c_{i-2}\} \cup V(P)]$ contains a path of order $m+1$ which, together with the 5-cycle in $G[\{a_1, a_2, c_i, c_{i+1}, c_{i+2}\}]$, forms a desired spanning subgraph.

Case 3. $3 \le e(A, C) \le 4$.
We have $d(p_1, C) = 5$, $d(p_{m-1}, C) \ge 4$ and $d(a_1, C) \ge 2$, and hence $|N'(p_1, C) \cap N(p_{m-1}, C) \cap N(a_1, C)| \ge (5+4+2) - 10 > 0$. Therefore arguing as in Subcase 2.1, we can find desired subgraphs. □

Claim 2.5. *Let $m \ge 1$ and $l \ge 1$, and let P, Q be subgraphs of a graph G such that $V(G) = V(P) \dot\cup V(Q)$, $P \cong P_m$ and $Q \cong P_l$. Write $P = p_1 p_2 \cdots p_m$ and $Q = q_1 q_2 \cdots q_l$, and suppose that $d(p_1, P) + d(q_1, P) \ge m$. Then $G \supseteq P_{m+l}$.*

Proof. Set $N^-(p_1, P) = \{v^- \mid v \in N(p_1, P)\}$. If $q_1 p_m \in E(G)$, then $p_1 p_2 \cdots p_m q_1 q_2 \cdots q_l$ is a path with the desired property. Thus we may assume that $N(q_1, P) \subseteq \{p_1, p_2, \cdots, p_{m-1}\}$. Then since $N^-(p_1, P) \subseteq \{p_1, p_2, \cdots, p_{m-1}\}$, and since $|N^-(p_1, P)| + d(q_1, P) = d(p_1, P) + d(q_1, P) \ge m$ by assumption, we get $N^-(p_1, P) \cap N(q_1, P) \ne \emptyset$. Take $p_i \in N^-(p_1, P) \cap N(q_1, P)$. Then $q_l q_{l-1} \cdots q_1 p_i p_{i-1} \cdots p_1 p_{i+1} p_{i+2} \cdots p_m$ is a desired path (see bold edges in Figure 3). □

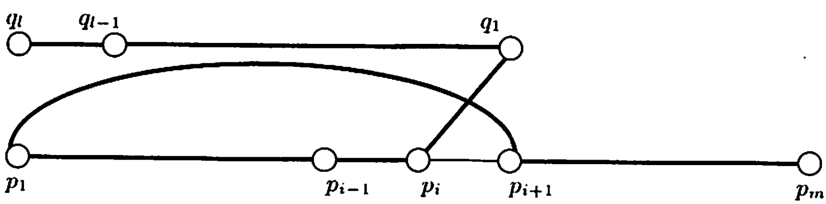

Figure 3: $q_l q_{l-1} \ldots q_1 p_i p_{i-1} \ldots p_1 p_{i+1} \ldots p_m$ is a path of order $m+l$

We now prove the main lemma of this section.

Lemma 2.6. *Let $k \geq 1$ and $s \geq 5$, and let G be a graph with $|V(G)| = 5k + 2s$ and $\sigma_2(G) \geq 6k + 2s$. Then the following two statements hold.*

(i) *If $G \supseteq kC_5$, then $G \supseteq kC_5 \cup P_{2s}$.*

(ii) *If $G \supseteq (k+1)C_5$, then $G \supseteq (k+1)C_5 \cup P_{2s-5}$.*

Proof. We prove (i) and (ii) simultaneously. Let $\epsilon \in \{0, 1\}$ and $g(s,\epsilon) = \lfloor \frac{2s-5\epsilon}{2} \rfloor$. Note that $g(s,\epsilon) \geq 2$. Suppose that $G \supseteq (k+\epsilon)C_5$, and let $\mathcal{C} = \{C^1, C^2, \cdots, C^{k+\epsilon}\}$ be a collection of $k+\epsilon$ vertex-disjoint 5-cycles in G. First we show that $G \supseteq (k+\epsilon)C_5 \cup g(s,\epsilon)K_2$. For this purpose, we suppose that we have chosen $\mathcal{C}$ so that the maximum number of independent edges in $G - \cup_{i=1}^{k+\epsilon} V(C^i)$ is as large as possible, and let $M = \{e_1, e_2, \cdots, e_{g'(s,\epsilon)}\}$ be a maximum collection of independent edges in $G - \cup_{i=1}^{k+\epsilon} V(C^i)$. What we want to show is $g'(s,\epsilon) = g(s,\epsilon)$. By way of contradiction, suppose that $g'(s,\epsilon) < g(s,\epsilon)$. Set $L = V(G) - \cup_{i=1}^{k+\epsilon} V(C^i) - \cup_{i=1}^{g'(s,\epsilon)} V(e_i)$. Note that $|L| = (2s - 5\epsilon) - 2g'(s,\epsilon) \geq 2$. By the maximality of M, we have $e(L) = 0$, which implies that $\sum_{v \in L} d(v) \geq \frac{|L|}{2}\sigma_2(G)$. On the other hand, by (ii) and (iii) of Claim 2.2,

$$\begin{aligned}\sum_{v\in L} d(v) &= \{\sum_{v\in L} d(v, \cup_{i=1}^{k+\epsilon} V(C^i)) + d(v, \cup_{i=1}^{g'(s,\epsilon)} V(e_i))\} + 2e(L)\\ &\leq \frac{|L|}{2}\{6(k+\epsilon) + 2g'(s,\epsilon)\} + 0\\ &< \frac{|L|}{2}(6k+2s)\\ &\leq \frac{|L|}{2}\sigma_2(G),\end{aligned}$$

which is a contradiction. Thus $g'(s,\epsilon) = g(s,\epsilon)$, and hence $G \supseteq (k+\epsilon)C_5 \cup g(s,\epsilon)K_2$.

Next we show that $G \supseteq (k+\epsilon)C_5 \cup P_3$. By way of contradiction, suppose that G does not contain $(k+\epsilon)C_5 \cup P_3$. Then with $\mathcal{C}$ and M as above, we have $E(G - \cup_{i=1}^{k+\epsilon} V(C^i)) = M$ (because otherwise $G - \cup_{i=1}^{k+\epsilon} V(C^i)$ contains P_3). This in particular implies that there is no edge between $V(e_1)$ and $V(e_2)$, and hence $\sum_{v \in V(e_1)\cup V(e_2)} d_G(v) \geq 2\sigma_2(G)$. On the other hand, applying Claim 2.4 with $m = 3$, we obtain

$$\begin{aligned}\sum_{v\in V(e_1)\cup V(e_2)} d(v) &= \{\sum_{v\in V(e_1)\cup V(e_2)} d(v, \cup_{i=1}^{k+\epsilon} V(C^i)) + d(v, \cup_{i=1}^{g(s,\epsilon)} V(e_i))\}\\ &< 2(6k+10)\\ &\leq 2\sigma_2(G),\end{aligned}$$

which is a contradiction. Thus $G \supseteq (k+\epsilon)C_5 \cup P_3$.

Finally, we show that $G \supseteq (k+\epsilon)C_5 \cup P_{2s-5\epsilon}$. Suppose now that we have chosen $\mathcal{C} = \{C^1, C^2, \cdots, C^{k+\epsilon}\}$ so that a longest path in $G - \cup_{i=1}^{k+\epsilon} V(C^i)$ is as long as possible. Let $V(\mathcal{C}) = \cup_{i=1}^{k+\epsilon} V(C^i)$, and let $P = p_0 p_1 \cdots p_{m-2} p_{m-1}$ be a longest path in $G - V(\mathcal{C})$. By way of contradiction, suppose that $m < 2s - 5\epsilon$, i.e., $V(P) \neq V(G) - V(\mathcal{C})$. Take $v \in V(G) - V(\mathcal{C}) - V(P)$, and let $F = V(G) - V(\mathcal{C}) - V(P) - \{v\}$ (it is possible that $F = \emptyset$ and, in the case where $F = \emptyset$, we take $N_G(x, F) = \emptyset$ and $d_G(x, F) = 0$ for $x \in V(G)$). Note that $m \geq 3$ and $|F| = 2s - 5\epsilon - m - 1$. Applying Claim 2.5 with $l = 1$, we obtain

$$d(v, P) + d(p_0, P) \leq m - 1 \qquad (1)$$

by the maximality of P. On the other hand, it immediately follows from the maximality of P that

$$d(p_0, F \cup \{v\}) = 0, \qquad (2)$$

and we clearly have

$$d(v, F) \leq |F| = 2s - 5\epsilon - m - 1. \qquad (3)$$

Since

$$\begin{aligned} d(v) + d(p_0) = {} & d(v, V(\mathcal{C})) + d(p_0, V(\mathcal{C})) + (d(v, P) + d(p_0, P)) + d(v, F) \\ & + d(p_0, F \cup \{v\}) \end{aligned}$$

and $vp_0 \notin E(G)$, it follows from (1), (2), (3) that

$$\begin{aligned} d(v, V(\mathcal{C})) + d(p_0, V(\mathcal{C})) &\geq \sigma_2(G) - \{(d(v, P) + d(p_0, P)) + d(v, F) \\ &\qquad + d(p_0, F \cup \{v\})\} \\ &\geq (6k + 2s) - \{(m-1) + (2s - 5\epsilon - m - 1) + 0\} \\ &= 6k + 5\epsilon + 2 \\ &> 6(k + \epsilon). \end{aligned}$$

Hence, there exists i such that $d(v, C^i) + d(p_0, C^i) \geq 7$. We may assume that $i = 1$. By the maximality of m, $N(p_0, C^1) \cap N'(v, C^1) = \emptyset$. Hence by (i) of Claim 2.2, we see that $N'(p_0, C^1) \cap N(v, C^1) \neq \emptyset$. Take $c \in N'(p_0, C^1) \cap N(v, C^1)$. Now set $C' = G[(V(C^1) - \{c\}) \cup \{p_0\}] \supseteq C_5$, $P' = p_1 p_2 \cdots p_{m-1}$ and $\mathcal{C}' = (\mathcal{C} - \{C^1\}) \cup \{C'\}$. Also set $V(\mathcal{C}') = (V(\mathcal{C}) - V(C^1)) \cup V(C')$. Then by Claim 2.4, it follows from the maximality of m that

$$\sum_{u \in \{c, v, p_1, p_{m-1}\}} d(u, V(\mathcal{C}')) \leq 12(k + \epsilon).$$

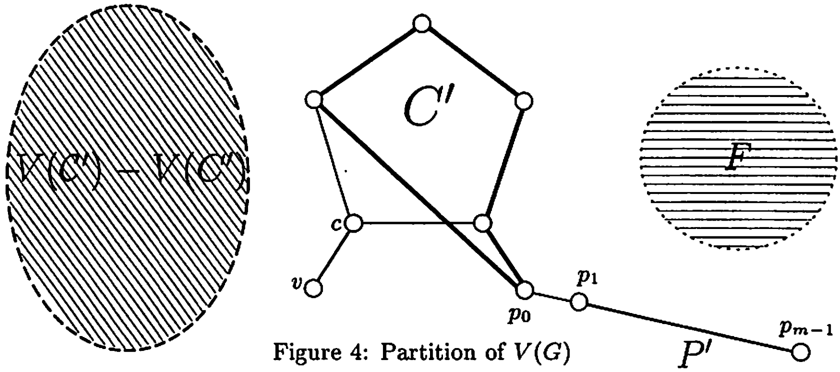

Figure 4: Partition of $V(G)$

By Claim 2.5, we have

$$d(c, P') + d(p_1, P') \leq |V(P')| - 1 = m - 2$$

and

$$d(v, P') + d(p_{m-1}, P') \leq |V(P')| - 1 = m - 2.$$

Consequently,

$$\sum_{u \in \{c, v, p_1, p_{m-1}\}} d(u,\ V(C') \cup V(P')) \leq 12(k + \epsilon) + 2(m - 2). \tag{4}$$

By the maximality of m, we also have

$$e(\{c, v\},\ \{p_1, p_{m-1}\}) = 0 \tag{5}$$

and

$$N(c, F) \cap N(p_1, F) = \emptyset,\ \ N(v, F) \cap N(p_{m-1}, F) = \emptyset. \tag{6}$$

Since (6) implies $d(c, F) + d(p_1, F) \leq |F|$ and $d(v, F) + d(p_{m-1}, F) \leq |F|$, it follows from (5) that

$$\begin{aligned} \sum_{u \in \{c, v, p_1, p_{m-1}\}} d(u, F \cup \{c, v\}) &= \sum_{u \in \{c, v, p_1, p_{m-1}\}} (d(u, F) + d(u, \{c, v\})) \\ &\leq 2|F| + 2 \\ &= 4s - 10\epsilon - 2m. \end{aligned} \tag{7}$$

By (4) and (7),

$$\begin{aligned} \sum_{u \in \{c, v, p_1, p_{m-1}\}} d(u) = \{ &\sum_{u \in \{c, v, p_1, p_{m-1}\}} d(u,\ V(C') \cup V(P')) \\ &+ d(u,\ F \cup \{c,\ v\})\} \\ &\leq \{12(k + \epsilon) + 2(m - 2)\} + (4s - 10\epsilon - 2m) \\ &= 12k + 4s + 2\epsilon - 4. \end{aligned}$$

Since (5) implies that $\sum_{u\in\{c,v,p_1,p_{m-1}\}} d(u) \geq 2\sigma_2(G)$, this contradicts the assumption that $\sigma_2(G) \geq 6k+2s$, and this contradiction completes the proof of Lemma 2.6. □

3 Existence of Two Disjoint 5-Cycles

In this section, we prove several results concerning the existence of two disjoint 5-cycles. The main result of this section is Lemma 3.10. We start with sufficient conditions for the existence of a 5-cycle.

Claim 3.1. *Let $s = 5$ or 6, and let A, P be subgraphs of a graph G such that $V(G) = V(A) \dot{\cup} V(P)$, $A \cong K_1$ and $P \cong P_s$. Write $V(A) = \{a\}$ and $P = p_1 \cdots p_s$. Then the following hold.*

(i) *If $s = 5$, $d(a, P) \geq 3$ and $ap_3 \notin E(G)$, then $G \supseteq C_5$.*

(ii) *If $s = 5$ and $d(a, P) \geq 4$, then $G \supseteq C_5$.*

(iii) *If $s = 6$ and $d(a, P) \geq 4$, then $G \supseteq C_5$.*

Proof. (i) Since $d(a, P) \geq 3$ and $ap_3 \notin E(G)$, we have $N(a, P) \supseteq \{p_1, p_4\}$ or $N(a, P) \supseteq \{p_2, p_5\}$, which implies that $G \supseteq C_5$.

(ii) As in (i), we have $N(a, P) \supseteq \{p_1, p_4\}$ or $N(a, P) \supseteq \{p_2, p_5\}$, which implies that $G \supseteq C_5$.

(iii) We have $N(a, P) \supseteq \{p_1, p_4\}$, $N(a, P) \supseteq \{p_2, p_5\}$ or $N(a, P) \supseteq \{p_3, p_6\}$, which implies that $G \supseteq C_5$. □

Claim 3.2. *Let $s = 3, 4$ or 5, and let A, P be subgraphs of a graph G such that $V(G) = V(A) \dot{\cup} V(P)$, $A \cong K_2$ and $P \cong P_s$. Write $V(A) = \{a_1, a_2\}$ and $P = p_1 \cdots p_s$. Then the following hold.*

(i) *If $s = 3$ and $e(A, \{p_1, p_3\}) \geq 3$, then $G \supseteq C_5$.*

(ii) *If $s = 4$ and $e(A, P) \geq 5$, then $G \supseteq C_5$.*

(iii) *If $s = 5$ and $e(A, P) \geq 6$, then $G \supseteq C_5$.*

Proof. (i) In view of Claim 2.2 (iii), we may assume $a_1p_1, a_2p_3 \in E(G)$. Now $a_1p_1p_2p_3a_2a_1$ is a 5-cycle in G.

(ii) We have $e(A, \{p_1, p_3\}) \geq 3$ or $e(A, \{p_2, p_4\}) \geq 3$. We may assume $e(A, \{p_1, p_3\}) \geq 3$. Then by (i), $G \supseteq C_5$.

(iii) We may assume $d(a_1,P) \geq d(a_2,P)$, so $d(a_1,P) \geq 3$. If $d(a_1,P) \geq 4$, then $G \supseteq C_5$ by Claim 3.1 (ii). Thus we may assume $d(a_1,P) = 3$. Then $d(a_1,P) = d(a_2,P) = 3$. In view of Claim 3.1 (i), we may assume $p_3 \in N(a_1,P)$ and $p_3 \in N(a_2,P)$. If $p_1 \in N(a_1,P) \cup N(a_2,P)$, then $G[\{a_1,a_2,p_1,p_2,p_3\}] \supseteq C_5$. Thus we may assume $p_1 \notin N(a_1,P) \cup N(a_2,P)$. Similarly we may assume $p_5 \notin N(a_1,P) \cup N(a_2,P)$. Now we have $N(a_1,P) = N(a_2,P) = \{p_2, p_3, p_4\}$, and hence $G[\{a_1, a_2, p_2, p_3, p_4\}] \supseteq C_5$. □

Claim 3.3. *Let $s = 4$ or 6, and let A, P, C be subgraphs of a graph G such that $V(G) = V(A) \dot\cup V(P) \dot\cup V(C)$, $A \cong K_2$, $P \cong P_s$ and $C \cong C_5$. Write $V(A) = \{a_1, a_2\}$, $P = p_1p_2 \cdots p_s$ and $C = c_1c_2c_3c_4c_5c_1$. Then the following hold.*

(i) *If $s = 4$, $e(A,C) \geq 7$, and $e(P,C) \geq 13$, then $G \supseteq 2C_5$.*

(ii) *If $s = 6$, $e(A,C) \geq 7$, and $e(P,C) \geq 19$, then $G \supseteq 2C_5$.*

Proof. By Claim 2.3 (ii), there exists j such that $G[\{c_j, c_{j+1}, c_{j+2}, a_1, a_2\}] \supseteq C_5$ and $G[\{c_j, c_{j-1}, c_{j-2}, a_1, a_2\}] \supseteq C_5$.
(i) We have $e(\{c_{j-2}, c_{j-1}, c_{j+1}, c_{j+2}\},P) = e(C,P) - d(c_j,P) \geq 13-4 = 9$. Hence $e(\{c_{j-2}, c_{j-1}\},P) \geq 5$ or $e(\{c_{j+1}, c_{j+2}\},P) \geq 5$ holds. By symmetry, we may suppose that $e(\{c_{j-2}, c_{j-1}\},P) \geq 5$. Then by Claim 3.2 (ii), $G[\{c_{j-2}, c_{j-1}\} \cup V(P)]$ contains a 5-cycle, which is disjoint from the 5-cycle in $G[\{c_j, c_{j+1}, c_{j+2}, a_1, a_2\}]$.

(ii) As in (i), $e(\{c_{j-2}, c_{j-1}, c_{j+1}, c_{j+2}\},P) = e(C,P) - d(c_j,P) \geq 13$. Hence there exists $x \in \{c_{j-2}, c_{j-1}, c_{j+1}, c_{j+2}\}$ such that $d(x,P) \geq \lceil \frac{13}{4} \rceil = 4$. We may assume that $x \in \{c_{j-2}, c_{j-1}\}$. Now by Claim 3.1 (iii), $G[\{x\} \cup V(P)]$ contains a 5-cycle, which is disjoint from the 5-cycle in $G[\{c_j, c_{j+1}, c_{j+2}, a_1, a_2\}]$. □

Claim 3.4. *Let P, C be subgraphs of a graph G such that $V(G) = V(P) \dot\cup V(C)$, $P \cong P_3$ and $C \cong C_5$, and write $P = p_1p_2p_3$ and $C = c_1c_2c_3c_4c_5c_1$.*

(i) *If $d(p_1,C) + d(p_3,C) \geq 5$, $d(p_1,C) > 0$ and $d(p_3,C) > 0$, then there exists $i \in \{1, 2, 3, 4, 5\}$ such that $G[\{p_1, p_2, p_3, c_i, c_{i+1}\}] \supseteq C_5$.*

(ii) *If $d(p_1,C) + d(p_3,C) \geq 7$, then there exists $i \in \{1, 2, 3, 4, 5\}$ such that $G[\{c_i, c_{i+1}, p_1, p_2, p_3\}] \supseteq C_5$ and $G[\{c_{i-1}, c_i, p_1, p_2, p_3\}] \supseteq C_5$.*

Proof. (i) We may assume that $d(p_1,C) \geq d(p_3,C)$, so $d(p_1,C) \geq 3$. It suffices to show that $N^{\pm}(p_1,C) \cap N(p_3,C) \neq \emptyset$. Now if $d(p_1,C) = 3$, then $|N^{\pm}(p_1,C)| \geq 4$, and hence $|N^{\pm}(p_1,C) \cap N(p_3,C)| \geq (4+2) - 5 > 0$; if

$d(p_1,C) \geq 4$, then $|N^{\pm}(p_1,C)| = 5$, and hence $|N^{\pm}(p_1,C) \cap N(p_3,C)| \geq (5+1)-5 > 0$.

(ii) We may assume that $d(p_1,C) \geq d(p_3,C)$, so $d(p_1,C) \geq 4$. It suffices to show that $N'(p_1,C) \cap N(p_3,C) \neq \emptyset$. Now if $d(p_1,C) = 5$, then $|N'(p_1,C) \cap N(p_3,C)| \geq 5+2-5 > 0$; if $d(p_1,C) = 4$, then $|N'(p_1,C)| = 3$ by Claim 2.1 (ii), and hence $|N'(p_1,C) \cap N(p_3,C)| \geq 3+3-5 > 0$. □

Claim 3.5. *Let P, Q, C be subgraphs of a graph G such that $V(G) = V(P) \dot\cup V(Q) \dot\cup V(C)$, $P \cong P_5$, $Q \cong P_5$ and $C \cong C_5$, and write $P = p_1p_2p_3p_4p_5$, $Q = q_1q_2q_3q_4q_5$ and $C = c_1c_2c_3c_4c_5c_1$. Then the following hold.*

(i) *If $N(p_1,C) \cap N(p_4,C) \neq \emptyset$ and $e(C,Q) \geq 16$, then $G \supseteq 2C_5$.*

(ii) *If there exists $m \in \{0, 1, 2, 3\}$ such that $e(P,C) \geq 12+m$, $d(p_3,C) \leq 1+m$ and $e(C,Q) \geq 16$, then $G \supseteq 2C_5$.*

Proof. (i) Let $c_i \in N(p_1,C) \cap N(p_4,C)$. Since $e(V(C) - \{c_i\},Q) \geq 16-5 = 11$, we have $e(\{c_{i-1}, c_{i-2}\},Q) \geq 6$ or $e(\{c_{i+1}, c_{i+2}\},Q) \geq 6$. By symmetry, we may assume that $e(\{c_{i-1}, c_{i-2}\},Q) \geq 6$. Then by Claim 3.2 (iii), $G[\{c_{i-1}, c_{i-2}\} \cup V(Q)]$ contains a 5-cycle, which is disjoint from the 5-cycle $c_ip_1p_2p_3p_4c_i$.

(ii) Since $d(p_3,C) \leq 1+m$, $e(\{p_1, p_2, p_4, p_5\},C) \geq (12+m)-(1+m) = 11$, which implies that either $e(\{p_1, p_4\},C) \geq 6$ or $e(\{p_2, p_5\},C) \geq 6$ holds. By symmetry, we may assume that $e(\{p_1, p_4\},C) \geq 6$, which implies that $N(p_1,C) \cap N(p_4,C) \neq \emptyset$. Since $e(C,Q) \geq 16$, the desired conclusion now follows immediately from (i). □

Claim 3.6. *Let P, Q, C be subgraphs of a graph G such that $V(G) = V(P) \dot\cup V(Q) \dot\cup V(C)$, where $P \cong P_3$, $Q \cong P_5$ and $C \cong C_5$. Write $P = p_1p_2p_3$, $Q = q_1q_2q_3q_4q_5$ and $C = c_1c_2c_3c_4c_5c_1$, and suppose that $e(C,Q) \geq 18$, $d(p_1,C) + d(p_3,C) \geq 5$, $d(p_1,C) > 0$ and $d(p_3,C) > 0$. Then $G \supseteq 2C_5$.*

Proof. By Claim 3.4 (i), there exists $1 \leq j \leq 5$ such that $G[\{p_1,p_2,p_3,c_j, c_{j+1}\}] \supseteq C_5$. If $\max\{d(c_{j+2},Q), d(c_{j+3},Q), d(c_{j+4},Q)\} \geq 4$ or $\max\{d(c_{j+2},Q) + d(c_{j+3},Q), d(c_{j+3},Q) + d(c_{j+4},Q)\} \geq 6$, then by Claim 3.1 (ii) or Claim 3.2 (iii), $G[\{c_{j+2},c_{j+3},c_{j+4}\} \cup V(Q)\}]$ contains a 5-cycle, which is disjoint from the 5-cycle in $G[\{p_1,p_2,p_3,c_j,c_{j+1}\}]$. Thus we may assume that $\max\{d(c_{j+2},Q),d(c_{j+3},Q),d(c_{j+4},Q)\} \leq 3$ and $\max\{d(c_{j+2},Q)+d(c_{j+3},Q),d(c_{j+3},Q) + d(c_{j+4},Q)\} \leq 5$. This implies $d(c_{j+2},Q) + d(c_{j+3},Q) + d(c_{j+4},Q) \leq 8$. Consequently, from the assumption that $e(Q,C) \geq 18$, it follows that $d(c_j,Q) = d(c_{j+1},Q) = 5$ and $d(c_{j+2},Q) + d(c_{j+3},Q) + d(c_{j+4},Q) = 8$, and hence $d(c_{j+2},Q) = 3$, $d(c_{j+3},Q) = 2$ and $d(c_{j+4},Q) = 3$. In view of Claim 3.1 (i), we may assume $q_3 \in N(c_{j+2},Q) \cap N(c_{j+4},Q)$.

Now if $(N(c_{j+2}, Q) \cup N(c_{j+4}, Q)) \cap \{q_2, q_4\} \neq \emptyset$. then $G[\{c_{j+2}, c_{j+3}, c_{j+4}, q_2, q_3, q_4\}] \supseteq C_5$, which is disjoint from the 5-cycle in $G[\{p_1, p_2, p_3, c_j, c_{j+1}\}]$. Thus we may assume $N(c_{j+2}, Q) = N(c_{j+4}, Q) = \{q_1, q_3, q_5\}$. Then $c_j q_1 q_2 q_3 q_4 c_j$ and $c_{j+1} c_{j+2} c_{j+3} c_{j+4} q_5 c_{j+1}$ are vertex-disjoint 5-cycles. □

Claim 3.7. *Let P, Q, C be subgraphs of a graph G such that $V(G) = V(P) \dot{\cup} V(Q) \dot{\cup} V(C)$, where $P \cong P_3$, $Q \cong P_5$ and $C \cong C_5$. Write $P = p_1 p_2 p_3$, $Q = q_1 q_2 q_3 q_4 q_5$ and $C = c_1 c_2 c_3 c_4 c_5 c_1$, and suppose that $e(C, Q) \geq 16$. Then the following hold.*

(i) *If there exists i such that $G[\{c_i, c_{i+1}, p_1, p_2, p_3\}] \supseteq C_5$ and $G[\{c_i, c_{i-1}, p_1, p_2, p_3\}] \supseteq C_5$, then $G \supseteq 2C_5$.*

(ii) *If there exists i such that $G[\{c_i, c_{i+1}, p_1, p_2, p_3\}] \supseteq C_5$ and $G[\{c_{i+2}, c_{i+3}, p_1, p_2, p_3\}] \supseteq C_5$, then $G \supseteq 2C_5$.*

(iii) *If $\max\{d(p_1, C), d(p_3, C)\} \geq 4$ and $\min\{d(p_1, C), d(p_3, C)\} \geq 2$, then $G \supseteq 2C_5$.*

Proof. (i) Since $e(\{c_{i-1}, c_{i-2}, c_{i+1}, c_{i+2}\}, Q) = e(C, Q) - d(c_i, Q) \geq 11$, $e(\{c_{i-1}, c_{i-2}\}, Q) \geq 6$ or $e(\{c_{i+1}, c_{i+2}\}, Q) \geq 6$ holds. By symmetry, we may assume that $e(\{c_{i-1}, c_{i-2}\}, Q) \geq 6$. Then by Claim 3.2 (iii), $G[\{c_{i-1}, c_{i-2}\} \cup V(Q)]$ contains a 5-cycle, which is disjoint from the 5-cycle in $G[\{c_i, c_{i+1}, p_1, p_2, p_3\}]$.

(ii) Since $e(\{c_i, c_{i+1}, c_{i+2}, c_{i+3}\}, Q) = e(C, Q) - d(c_{i+4}, Q) \geq 11$, $e(\{c_i, c_{i+1}\}, Q) \geq 6$ or $e(\{c_{i+2}, c_{i+3}\}, Q) \geq 6$ holds. By symmetry, we may assume that $e(\{c_i, c_{i+1}\}, Q) \geq 6$. Then by Claim 3.2 (iii), $G[\{c_i, c_{i+1}\} \cup V(Q)]$ contains a 5-cycle, which is disjoint from the 5-cycle in $G[\{c_{i+2}, c_{i+3}, p_1, p_2, p_3\}]$.

(iii) By symmetry, we may assume that $d(p_1, C) \geq 4$ and $d(p_3, C) \geq 2$. Now we may assume $N(p_1, C) \supseteq \{c_2, c_3, c_4, c_5\}$. Then $N'(p_1, C) \supseteq \{c_1, c_3, c_4\}$. Since $d(p_3, C) \geq 2$, $N(p_3, C) \cap N'(p_1, C) \neq \emptyset$ or $N(p_3, C) = \{c_2, c_5\}$ holds. If $N(p_3, C) \cap N'(p_1, C) \neq \emptyset$, then P and C satisfy the assumption of (i), and hence the desired conclusion follows from (i); if $N(p_3, C) = \{c_2, c_5\}$, P and C satisfy the assumption of (ii) with $i = 2$, and hence the desired conclusion follows from (ii). □

Claim 3.8. *Let A, P, C be subgraphs of a graph G such that $V(G) = V(A) \dot{\cup} V(P) \dot{\cup} V(C)$, $A \cong K_2$, $P \cong P_5$ and $C \cong C_5$. Write $V(A) = \{a_1, a_2\}$, $P = p_1 p_2 p_3 p_4 p_5$ and $C = c_1 c_2 c_3 c_4 c_5 c_1$, and suppose that $e(C, P) \geq 16$. Then the following hold.*

(i) *If there exists i such that $G[\{c_i, c_{i+1}, c_{i+2}, a_1, a_2\}] \supseteq C_5$ and $G[\{c_i, c_{i-1}, c_{i-2}, a_1, a_2\}] \supseteq C_5$, then $G \supseteq 2C_5$.*

(ii) *If* $\max\{d(a_1,C),d(a_2,C)\} \geq 4$ *and* $\min\{d(a_1,C),d(a_2,C)\} \geq 2$, *then* $G \supseteq 2C_5$.

Proof. (i) Since $e(\{c_{i-1}, c_{i-2}, c_{i+1}, c_{i+2}\}, P) = e(C,P) - d(c_i,P) \geq 11$, $e(\{c_{i-1}, c_{i-2}\},P) \geq 6$ or $e(\{c_{i+1}, c_{i+2}\},P) \geq 6$ holds. By symmetry, we may assume that $e(\{c_{i-1}, c_{i-2}\},P) \geq 6$. Then by Claim 3.2 (iii), $G[\{c_{i-1}, c_{i-2}\} \cup V(P)]$ contains a 5-cycle, which is disjoint from the 5-cycle in $G[\{c_i, c_{i+1}, c_{i+2}, a_1, a_2\}]$.

(ii) By symmetry, we may assume that $d(a_1,C) \geq 4$ and $d(a_2,C) \geq 2$. We may also assume $N(a_1,C) \supseteq \{c_2,c_3,c_4,c_5\}$. Then $N'(a_1,C) \supseteq \{c_1,c_3,c_4\}$. If $N(a_2,C) \cap \{c_1,c_2,c_5\} \neq \emptyset$, then letting $c_i \in N(a_2,C) \cap \{c_1,c_2,c_5\}$, we see that $G[\{c_i,c_{i+1},c_{i+2},a_1,a_2\}] \supseteq C_5$ and $G[\{c_i,c_{i-1},c_{i-2},a_1,a_2\}] \supseteq C_5$, and hence $G \supseteq 2C_5$ by (i). Thus we may assume $N(a_2,C) \cap \{c_1, c_2, c_5\} = \emptyset$. Then $N(a_2,C) = \{c_3, c_4\}$. Since $e(C,P) \geq 16$, there exists l such that $d(c_l,P) \geq \lceil \frac{16}{5} \rceil = 4$. Then by Claim 3.1 (ii), $G[\{c_l\} \cup V(P)]$ contains a 5-cycle. If $c_l \in \{c_1,c_3,c_4\}$, then $a_1c_{l-1}c_{l-2}c_{l+2}c_{l+1}a_1$ is a 5-cycle, which is disjoint from the 5-cycle in $G[\{c_l\} \cup V(P)]$. Thus we may assume $c_l \in \{c_2,c_5\}$. Now if $c_l = c_2$, then $a_1c_5c_4c_3a_2a_1$ is a 5-cycle, which is disjoint from the 5-cycle in $G[\{c_l\} \cup V(P)]$; if $c_l = c_5$, then $a_1c_2c_3c_4a_2a_1$ is a 5-cycle, which is disjoint from the 5-cycle in $G[\{c_l\} \cup V(P)]$. □

Claim 3.9. *Let* P, C *be subgraphs of a graph* G *such that* $V(G) = V(P) \dot\cup V(C)$, $P \cong P_5$ *and* $C \cong C_5$. *Write* $P = p_1p_2p_3p_4p_5$ *and* $C = c_1c_2c_3c_4c_5c_1$, *and suppose that* $e(P,C) \geq 20$, $d(p_1,C) > 0$ *and* $d(p_5,C) > 0$. *Then* $G \supseteq 2C_5$.

Proof. We first prove four subclaims.

Subclaim A. *If* $d(p_1,C) + d(p_2,C) \geq 9$, $d(p_3,C) + d(p_5,C) \geq 5$ *and* $d(p_3,C) > 0$, *then* $G \supseteq 2C_5$.

Proof. By Claim 3.4 (i), there exists j such that $G[\{p_3,p_4,p_5,c_j,c_{j+1}\}] \supseteq C_5$. Since $d(p_1,C) + d(p_2,C) \geq 9$, we have $e(\{p_1,p_2\},\{c_{j-1},c_{j+2}\}) \geq 3$. Hence by Claim 3.2 (i), $G[\{p_1, p_2, c_{j-1}, c_{j-2}, c_{j+2}\}]$ contains a 5-cycle, which is disjoint from the 5-cycle in $G[\{p_3, p_4, p_5, c_j, c_{j+1}\}]$. □

Subclaim B. *If* $d(p_1,C) + d(p_2,C) \geq 8$ *and* $d(p_3,C) + d(p_5,C) \geq 7$, *then* $G \supseteq 2C_5$.

Proof. By Claim 3.4 (ii), there exists j such that $G[\{c_j, c_{j+1}, p_3, p_4, p_5\}] \supseteq C_5$ and $G[\{c_{j-1}, c_j, p_3, p_4, p_5\}] \supseteq C_5$. Since $d(p_1,C) + d(p_2,C) \geq 8$, we have $e(\{p_1,p_2\},\{c_{j-1},c_{j+2}\}) \geq 3$ or $e(\{p_1,p_2\},\{c_{j-2},c_{j+1}\}) \geq 3$. We may assume $e(\{p_1,p_2\},\{c_{j-1},c_{j+2}\}) \geq 3$. Then by Claim 3.2 (i), $G[\{p_1, p_2, c_{j-1},$

$c_{j-2}, c_{j+2}\}]$ contains a 5-cycle, which is disjoint from the 5-cycle in $G[\{c_j, c_{j+1}, p_3, p_4, p_5\}]$. □

Subclaim C. *If $d(p_1, C) + d(p_2, C) \geq 7$ and $d(p_3, C) + d(p_5, C) \geq 8$, then $G \supseteq 2C_5$.*

Proof. By Claim 2.3 (ii), there exists j such that $G[\{c_j, c_{j-1}, c_{j-2}, p_1, p_2\}] \supseteq C_5$ and $G[\{c_j, c_{j+1}, c_{j+2}, p_1, p_2\}] \supseteq C_5$. Since $d(p_3, C) + d(p_5, C) \geq 8$, we have $e(\{p_3, p_5\}, \{c_{j-1}, c_{j-2}\}) \geq 3$ or $e(\{p_3, p_5\}, \{c_{j+1}, c_{j+2}\}) \geq 3$. We may assume $e(\{p_3, p_5\}, \{c_{j-1}, c_{j-2}\}) \geq 3$. Then by Claim 3.2 (i), $G[\{c_{j-1}, c_{j-2}, p_3, p_4, p_5\}]$ contains a 5-cycle, which is disjoint from the 5-cycle in $G[\{c_j, c_{j+1}, c_{j+2}, p_1, p_2\}]$. □

Subclaim D. *If $d(p_1, C) + d(p_2, C) \geq 5$, $d(p_2, C) > 0$ and $d(p_3, C) + d(p_5, C) \geq 9$, then $G \supseteq 2C_5$.*

Proof. By Claim 2.3 (i), there exists j such that $G[\{c_j, c_{j+1}, c_{j+2}, p_1, p_2\}] \supseteq C_5$. Since $d(p_3, C) + d(p_5, C) \geq 9$, we have $e(\{p_3, p_5\}, \{c_{j-1}, c_{j-2}\}) \geq 3$. Hence by Claim 3.2 (i), $G[\{c_{j-1}, c_{j-2}, p_3, p_4, p_5\}]$ contains a 5-cycle, which is disjoint from the 5-cycle in $G[\{c_j, c_{j+1}, c_{j+2}, p_1, p_2\}]$. □

We return to the proof of Claim 3.9. We distinguish two cases whether $d(p_3, C) > 0$ or $d(p_3, C) = 0$.

Case 1. $d(p_3, C) > 0$.
By symmetry, we may assume $d(p_2, C) \geq d(p_4, C)$. Then $d(p_2, C) > 0$. Since $e(P, C) \geq 20$, $d(p_1, C) + d(p_2, C) + d(p_3, C) + d(p_5, C) \geq 15$. This in particular implies that we have $d(p_1, C) + d(p_2, C) \geq 5$ and $d(p_3, C) + d(p_5, C) \geq 5$. Thus if $d(p_1, C) + d(p_2, C) \geq 9$, then $G \supseteq 2C_5$ by Subclaim A. If $d(p_1, C) + d(p_2, C) = 8$, then $d(p_3, C) + d(p_5, C) \geq 7$, and hence $G \supseteq 2C_5$ by Subclaim B. If $d(p_1, C) + d(p_2, C) = 7$, then $d(p_3, C) + d(p_5, C) \geq 8$, and hence $G \supseteq 2C_5$ by Subclaim C. Finally if $5 \leq d(p_1, C) + d(p_2, C) \leq 6$, then $d(p_3, C) + d(p_5, C) \geq 9$, and hence $G \supseteq 2C_5$ by Subclaim D.

Case 2. $d(p_3, C) = 0$.
We have $d(p_1, C) = d(p_2, C) = d(p_4, C) = d(p_5, C) = 5$. Thus for any j, $p_1p_2p_3p_4c_jp_1$ and $p_5c_{j+1}c_{j+2}c_{j-2}c_{j-1}p_5$ are disjoint 5-cycles. □

Lemma 3.10. *Let $s \geq 5$, and let P, C be subgraphs of a graph G such that $V(G) = V(P) \dot\cup V(C)$, $P \cong P_{2s}$ and $C \cong C_5$. Write $P = p_1p_2 \cdots p_{2s-1}p_{2s}$ and $C = c_1c_2c_3c_4c_5c_1$, and suppose that $e(P, C) \geq 6s + 1$. Then $G \supseteq 2C_5$.*

Proof. Let $P^{(1)} = p_1p_2 \cdots p_s$ and $P^{(2)} = p_{s+1}p_{s+2} \cdots p_{2s}$. We may assume that $e(P^{(1)}, C) \leq e(P^{(2)}, C)$. Then $e(P^{(2)}, C) \geq 3s + 1$. We first show that the lemma holds when $s = 5$. We consider two cases separately.

Case 1. $e(P^{(2)}, C) \geq 20$.
If $d(p_6, C) > 0$ and $d(p_{10}, C) > 0$, then the desired conclusion immediately follows from Claim 3.9. Thus we may assume either $d(p_6, C) = 0$ or $d(p_{10}, C) = 0$ holds. Then there exists $6 \leq i \leq 7$ such that $d(p_i, C) = d(p_{i+1}, C) = d(p_{i+2}, C) = d(p_{i+3}, C) = 5$. Since $e(P^{(1)}, C) \geq 11$, there exists $1 \leq j \leq 5$ such that $d(p_j, C) \geq \lceil \frac{11}{5} \rceil = 3$. Then $|N'(p_j, C)| \geq 1$ by Claim 2.1 (iii). Let $c_l \in N'(p_j, C)$. Then $p_j c_{l+1} c_{l+2} c_{l-2} c_{l-1} p_j$ and $p_i p_{i+1} p_{i+2} p_{i+3} c_l p_i$ are disjoint 5-cycles.

Case 2. $16 \leq e(P^{(2)}, C) \leq 19$.
Write $e(P^{(2)}, C) = 19 - m$. Then $0 \leq m \leq 3$ and $e(P^{(1)}, C) \geq 12 + m$. If $4 \leq d(p_3, C)$, then there exists $j \in \{1, 2, 4, 5\}$ such that $d(p_j, C) \geq 2$ because $\lceil \frac{e(P^{(1)}, C) - d(p_3, C)}{4} \rceil \geq 2$. Hence by Claim 3.7 (iii) or Claim 3.8 (ii), we obtain $G \supseteq 2C_5$. If $d(p_3, C) \leq 1 + m$, then $G \supseteq 2C_5$ by Claim 3.5 (ii). Thus we may assume $2 + m \leq d(p_3, C) \leq 3$. This implies $0 \leq m \leq 1$, and hence $e(P^{(2)}, C) \geq 18$, $e(P^{(1)}, C) \geq 12$ and $d(p_3, C) \geq 2$. If $d(p_1, C) + d(p_3, C) \geq 5$ or $d(p_3, C) + d(p_5, C) \geq 5$, then $G \supseteq 2C_5$ by Claim 3.6. Thus we may assume $d(p_1, C) + d(p_3, C) \leq 4$ and $d(p_3, C) + d(p_5, C) \leq 4$. Consequently, $d(p_2, C) + d(p_4, C) = e(P^{(1)}, C) - (d(p_1, C) + d(p_3, C) + d(p_5, C)) \geq 12 - 6 = 6$, and we therefore obtain $G \supseteq 2C_5$ by Claim 3.6. This completes the proof of the lemma for $s = 5$.

Next we consider the case $s = 6$. If $\sum_{i=3}^{12} d(p_i, C) \geq 31$, then the desired conclusion follows from the case $s = 5$. Thus we may assume $\sum_{i=3}^{12} d(p_i, C) \leq 30$. Then $d(p_1, C) + d(p_2, C) \geq e(P, C) - \sum_{i=3}^{12} d(p_i, C) \geq 37 - 30 = 7$. Since $e(P^{(2)}, C) \geq 19$, we now obtain $G \supseteq 2C_5$ by Claim 3.3 (ii). Thus the lemma holds for $s = 6$.

Now we complete the proof of the lemma by induction on s. Thus let $s \geq 7$, and assume that the lemma holds for $s - 2$ and $s - 1$. If $\sum_{i=3}^{2s} d(p_i, C) \geq 6(s - 1) + 1$, then $G \supseteq 2C_5$ by the induction hypothesis. Thus we may assume $\sum_{i=3}^{2s} d(p_i, C) \leq 6(s - 1)$. Then $d(p_1, C) + d(p_2, C) \geq e(P, C) - \sum_{i=3}^{2s} d(p_i, C) \geq (6s + 1) - 6(s - 1) = 7$. Similarly, we may assume $\sum_{i=1}^{2s-4} d(p_i, C) \leq 6(s - 2)$, which implies that $\sum_{i=2s-3}^{2s} d(p_i, C) \geq 6s + 1 - 6(s - 2) = 13$. Therefore we obtain $G \supseteq 2C_5$ by Claim 3.3 (i). This completes the proof. □

4 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. A graph G is called *pancyclic* if $G \supseteq C_l$ for each $3 \leq l \leq |V(G)|$. We recall that $T_2(n)$ stands for the complete bipartite graph of order n whose color classes are as equal as possible; that is to say, $T_2(n) = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. We make use of the following three theorems in the proof of Theorem 1.1.

Theorem 4.1 (Dirac [6]). *Let G be a graph of order n with $\delta(G) \geq \frac{n}{2}$. Then G is a hamilton graph.*

Theorem 4.2 (Bondy [3]). *Let G be a hamilton graph of order n with $e(G) \geq \frac{n^2}{4}$. Then either G is pancyclic or n is even and $G \cong K_{\frac{n}{2},\frac{n}{2}}$.*

Theorem 4.3 (Woodall [14]). *Let G be a graph of order n with $e(G) > e(T_2(n)) = \lfloor \frac{n^2}{4} \rfloor$. Then $G \supseteq C_{2r+1}$ for every $1 \leq r \leq \lfloor \frac{(n+1)}{4} \rfloor$.*

We first deduce the following result from Theorems 4.1, 4.2 and 4.3.

Lemma 4.4. *Let $r \geq 2$ and $n \geq 4r$, and let G be a graph of order n. Suppose that $G \not\supseteq C_{2r+1}$ and $e(G) \geq \lfloor \frac{n^2}{4} \rfloor$. Then $G \cong T_2(n)$.*

Proof. Write $V(G) = \{v_1 . v_2, \cdots, v_n\}$. Suppose that $G \not\cong T_2(n)$. We may assume that $d(v_1) = \delta(G)$. If $\delta(G) \geq \frac{n}{2}$, then G is hamiltonian by Theorem 4.1, and hence $G \supseteq C_{2r+1}$ by Theorem 4.2. Thus we may assume that $\delta(G) \leq \frac{n-1}{2}$, so $d(v_1) \leq \frac{n-1}{2}$. First we show that the case n is even, and let $n = 2s$. Then $|V(G - \{v_1\})| = 2s - 1 \geq 4r - 1$, and $e(G - \{v_1\}) \geq s^2 - s + 1 > e(T_2(2s-1))$. Hence by Theorem 4.3, $G \supseteq G - \{v_1\} \supseteq C_{2r+1}$. Finally we consider the case n is odd, and let $n = 2s+1$. Then $|V(G - \{v_1\})| = 2s \geq 4r$ and $e(G - \{v_1\}) \geq s(s+1) - s = s^2$. Hence by the case n is even, $G - \{v_1\} \cong K_{s,s}$ and $d(v_1) = s$. By the assumption that $G \not\cong T_2(n)$, v_1 is adjacent to some vertex in both color classes of $G - \{v_1\}$, we therefore obtain $G \supseteq C_{2r+1}$. □

It is worth mentioning that the assumption $n \geq 4r$ in Lemma 4.4 is sharp as the following example shows. Let H be the graph $K_1 + (K_{2r-1} \cup K_{n-2r})$ of order $n \leq 4r - 1$. Then $H \not\supseteq C_{2r+1}$, and $e(H) > e(T_2(n)) = \lfloor \frac{n^2}{4} \rfloor$ when $n \leq 4r - 3$ and $e(H) = e(T_2(n)) = \lfloor \frac{n^2}{4} \rfloor$ when $4r - 2 \leq n \leq 4r - 1$.

In order to state Claim 4.5, for each $s \geq 4$, we define a graph H_s of order $2s + 5$ as follows (see Figure 5):

(i) $V(H_s) = \{c_1, c_2, c_3, c_4, c_5\} \cup \{a_1, a_3, a_5, \cdots, a_{2s-1}\} \cup \{a_2, a_4, a_6, \cdots, a_{2s}\}$ (let $A_1 = \{a_1, a_3, a_5, \cdots, a_{2s-1}\}$ and $A_2 = \{a_2, a_4, a_6, \cdots, a_{2s}\}$);
(ii) $\{c_1, c_2, c_3, c_4, c_5\}$ induces a 5-cycle $c_1c_2c_3c_4c_5c_1$;
(iii) $A_1 \cup A_2$ induces a complete bipartite graph $K_{s,s}$ with bipartition (A_1, A_2);
(iv) $N_{H_s}(c_1, A_1 \cup A_2) = A_1 \cup A_2$, $N_{H_s}(c_2, A_1 \cup A_2) = N_{H_s}(c_4, A_1 \cup A_2) = A_1$ and $N_{H_s}(c_3, A_1 \cup A_2) = N_{H_s}(c_5, A_1 \cup A_2) = A_2$.

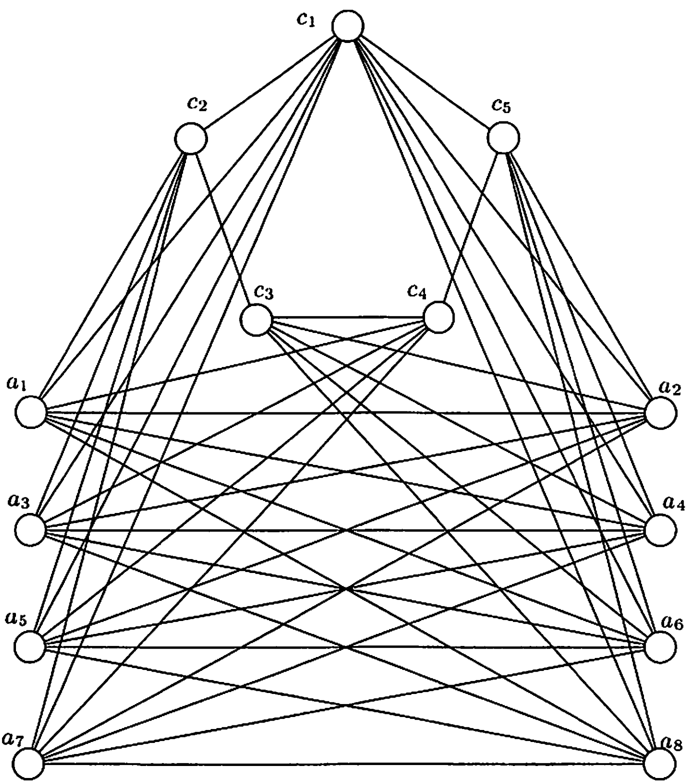

Figure 5: H_4

Claim 4.5. *Let $s \geq 4$, and let A,C be subgraphs of a graph G such that $V(G) = V(A)\dot{\cup}V(C)$, $A \cong K_{s,s}$ and $C \cong C_5$. Let (A_1, A_2) be the bipartition of A and write $C = c_1c_2c_3c_4c_5c_1$, and suppose that $e(A, C) \geq 6s$ and $G \not\supseteq 2C_5$. Then there exist $m \in \{1, 2, 3, 4, 5\}$ and $l \in \{1, 2\}$ such that $N(c_m, A) = A_1 \cup A_2$, $N(c_{m+1}, A) = N(c_{m+3}, A) = A_l$ and $N(c_{m+2}, A) = N(c_{m+4}, A) = A_{3-l}$ (so $G \supseteq H_s$).*

Proof. Write $A_1 = \{a_1, a_3, a_5, \cdots, a_{2s-1}\}$ and $A_2 = \{a_2, a_4, a_6, \cdots, a_{2s}\}$. For $v \in V(C)$, define

$$\delta'(v) = \min\{d(v, A_1), d(v, A_2)\}.$$

Then $\delta'(v) \geq d(v, A) - s$.

Subclaim E. *If there exist $u, v \in V(C)$ with $u \neq v$, and there exist $a_{i_1}, a_{j_1} \in A_1$ and $a_{i_2}, a_{j_2} \in A_2$ with $a_{i_1} \neq a_{j_1}$ and $a_{i_2} \neq a_{j_2}$ such that*

$a_{i_1}, a_{i_2} \in N(u, A)$ *and* $a_{j_1}, a_{j_2} \in N(v, A)$, *then* $G \supseteq 2C_5$.

Proof. Take $a_{i_3}, a_{j_3} \in A_1 - \{a_{i_1}, a_{j_1}\}$ and $a_{i_4}, a_{j_4} \in A_2 - \{a_{i_2}, a_{j_2}\}$ with $a_{i_3} \neq a_{j_3}$ and $a_{i_4} \neq a_{j_4}$. Then $ua_{i_1}a_{i_4}a_{i_3}a_{i_2}u$ and $va_{j_1}a_{j_4}a_{j_3}a_{j_2}v$ are disjoint 5-cycles. □

Subclaim F. *If there exist* $u, v \in V(C)$ *with* $u \neq v$ *such that* $\delta'(u) \geq 2$ *and* $\delta'(v) \geq 1$, *then* $G \supseteq 2C_5$.

Proof. This follows immediately from Subclaim E and the definition of $\delta'(v)$. □

We return to the proof of Claim 4.5. By symmetry, we may assume that

$$d(c_1, A) = \max\{d(c_1, A),\ d(c_2, A),\ d(c_3, A),\ d(c_4, A),\ d(c_5, A)\}.$$

Since $\frac{e(A,C)}{5} \geq \frac{6s}{5} > s$, we have $d(c_1, A) \geq s + 1$. We divide the proof into two cases according to the value of $d(c_1, A)$.

Case 1. $d(c_1, A) = 2s$.
In view of Subclaim F, we have $\delta'(c_i) = 0$ for each $2 \leq i \leq 5$, and hence $d(c_i, A) \leq s$ for each $2 \leq i \leq 5$. Since $d(c_2, A) + d(c_3, A) + d(c_4, A) + d(c_5, A) = e(A, C) - d(c_1, A) \geq 4s$, this implies that for each $2 \leq i \leq 5$, $d(c_i, A) = s$ and we have $N(c_i, A) = A_1$, or $N(c_i, A) = A_2$. Suppose that there exist $i \in \{2, 4\}$ and $j \in \{3, 5\}$ such that $N_G(c_i, A) = N_G(c_j, A)$. By symmetry of the roles of A_1 and A_2, we may assume that $N_G(c_i, A) = N_G(c_j, A) = A_1$. Now if $(i, j) = (2, 5)$, then $c_1a_1a_2a_3a_4c_1$ and $c_2c_3c_4c_5a_5c_2$ are disjoint 5-cycles; if $(i, j) \in \{(2, 3), (4, 3), (4, 5)\}$, then $c_1a_1a_2a_3a_4c_1$ and $c_ic_ja_5a_6a_7c_i$ are disjoint 5-cycles. This contradicts the assumption that $G \supseteq 2C_5$. Thus $N_G(c_i, A) \neq N_G(c_j, A)$ for each $i \in \{2, 4\}$ and each $j \in \{3, 5\}$, which implies the desired conclusion.

Case 2. $s + 1 \leq d(c_1, A) \leq 2s - 1$.
Since $d(c_2, A) + d(c_3, A) + d(c_4, A) + d(c_5, A) \geq 4s + 1$, there exists l with $2 \leq l \leq 5$ such that $d(c_l, A) \geq s + 1$. Thus $\delta'(c_1) \leq 1$ and $\delta'(c_l) \leq 1$ by Subclaim F, we see that $d(c_1, A) = d(c_l, A) = s + 1$. Since $e(A, C) \geq 6s \geq 5(s + 1) - 1$, this implies that there exists m with $2 \leq m \leq 5$ such that $d(c_j, A) = s + 1$ for each $m + 1 \leq j \leq m + 4$. By Subclaim F, $\delta'(c_j) = 1$ for each $m + 1 \leq j \leq m + 4$. By Subclaim E and by symmetry, we may assume that $N(c_j, A) \supseteq A_1$ for each $m + 1 \leq j \leq m + 4$. Then $c_{m+1}c_{m+2}a_1a_2a_3c_{m+1}$ and $c_{m+3}c_{m+4}a_5a_6a_7c_{m+3}$ are disjoint 5-cycles. This contradicts the assumption that $G \not\supseteq 2C_5$, which completes the claim. □

Claim 4.6. *Let* $m \geq 8$, *and let* P *be a subgraph of a graph* G *such that* $P \cong P_m$. *Write* $P = p_1p_2 \cdots p_m$, *and suppose that* $G[V(P)] \not\supseteq C_5$. *Then*

$\sum_{i=1}^{m} d(p_i) \geq \frac{m}{2}\sigma_2(G)$.

Proof. We proceed by induction on m. Since $G[V(P)] \not\supseteq C_5$, $p_ip_{i+4} \notin E(G)$ for each $1 \leq i \leq m-4$. Hence if $m = 8$, we obtain $\sum_{i=1}^{m} d(p_i) = \sum_{i=1}^{4}(d(p_i)+d(p_{i+4})) \geq 4\sigma_2(G)$. Thus let $m \geq 9$, and assume that the claim holds for $m-1$, If $p_1p_m \in E(G)$, then $p_{m-3}p_1, p_{m-2}p_2, p_{m-1}p_3, p_mp_4 \notin E(G)$, and hence

$$\begin{aligned}\sum_{i=1}^{m} d(p_i) &= \frac{\sum_{i=1}^{m-4}(d(p_i)+d(p_{i+4})) + \sum_{j=1}^{4}(d(p_{m-4+j})+d(p_j))}{2} \\ &\geq \frac{(m-4)\sigma_2(G)+4\sigma_2(G)}{2} \\ &= \frac{m}{2}\sigma_2(G).\end{aligned}$$

Thus we may assume that $p_1p_m \notin E(G)$. Then by the induction hypothesis,

$$\begin{aligned}\sum_{i=1}^{m} d(p_i) &= \frac{\{\sum_{i=1}^{m-1} d(p_i)\} + \{\sum_{j=2}^{m} d(p_j)\} + (d(p_1)+d(p_m))}{2} \\ &\geq \frac{\frac{m-1}{2}\sigma_2(G) + \frac{m-1}{2}\sigma_2(G) + \sigma_2(G)}{2} \\ &= \frac{m}{2}\sigma_2(G).\end{aligned}$$

□

Lemma 4.7. *Let $k \geq 1$ and $s \geq 5$, and let G be a graph of order $5k+2s$ with $\sigma_2(G) \geq 6k+2s$. Suppose that $G \supseteq kC_5 \cup P_{2s}$ and $G \not\supseteq (k+1)C_5$. Then $\overline{K_k} + K_{2k+s,2k+s} \subseteq G \subseteq K_k + K_{2k+s,2k+s}$.*

Proof. Let $\mathcal{C} = \{C^1, C^2, \cdots, C^k\}$ be a collection of k vertex-disjoint 5-cycles in G such that $G - V(\mathcal{C})$ has a hamilton path, where $V(\mathcal{C}) = \cup_{i=1}^{k} V(C^i)$. Let $A = a_1a_2\cdots a_{2s}$ be a hamilton path in $G - V(\mathcal{C})$. Since $G[V(A)] \not\supseteq C_5$, it follows from Claim 4.6 that $\sum_{l=1}^{2s} d(a_l) \geq s \cdot \sigma_2(G) \geq s(6k+2s)$. On the other hand, by Lemma 3.10 and Lemma 4.4,

$$\begin{aligned}\sum_{l=1}^{2s} d(a_l) &= \{\sum_{l=1}^{2s} d(a_l, V(\mathcal{C}))\} + 2e(V(A)) \\ &= \{\sum_{i=1}^{k} e(A, C^i)\} + 2e(V(A)) \\ &\leq s(6k+2s).\end{aligned}$$

Hence $e(A, C^i) = 6s$ for each $1 \leq i \leq k$, and $e(V(A)) = s^2$. In view of Lemma 4.4, this implies that $G[V(A)] \cong K_{s,s}$ with bipartition (A_1, A_2), where $A_1 = \{a_1, a_3, a_5, \cdots, a_{2s-1}\}$ and $A_2 = \{a_2, a_4, a_6, \cdots, a_{2s}\}$. From Claim 4.5, it also follows that for each $1 \leq i \leq k$, we can write $C^i = c_1^{(i)}c_2^{(i)}c_3^{(i)}c_4^{(i)}c_5^{(i)}c_1^{(i)}$ so that the following hold:

(i) $N(c_1^{(i)}, A) = A_1 \cup A_2$;
(ii) $N(c_2^{(i)}, A) = N(c_4^{(i)}, A) = A_1$;
(iii) $N(c_3^{(i)}, A) = N(c_5^{(i)}, A) = A_2$.

Since $G[V(A)] \cong K_{s,s}$, we see from (i), (ii), (iii) that $d(a_l) = 3k + s$ for every $1 \leq l \leq 2s$. Let $R = \cup_{i=1}^{k}\{c_1^{(i)}\}$, $B_1 = \cup_{i=1}^{k}\{c_3^{(i)}, c_5^{(i)}\}$ and $B_2 = \cup_{i=1}^{k}\{c_2^{(i)}, c_4^{(i)}\}$.

Claim G. *Both B_1 and B_2 are independent sets.*

Proof. Suppose that B_1(resp. B_2) is not an independent set. There are two possibilities.

Case 1. There exists j with $1 \leq j \leq k$ such that $c_3^{(j)}c_5^{(j)} \in E(G)$ (resp. $c_2^{(j)}c_4^{(j)} \in E(G)$).
In this case, $\{c_3^{(j)}c_5^{(j)}a_2a_3a_4c_3^{(j)}, c_1^{(j)}a_5a_6a_7a_8c_1^{(j)}\} \cup (\mathcal{C} - \{C^j\})$ (resp. $\{c_2^{(j)}c_4^{(j)} a_1a_2a_3c_2^{(j)}, c_1^{(j)}a_5a_6a_7a_8c_1^{(j)}\} \cup (\mathcal{C} - \{C^j\})$) forms a collection of $k+1$ vertex-disjoint 5-cycles, a contradiction.

Case 2. There exist j_1, j_2 with $1 \leq j_1 < j_2 \leq k$, and there exist p, q with $1 \leq p, q \leq 2$ such that $c_{2p+1}^{(j_1)}c_{2q+1}^{(j_2)} \in E(G)$ (resp. $c_{2p}^{(j_1)}c_{2q}^{(j_2)} \in E(G)$).
In this case, $\{c_{2p+1}^{(j_1)}c_{2q+1}^{(j_2)}a_2a_3a_4c_{2p+1}^{(j_1)}, c_1^{(j_1)}a_1a_6a_5c_2^{(j_1)}c_1^{(j_1)}, c_1^{(j_2)}a_7a_8a_9a_{10}c_1^{(j_2)}\}$ $\cup (\mathcal{C} - \{C^{j_1}, C^{j_2}\})$ (resp. $\{c_{2p}^{(j_1)}c_{2q}^{(j_2)}a_1a_2a_3c_{2p}^{(j_1)}, c_1^{(j_1)}a_4a_5a_6c_5^{(j_1)}c_1^{(j_1)}, c_1^{(j_2)}a_7 a_8a_9a_{10}c_1^{(j_2)}\} \cup (\mathcal{C} - \{C^{j_1}, C^{j_2}\})$) forms a collection of $k+1$ vertex-disjoint 5-cycles, a contradiction. □

Claim G implies the following facts:

(i') For all i, $N(c_2^{(i)}, V(\mathcal{C})) \subseteq R \cup B_1 = \cup_{i=1}^{k}\{c_1^{(i)}, c_3^{(i)}, c_5^{(i)}\}$;
(ii') for all i, $N(c_3^{(i)}, V(\mathcal{C})) \subseteq R \cup B_2 = \cup_{i=1}^{k}\{c_1^{(i)}, c_2^{(i)}, c_4^{(i)}\}$;
(iii') for all i, $N(c_4^{(i)}, V(\mathcal{C})) \subseteq R \cup B_1 = \cup_{i=1}^{k}\{c_1^{(i)}, c_3^{(i)}, c_5^{(i)}\}$;
(iv') for all i, $N(c_5^{(i)}, V(\mathcal{C})) \subseteq R \cup B_2 = \cup_{i=1}^{k}\{c_1^{(i)}, c_2^{(i)}, c_4^{(i)}\}$.

On the other hand, for each $1 \leq i \leq k$, since $c_2^{(i)}a_2$, $c_4^{(i)}a_2$, $c_3^{(i)}a_1$, $c_5^{(i)}a_1 \notin$

$E(G)$ and $d(a_1) = d(a_2) = 3k + s$, we obtain $d(c_2^{(i)}) \geq \sigma_2(G) - d(a_2) \geq (6k + 2s) - (3k + s) = 3k + s$, $d(c_3^{(i)}) \geq 3k + s$, $d(c_4^{(i)}) \geq 3k + s$ and $d(c_5^{(i)}) \geq 3k + s$. Therefore it follows from (ii),(iii),(i'),(ii'),(iii') and (iv') that for every $1 \leq i \leq k$, we have

$$N(c_2^{(i)}) = N(c_4^{(i)}) = R \cup A_1 \cup B_1 \text{ and } N(c_3^{(i)}) = N(c_5^{(i)}) = R \cup A_2 \cup B_2. \quad (8)$$

It immediately follows from (i) and (8) that $N(c_1^{(i)}) \supseteq A_1 \cup A_2 \cup B_1 \cup B_2$ for all $1 \leq i \leq k$. Hence $G = G[R] + G[\cup_{i=1}^{2}(A_i \cup B_i)]$. Since (8) also implies that $G[\cup_{i=1}^{2}(A_i \cup B_i)]$ is a complete bipartite graph $K_{2k+s,2k+s}$ with bipartition $(A_1 \cup B_1,\ A_2 \cup B_2)$, and since we clearly have $\overline{K_k} \subseteq G[R] \subseteq K_k$, this completes the proof of Lemma 4.7. □

We are now ready to prove Theorem 1.1. We restate it here in the following form (note that $\overline{K_k} + K_{2k+s,2k+s} \subseteq G \subseteq K_k + K_{2k+s,2k+s}$ is equivalent to $G \subseteq K_k + K_{2k+s,2k+s}$ under the assumption of Theorem 1.1).

Theorem 1.1. *Let $k \geq 1, s \geq 5$ be integers, and let G be a graph of order $5k + 2s$ such that $\sigma_2(G) \geq 6k + 2s$ and $G \not\subseteq K_k + K_{2k+s,2k+s}$. Then $G \supseteq (k+1)C_5 \cup P_{2s-5}$.*

Proof of Theorem 1.1. Suppose that the statement is false, and let G be an edge maximal counterexample. Then $G \supseteq kC_5$, and hence $G \supseteq kC_5 \cup P_{2s}$ by Lemma 2.6 (i). Since $G \not\subseteq K_k + K_{2k+s,2k+s}$ by the assumption, this together with Lemma 4.7 implies $G \supseteq (k + 1)C_5$, and we therefore obtain $G \supseteq (k+1)C_5 \cup P_{2s-5}$ by Lemma 2.6 (ii). This contradicts the assumption that G is a counterexample, and this contradiction completes the proof of Theorem 1.1. □

5 Proof of Corollary 1.3

In this short section, we prove Corollary 1.3. We start with the following simple lemma, which is an easy consequence of Theorem 1.1.

Lemma 5.1. *Let $k \geq 1$ and $s \geq 4k^2 + 6k + 5$, and let G be a graph of order $n = 5k + 2s$ with $e(G) \geq e(K_k + K_{2k+s,2k+s})$. Suppose that $G \not\supseteq (k+1)C_5$. Then $G \supseteq K_k + K_{2k+3,2k+3}$.*

Proof. Suppose that $G \not\supseteq K_k + K_{2k+3,2k+3}$. We may assume that $\sigma_2(G) < \sigma_2(K_k + K_{2k+s,2k+s}) = 6k + 2s$. If not, $G \cong K_k + K_{2k+s,2k+s} \supseteq K_k + K_{2k+3,2k+3}$ or $G \supseteq (k + 1)C_5$ holds by Theorem 1.1, which is a contradiction. Set $G_n = G$. Then the same argument works for $G_{n-1} = G_n - \{a, b\}$ for any pair of nonadjacent vertices a and b of degree sum strictly less than

$\sigma_2(K_k + K_{2k+s,2k+s})$. In view of this fact, we get a sequence of graphs G_{n-m} of order $n-2m$ with at least $e(K_k + K_{2k+s-m,2k+s-m}) + m$ edges, where G_{n-m} is obtained from G_{n-m+1} by removing a pair of nonadjacent vertices of degree sum at most $\sigma_2(K_k + K_{2k+s-m+1,2k+s-m+1}) - 1$. Since $G \not\supseteq K_k + K_{2k+3,2k+3}$ and by Theorem 1.1, there exists a graph $G_{n-(s-2)}$ of order $5k+4$. Then $e(G_{n-(s-2)}) \geq e(K_k + K_{2k+2,2k+2}) + (s-2) > \frac{(5k+4)(5k+3)}{2} = e(K_{5k+4})$, which is a contradiction. This contradiction implies the desired conclusion. □

We are now ready to prove Corollary 1.3. We restate it here in the following equivalent form.

Corollary 1.3. *Let $k \geq 1$, and let G be a graph of order $n \geq 8k^2+17k+10$. Suppose that $G \not\supseteq (k+1)C_5$ and $e(G) \geq e(K_k + T_2(n-k))$. Then $G \cong K_k + T_2(n-k)$.*

Proof of Corollary 1.3. Suppose that $G \not\cong K_k + T_2(n-k)$. We first show the case where $n-5k$ is even, and let $n-5k = 2s$. By Lemma 5.1, G contains a subgraph H of order $5k+6$ such that $H \supseteq K_k + K_{2k+3,2k+3}$. We can write $V(H) = R \,\dot\cup\, B_1 \,\dot\cup\, B_2$ such that $R = \{r_1, \cdots, r_k\}$, $B_1 = \{b_1, b_3, \cdots, b_{4k+5}\}$ and $B_2 = \{b_2, b_4, \cdots, b_{4k+6}\}$, where $G[R] \cong K_k$ and $G[B_1 \cup B_2]$ contains a complete bipartite graph $K_{2k+3,2k+3}$ with bipartition (B_1, B_2). Set $G - V(H) = U$. Since $H \supseteq K_k + K_{2k+3,2k+3} \supseteq kC_5$, U does not contain a 5-cycle.

Claim H. *Both B_1 and B_2 are independent sets, i.e., $H \cong K_k + K_{2k+3,2k+3}$.*

Proof. Suppose that B_1(resp. B_2) is not an independent set. We may assume that $b_{4k+3}b_{4k+5} \in E(G)$ (resp. $b_{4k+4}b_{4k+6} \in E(G)$). Then $\cup_{i=1}^{k}\{r_i b_{4i-3}b_{4i-2}b_{4i-1}b_{4i}r_i\} \cup \{b_{4k+3}b_{4k+5}b_{4k+2}b_{4k+1}b_{4k+4}b_{4k+3}\}$ (resp. $\cup_{i=1}^{k}\{r_i b_{4i-3}b_{4i-2}b_{4i-1}b_{4i}r_i\} \cup \{b_{4k+4}b_{4k+6}b_{4k+1}b_{4k+2}b_{4k+3}b_{4k+4}\}$) forms a collection of $k+1$ vertex-disjoint 5-cycles, a contradiction. □

Claim I. *For any $v \in V(U)$,* $\min\{d(v, B_1), d(v, B_2)\} = 0$.

Proof. Suppose that there exists $v \in V(U)$ such that $\min\{d(v, B_1), d(v, B_2)\} \geq 1$. We may assume that $vb_{4k+5}, vb_{4k+6} \in E(G)$. Then $\cup_{i=1}^{k}\{r_i b_{4i-3}b_{4i-2}b_{4i-1}b_{4i}r_i\} \cup \{vb_{4k+5}b_{4k+2}b_{4k+3}b_{4k+6}v\}$ forms a collection of $k+1$ vertex-disjoint 5-cycles, a contradiction. □

Claim J. *For any $v \in V(U)$, $d(v, H) \leq 3k+3$ and equality holds if and only if either $N(v, H) = R \cup B_1$ or $N(v, H) = R \cup B_2$ holds.*

Proof. The claim follows immediately from Claim I. □

We return to the proof of Corollary 1.3. By Claim H, J and Lemma 4.4,

$$\begin{aligned} e(G) &= e(V(H)) + e(H,U) + e(V(U)) \\ &\leq \{\frac{k(k-1)}{2} + k(4k+6) + (2k+3)^2\} + (3k+3)(2s-6) + (s-3)^2 \\ &= \frac{k(k-1)}{2} + k(4k+2s) + (2k+s)^2 \\ &= e(K_k + T_2(n-k)). \end{aligned}$$

Hence $e(G) = e(K_k + T_2(n-k))$, $d(u,H) = 3k+3$ for each $u \in V(U)$, and $e(V(U)) = (s-3)^2$. In view of lemma 4.4, this implies that $U \cong K_{s-3,s-3}$ with bipartition (U_1, U_2).

Claim K. *For any edge $uv \in E(U)$, $N(u) \cap N(v) = R$.*

Proof. Suppose that there exists an edge $uv \in E(U)$ such that $N(u) \cap N(v) \neq R$. Since $d(x,H) = 3k+3$ for each $x \in V(U)$ and $U \cong K_{s-3,s-3}$, we may assume that $N(u) \cap N(v) = R \cup B_2$ by Claim J. Then $\cup_{i=1}^{k}\{r_i b_{4i-3} b_{4i-2} b_{4i-1} b_{4i} r_i\} \cup \{uvb_{4k+4}b_{4k+5}b_{4k+6}u\}$ forms a collection of $k+1$ vertex-disjoint 5-cycles, a contradiction. □

By symmetry and Claim J, we may assume that $N(u_1, H) = R \cup B_2$ for some $u_1 \in U_1$. Then using repeatedly Claim K, we conclude that $N(v_2, H) = R \cup B_1$ for each $v_2 \in U_2$ and $N(v_1, H) = R \cup B_2$ for each $v_1 \in U_1$. Therefore we obtain $G \cong K_k + T_2(n-k)$, which is a contradiction. This contradiction implies the case $n-5k$ is even.

Finally we consider the case $n-5k$ is odd. Let s be an integer so that $n-5k = 2s+1$. We may assume that $d(v) = \delta(G)$. If $\delta(G) \geq \delta(K_k + T_2(n-k))+1$ then, we see that $\delta(G - \{v\}) \geq \delta(K_k + T_2(n-1-k))$. Then by Theorem 1.1, $G - \{v\}$ contains a complete tripartite graph $\overline{K_k} + T_2(n-1-k)$. Since $\delta(G) \geq \delta(K_k + T_2(n-k))+1$, v is adjacent to at least one vertex in each of two large color classes of this complete tripartite graph. We clearly have that $G \supseteq (k+1)C_5$, which is a contradiction. Therefore we may assume that $\delta(G) \leq \delta(K_k + T_2(n-k))$. If $\delta(G) < \delta(K_k + T_2(n-k))$, then $e(G - \{v\}) > e(K_k + T_2(n-1-k))$, which implies that $G \supseteq G - \{v\} \supseteq (k+1)C_5$, a contradiction. Hence we may assume that $\delta(G) = \delta(K_k + T_2(n-k))$. Since $e(G - \{v\}) = e(G) - d(v) = e(K_k + T_2(n-1-k))$ and $G \not\supseteq (k+1)C_5$, we have $G - \{v\} \cong K_k + T_2(n-1-k)$. Now similarly to the case for $n-5k$ is odd and $\delta(G) \geq \delta(K_k + T_2(n-k))+1$, we see that $G \cong K_k + T_2(n-k)$. This contradicts the assumption that $G \not\cong K_k + T_2(n-k)$, and this contradiction completes the proof of Corollary 1.3. □

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