Large sets of λ-fold P_3-factors in $K_{v,v}$*

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Abstract: Let G be a finite graph and H be an subgraph of G. If $V(H) = V(G)$ then the subgraph H is called a *spanning subgraph* of G. A spanning subgraph H of G is called an *F-factor* if each component of H is isomorphic to F. Further if there exists a subgraph of G whose vertex set is $\lambda V(G)$ and can be partitioned into F-factors then it is called a *λ-fold F-factor* of G, denoted by $S_\lambda(1, F, G)$. A *large set* of λ-fold *F-factors* in G is a partition $\{\mathcal{B}_i\}_i$ of all subgraphs of G isomorphic to F, such that each $(X, \mathcal{B}_i)$ forms a λ-fold F-factor of G. In this paper, we investigate the *large set* of λ-fold *P_3-factors* in $K_{v,v}$ and obtain its existence spectrum.
key words: Large set; Hamilton cycle; P_3-factor; $LS_\lambda(1, P_3, K_{v,v})$

1 Introduction

A *complete multigraph* of order v and index λ, denoted by λK_v, is a graph with v vertices, where any two distinct vertices x and y are joined by λ edges $\{x, y\}$. Let $\lambda K_{n_1,n_2,\cdots,n_h}$ be a *complete multipartite graph* whose vertex set X consists of h disjoint sets $X_1, \cdots, X_h$, where $|X_i| = n_i$ and any two vertices x and y from different sets X_i and X_j are joined by exactly λ edges $\{x, y\}$.

Let $G = (V(G), E(G))$ be a finite graph. A subgraph H of G is called a *spanning subgraph* of G if $V(H) = V(G)$. Especially, H is called an *F-factor* if each component of H is isomorphic to a given graph F. Furthermore, if there exists a subgraph of G whose vertex set is $\lambda V(G)$ and can be partitioned into F-factors then it is called a *λ-fold F-factor* of G, denoted

*Research supported by NSFC Grant 10971051 and NSFH Grant A2007000230.

by $S_\lambda(1, F, G)$. A *λ-fold F-factorization* of G is a set of edge-disjoint *λ-fold F-factors* of G, whose edge sets partition the edges of G. For $\lambda = 1$, it is called an *F-factorization* of G. Particularly, if F is just an edge of G, then the F-factor is called a *one-factor* of G, and the corresponding F-factorization is called a *one-factorization* of G.

A *t-wise balanced design* $S_\lambda(t, K, v)$ is a pair $(X, \mathcal{B})$, where X is a v-set, K is a set of positive integers and $\mathcal{B}$ is a collection of subsets of X with size in K, called *blocks*, such that each t-subset of X appears exactly in λ blocks of $\mathcal{B}$. When $K = \{k\}$, it is called a *t-design* and briefly denoted by $S_\lambda(t, k, v)$. For $\lambda = 1$, the index 1 is often omitted. A $S(t, K, v)$ is called *separable* if it can be partitioned into some $S(t-1, k, v)$, where $k \in K$. An $S_\lambda(2, 3, v)$ is called a *triple system* of order v and index λ, briefly denoted by $TS(v, \lambda)$.

A *λ-parallel class* on a block design $(X, \mathcal{B})$ is a set of some blocks in $\mathcal{B}$, which forms a partition of λX. A 1-parallel class is simply called a *parallel class.* If the block set of a $TS(v, \lambda)$ can be partitioned into parallel classes, then it is called a *resolvable triple system* of order v and index λ and denoted by $RTS(v, \lambda)$. For $\lambda = 1$, $TS(v, \lambda)$ and $RTS(v, \lambda)$ are a Steiner triple system and a Kirkman triple system, respectively, which we briefly denote by $STS(v)$ and $KTS(v)$.

A *group divisible design, k-$GDD(g^m)$*, is a trio $(X, \mathcal{G}, \mathcal{B})$, where X is a set of order gm, $\mathcal{G}$ is a partition of X into g-subsets, called *groups*, $\mathcal{B}$ is a collection of k-subsets of X, called *blocks*, such that $|B \cap G| \le 1$ for each block $B \in \mathcal{B}$ and $G \in \mathcal{G}$, and every 2-subset of X belonging to different groups appears exactly in one block of $\mathcal{B}$. Furthermore, if the block set $\mathcal{B}$ can be partitioned into parallel classes, then it is called *resolvable* and denoted by *k-$RGDD(g^m)$*.

Lemma 1.1. *There exists a separable* $S(2, \{2, 3\}, v)$ *for* $v \equiv 0 \bmod 3$ *and* $v \ne 6, 12$.

Proof.

(1). For $v \equiv 3 \bmod 6$, there exists a $KTS(v) = (Z_v, \mathcal{B})$, where $\mathcal{B}$ consists of $\frac{v-1}{2}$ parallel classes $\mathcal{P}_i$, and each $\mathcal{P}_i$ is an $S(1, 3, v)$. So, the $KTS(v)$ is separable.

(2). For $v \equiv 0 \bmod 6$ and $v \ne 6, 12$, there exists a 3-$RGDD(2^{v/2})$ = $(Z_v, \mathcal{G}, \mathcal{B})$ from [1], where $\mathcal{B}$ consists of $\frac{v-2}{2}$ parallel classes $\mathcal{P}_i$, each $\mathcal{P}_i$ is an $S(1, 3, v)$, and $\mathcal{G}$ is an $S(1, 2, v)$. So, $(Z_v, \mathcal{G} \cup \mathcal{B})$ is a separable $S(2, \{2, 3\}, v)$ indeed. ∎

A *k-cycle*, denoted by $(x_1, x_2, \cdots, x_k)$, is a subgraph of K_v, which consists of k ($\le v$) distinct points $x_1, x_2, \cdots, x_k$ and k edges $\{x_1, x_2\}, \cdots, \{x_{k-1}, x_k\}, \{x_k, x_1\}$. When $k = v$, it is called a *Hamilton cycle* of K_v. A *k-cycle system* of order v and index λ, $CS(v, k, \lambda)$, is a collection $\mathcal{C}$ of k-cycles of

K_v, such that each edge of K_v appears exactly in λ members of $\mathcal{C}$. Especially, a $CS(v, v, 1)$ is called a *Hamilton cycle decomposition* of K_v.

Lemma 1.2.[1] For $n \geq 1$, there exist

a Hamilton cycle decomposition of K_{2n+1},
a Hamilton cycle decomposition of $K_{2n} \backslash \Gamma$, and
a one-factorization of K_{2n},

where Γ is a one-factor of K_{2n},

$K_{p,q}$-factorization of complete bipartite graph $K_{m,n}$ has been applied in many fields. Particularly, Yamamoto [9] applied it to construct $HUBMFS_2$ scheme. For the path graph P_k with k vertices, Ushio[7] completely solved the existence of P_3-*factorizations* of $K_{m,n}$. From then on, the existence problems of $K_{p,q}$-*factorization* of $K_{m,n}$ have been widely researched, see [2-6,8].

A *large set* of λ-fold F-*factors* in G, denoted by $LS_\lambda(1, F, G)$, is a partition $\{\mathcal{B}_i\}_i$ of all subgraphs of G isomorphic to F, such that each $(X, \mathcal{B}_i)$ forms a λ-fold F-*factor* of G. In this paper, we will discuss the existence of $LS_\lambda(1, P_3, K_{v,v})$ and obtain its spectrum.

2 Main Constructions

An $S_\lambda(1, P_3, K_{v,v})$ consists of $\frac{2\lambda v}{3}$ blocks, and an $LS_\lambda(1, P_3, K_{v,v})$ consists of $\frac{3v(v-1)}{2\lambda}$ disjoint $S_\lambda(1, P_3, K_{v,v})$s. The point set of $K_{v,v}$ is taken as $Z_v \cup \overline{Z}_v$. Suppose that an $S_\lambda(1, P_3, K_{v,v})$ consists of x P_3-blocks in the form $[a, \overline{c}, b]$, and y P_3-blocks in the form $[\overline{a}, c, \overline{b}]$, then

$$\begin{cases} 2x + y = \lambda v \\ x + 2y = \lambda v \end{cases} \Longrightarrow x = y = \frac{\lambda v}{3}.$$

Thus, there exists an $LS_\lambda(1, P_3, K_{v,v})$ only if $3|\lambda v$ and $\lambda|3\binom{v}{2}$. We need only to consider the following cases:

$$\lambda = 1, \ v \equiv 0 \bmod 3; \quad \lambda = 3, \ v \equiv 1, 2 \bmod 3.$$

2.1 Case $3|v$

Lemma 2.1. There exists an $LS(1, P_3, K_{v,v})$ for $v \equiv 0 \bmod 3$ and $v \neq 6, 12$.

Construction. By Lemma 1.1, there exists a separable $S(2, \{2, 3\}, v)$

$\{(Z_v, \mathcal{P}_h) : 1 \leq h \leq 3t + 1\}$ for $v = 6t + 3$,
$\{(Z_v, \mathcal{P}_h) : 1 \leq h \leq 3t - 1\} \cup \{(Z_v, \mathcal{Q})\}$ for $v = 6t \geq 18$,

where $(Z_v, \mathcal{Q})$ is a $S(1, 2, 6t)$, and $\mathcal{Q} = \{\{i, i+3t\} : i \in Z_{3t}\}$. Each $(Z_v, \mathcal{P}_h)$ is an $S(1, 3, v)$, which consists of $\frac{v}{3}$ 3-subsets $\{a_r, b_r, c_r\}, r \in Z_{\frac{v}{3}}$.

According to natural order $a_r < b_r < c_r$, we define the following three collections of ordered 3-tuples:

$\mathcal{P}_h^1 = \{(a_r, b_r, c_r) : r \in Z_{\frac{v}{3}}\}$, $\mathcal{P}_h^2 = \{(b_r, c_r, a_r) : r \in Z_{\frac{v}{3}}\}$,
$\mathcal{P}_h^3 = \{(c_r, a_r, b_r) : r \in Z_{\frac{v}{3}}\}$.

On each $\mathcal{P}_h$, define a mapping

$$\sigma_h : \ (a_r, b_r, c_r) \mapsto \ (a_{r+1}, b_{r+1}, c_{r+1}),$$

which induces a permutation on Z_v: $a_r \to a_{r+1}, b_r \to b_{r+1}, c_r \to c_{r+1}$, $r \in Z_{\frac{v}{3}}$. Thus, the cyclic group $\langle \sigma_h \rangle$ of order $\frac{v}{3}$ generated by σ_h divides all elements of Z_v into three orbits:

$$(a_0, a_1, \cdots, a_{\frac{v}{3}-1}),\ (b_0, b_1, \cdots, b_{\frac{v}{3}-1}), (c_0, c_1, \cdots, c_{\frac{v}{3}-1}).$$

If x and y in Z_v belong to the same orbit, then it is denoted by $x \in \mathcal{O}_h(y)$ or $y \in \mathcal{O}_h(x)$. Take the point set of $K_{v,v}$ as $Z_v \cup \overline{Z}_v$. Define the following collections of P_3-blocks on $K_{v,v}$, where $k \in Z_{\frac{v}{3}}, s = 1, 2, 3; 1 \le h \le \lfloor \frac{v-1}{2} \rfloor, v = 6t + 3$ or $6t$.

$$\mathcal{A}_{h,k}^s = \{[a, \overline{\sigma_h^k(a)}, b], [\overline{\sigma_h^k(b)}, c, \overline{\sigma_h^k(c)}] : (a, b, c) \in \mathcal{P}_h^s\},$$

$$\mathcal{B}_{h,k}^s = \{[a, \overline{\sigma_h^k(b)}, b], [\overline{\sigma_h^k(c)}, c, \overline{\sigma_h^k(a)}] : (a, b, c) \in \mathcal{P}_h^s\},$$

$$\mathcal{C}_{h,k}^s = \{[a, \overline{\sigma_h^k(c)}, b], [\overline{\sigma_h^k(a)}, c, \overline{\sigma_h^k(b)}] : (a, b, c) \in \mathcal{P}_h^s\}.$$

For $v = 6t$, define

$$\mathcal{D}_0 = \{[2i, \overline{2t+1+i}, 2i+3t], [\overline{2i+1}, 2t+i, \overline{2i+1+3t}], [2i+1, \overline{5t+1+i}, 2i+1+3t], [\overline{2i+2}, 5t+i, \overline{2i+2+3t}] : i \in Z_t\},$$

$$\mathcal{E}_0 = \{[2i, \overline{t+1+i}, 2i+3t], [\overline{2t+1+2i}, 2t+i, \overline{5t+1+2i}], [2i+1, \overline{4t+1+i}, 2i+1+3t], [\overline{2t+2+2i}, 5t+i, \overline{5t+2+2i}] : i \in Z_t\},$$

$$\mathcal{F}_0 = \{[2i, \overline{1+i}, 2i+3t], [\overline{t+1+2i}, 2t+i, \overline{4t+1+2i}], [2i+1, \overline{1+i+3t}, 2i+1+3t], [\overline{t+2+2i}, 5t+i, \overline{4t+2+2i}] : i \in Z_t\}.$$

Furthermore, denote $\mathcal{D}_x = \mathcal{D}_0 + x, \mathcal{E}_x = \mathcal{E}_0 + x$ and $\mathcal{F}_x = \mathcal{F}_0 + x$, where $x \in Z_{3t}$. Then each of $\mathcal{A}_{h,k}^s, \mathcal{B}_{h,k}^s, \mathcal{C}_{h,k}^s, \mathcal{D}_x, \mathcal{E}_x$ and $\mathcal{F}_x$ is an $S(1, P_3, K_{v,v})$. And the following collections form an $LS(1, P_3, K_{v,v})$ on $Z_v \cup \overline{Z}_v$ respectively:

$\{\mathcal{A}_{h,k}^s, \mathcal{B}_{h,k}^s, \mathcal{C}_{h,k}^s : k \in Z_{2t+1}, 1 \le h \le 3t+1, s = 1, 2, 3\}$, for $v = 6t + 3$;
$\{\mathcal{A}_{h,k}^s, \mathcal{B}_{h,k}^s, \mathcal{C}_{h,k}^s : k \in Z_{2t}, 1 \le h \le 3t-1, s = 1, 2, 3\} \cup \{\mathcal{D}_x, \mathcal{E}_x, \mathcal{F}_x : x \in Z_{3t}\}$, for $v = 6t$.

Proof. First, each $\mathcal{A}_{h,k}^s, \mathcal{B}_{h,k}^s, \mathcal{C}_{h,k}^s$ is just an $S(1, P_3, K_{v,v})$, and $\mathcal{D}_x, \mathcal{E}_x$ and $\mathcal{F}_x$ are also. For example, the point set of Z_v covered by $4t$ P_3-blocks in $\mathcal{D}_0$ is:

$$\{2i, 2i+3t, 2i+1, 2i+1+3t, 2t+i, 5t+i : i \in Z_t\} = [2t, 3t-1] \cup [5t, 6t-1] \cup [3t, 5t-2]_2 \cup [1, 2t-1]_2 \cup [3t+1, 5t-1]_2 \cup [0, 2t-2]_2 = [0, 6t-1].$$

Accordingly, the point set of $\overline{Z}_v$ covered by $4t$ P_3-blocks in $\mathcal{D}_0$ is:

$$\{\overline{2t+1+i}, \overline{5t+1+i}, \overline{2i+1}, \overline{2i+1+3t}, \overline{2i+2}, \overline{2i+2+3t} : i \in Z_t\} = [\overline{2t+1}, \overline{3t}] \cup [\overline{5t+1}, \overline{6t}] \cup [\overline{1}, \overline{2t-1}]_2 \cup [\overline{3t+1}, \overline{5t-1}]_2 \cup [\overline{2}, \overline{2t}]_2 \cup [\overline{3t+2}, \overline{5t}]_2 = [\overline{1}, \overline{6t}] = [\overline{0}, \overline{6t-1}].$$

For $v = 6t+3$, the total number of $\mathcal{A}^s_{h,k}, \mathcal{B}^s_{h,k}, \mathcal{C}^s_{h,k}$ is $9(2t+1)(3t+1)$; for $v = 6t \geq 18$, the total number of $\mathcal{A}^s_{h,k}, \mathcal{B}^s_{h,k}, \mathcal{C}^s_{h,k}, \mathcal{D}_x, \mathcal{E}_x, \mathcal{F}_x$ is $9 \cdot 2t(3t-1)+9t$, as expected. Below we only need to verify that each P_3-block in the form $T = [x, \overline{z}, y]$ or $T' = [\overline{x}, z, \overline{y}]$ appears in one $S(1, P_3, K_{v,v})$.

Case 1: $v = 6t+3$. We need only to consider P_3-blocks in the form T (T' is similar).

Since $\mathcal{P} = \{\mathcal{P}_h : 1 \leq h \leq 3t+1\}$ is a $KTS(6t+3)$, there exists a block $B \in \mathcal{P}$ which contains $\{x, y\}$. Let $B = \{x, y, z'\} \in \mathcal{P}_h$. Then for some $s \in \{1, 2, 3\}$, we have (x, y, z') or $(y, x, z') \in \mathcal{P}^s_h$. Furthermore, by the property of cyclic group $\langle \sigma_h \rangle$,

if $z \in \mathcal{O}_h(x)$, then $\exists\ k \in Z_{2t+1}$, such that $\sigma^k_h(x) = z$, so $T \in \mathcal{A}^s_{h,k}$;

if $z \in \mathcal{O}_h(y)$, then $\exists\ k \in Z_{2t+1}$, such that $\sigma^k_h(y) = z$, so $T \in \mathcal{B}^s_{h,k}$;

if $z \in \mathcal{O}_h(z')$, then $\exists\ k \in Z_{2t+1}$, such that $\sigma^k_h(z') = z$, so $T \in \mathcal{C}^s_{h,k}$.

Case 2: $v = 6t \geq 18$. When $y - x \neq 3t$, it is the same to the case $v = 6t+3$. So we only consider $y = x + 3t$, i.e.,

$$T = [x, \overline{z}, x+3t] \text{ or } T' = [\overline{x}, z, \overline{x+3t}],$$

where $\{x, x+3t\} \in \mathcal{Q}$, i.e., each pair $\{x, x+3t\}$ is not contained in any $\mathcal{P}_h$ $(1 \leq h \leq 3t-1)$. We need only to verify that the set of ordered difference $d = z - x$ (or $\overline{d} = z - y$) covered by the blocks in the form T (or T') of $\{\mathcal{D}_x, \mathcal{E}_x, \mathcal{F}_x : x \in Z_{3t}\}$ are both $Z_{3t} \cup \overline{Z}_{3t}$. In fact,

$z - x : \{2t+1-i, t+1-i, 1-i : 0 \leq i \leq t-1\} =$
$[t+2, 2t+1] \cup [2, t+1] \cup [2t+2, 3t-1] \cup [0, 1] = [0, 3t-1]$;

$z - y : \{\overline{2t-i}, \overline{t-i}, \overline{-i} : 0 \leq i \leq t-1\} =$
$[\overline{t+1}, \overline{2t}] \cup [\overline{1}, \overline{t}] \cup [\overline{2t+1}, \overline{3t-1}] \cup [\overline{0}] = [\overline{0}, \overline{3t-1}]$;

$z - x : \{2t-1-i, -1-i, t-1-i : 0 \leq i \leq t-1\} =$
$[t, 2t-1] \cup [2t, 3t-1] \cup [0, t-1] = [0, 3t-1]$;

$z - y : \{\overline{2t-2-i}, \overline{-2-i}, \overline{t-2-i} : 0 \leq i \leq t-1\} =$
$[\overline{t-1}, \overline{2t-2}] \cup [\overline{2t-1}, \overline{3t-2}] \cup [\overline{0}, \overline{t-2}] \cup [\overline{3t-1}] = [\overline{0}, \overline{3t-1}]$. ■

Lemma 2.2. *There exists an* $LS(1, P_3, K_{6,6})$.

Proof. By Lemma 1.2., there exists an one-factorization $\{F_k : 1 \leq k \leq 5\}$ of K_6 on I_6. Each one-factor F_k consists of three pairs $P_{k,i} = \{p^1_{k,i}, p^2_{k,i}\}$, $i \in Z_3$. Take the point set as $I_6 \times I_6$. In $K_{6,6}$, denote the P_3-block with single point on the left or on the right by $[p, P]$ (*or* $[P, p]$). For $i, j \in Z_3$, $1 \leq k \leq 5$, define the following collection of P_3-blocks:

$$\mathcal{A}^k_{i,j} = \{[p^1_{k,i}, P_{k,j}], [p^2_{k,i}, P_{k,j+1}], [P_{k,i+1}, p^1_{k,j+2}], [P_{k,i+2}, p^2_{k,j+2}]\}.$$

Obviously, for certain k, i, j it is an $S(1, P_3, K_{6,6})$. Then,

$$\bigcup_{i \in Z_3} \{p^1_{k,i}, p^2_{k,i}\} = \bigcup_{j \in Z_3} \{p^1_{k,j+2}, p^2_{k,j+2}\} = I_6.$$

For certain k, when i, j run over Z_3, $P_{k,j}, P_{k,j+1}, P_{k,i+1}$ and $P_{k,i+2}$ covers the 2-subsets of F_k once, respectively. Then, for $i, j \in Z_3$ and $1 \leq k \leq 5$, all blocks of $\mathcal{A}^k_{i,j}$ just cover all P_3-blocks of $K_{6,6}$. It is shown that $\{\mathcal{A}^k_{i,j} : i, j \in Z_3, 1 \leq k \leq 5\}$ forms an $LS(1, P_3, K_{6,6})$, which consists of $3 \times 3 \times 5 = 45$

$S(1, P_3, K_{6,6})$s. ■

Lemma 2.3. *There exists an* $LS(1, P_3, K_{12,12})$.

Proof. By Lemma 1.2., there exists an one-factorization $\{F_k : 1 \le k \le 11\}$ of K_{12} on I_{12}. Each one-factor F_k consists of six pairs $P_{k,i} = \{p^1_{k,i}, p^2_{k,i}\}$, $i \in Z_6$. For each one-factor F_k and $i \in Z_6, j = 0, 1, 2$, take the point set as $I_{12} \times I_{12}$, define the following collection of P_3-blocks:

$$\mathcal{A}^k_{i,j} = \{[p^1_{k,i}, P_{k,j}], [p^2_{k,i}, P_{k,j+1}], [p^1_{k,i+1}, P_{k,j+3}], [p^2_{k,i+1}, P_{k,j+4}], [P_{k,i+2}, p^1_{k,4j+2}], [P_{k,i+3}, p^2_{k,4j+2}], [P_{k,i+4}, p^1_{k,4j+5}], [P_{k,i+5}, p^2_{k,4j+5}]\}.$$

Since for any $j = 0, 1, 2$ we have $\{j, j+1, j+3, j+4, 4j+2, 4j+5\} = Z_6$, it is easy to see that each $\mathcal{A}^k_{i,j}$ is just an $S(1, P_3, K_{12,12})$ on $I_{12} \times I_{12}$. For certain k,

$$\bigcup_{i\in Z_6} \{p^1_{k,i}\} = \bigcup_{i\in Z_6} \{p^1_{k,i+1}\} = \bigcup_{j\in Z_3} \{p^1_{k,4j+2}, p^1_{k,4j+5}\} = H = \tfrac{1}{2} I_{12};$$

$$\bigcup_{i\in Z_6} \{p^2_{k,i}\} = \bigcup_{i\in Z_6} \{p^2_{k,i+1}\} = \bigcup_{i\in Z_6} \{p^2_{k,4j+2}, p^2_{k,4j+5}\} = I_{12} \backslash H.$$

When i runs over Z_6, $P_{k,i+2}, P_{k,i+3}, P_{k,i+4}$ and $P_{k,i+5}$ cover all 2-subsets of F_k once. When j runs over $\{0, 1, 2\}$, $P_{k,j} \cup P_{k,j+3}$ and $P_{k,j+1} \cup P_{k,j+4}$ cover all 2-subsets of F_k once also.

So, for $i \in Z_6, j = 0, 1, 2$ and $1 \le k \le 11$, all blocks of each $\mathcal{A}^k_{i,j}$ just cover all P_3-blocks of $K_{12,12}$. It is showed that $\{\mathcal{A}^k_{i,j} : i \in Z_6, 1 \le k \le 11, j = 0, 1, 2\}$ can form an $LS(1, P_3, K_{12,12})$, which consists of $6 \times 3 \times 11 = 198$ $S(1, P_3, K_{12,12})$s. ■

Theorem 2.4. *There exists an* $LS(1, P_3, K_{v,v})$ *if and only if* $3|v$.

Proof. Combining Lemma 2.1., Lemma 2.2. and Lemma 2.3., we complete the proof. ■

2.2 Case $3 \not| v$

Theorem 2.5. *There exists an* $LS_3(1, P_3, K_{v,v})$ *for* $v \equiv 1, 2 \bmod 3$ *and* $v \ge 2$.

Construction. By Lemma 1.2., let $\{\Omega_k : 1 \le k \le \lfloor \frac{v-1}{2} \rfloor\}$ be Hamilton cycle decomposition on K_v (odd v) or $K_v \backslash \Theta$ (even v), where each Ω_k is a Hamilton cycle, whereas $\Theta = \{\{e_i, f_i\} : 0 \le i \le \frac{v-2}{2}\}$ is an one-factor of K_v for even v. Take the point set of $K_{v,v}$ as $Z_v \cup \overline{Z}_v$. For each Hamilton cycle $\Omega_k = (a_0, a_1, \cdots, a_{v-1})$ and $d \in Z_v$, define the collections of P_3-blocks

$$\mathcal{A}^k_d = \{[\overline{a}_{i+d}, a_i, \overline{a}_{i+d+1}], [a_{i+d}, \overline{a}_i, a_{i+d+1}] : i \in Z_v\}.$$

Furthermore, for even v and $0 \le r \le \frac{v-2}{2}$, define

$$\mathcal{B}_r = \{[\overline{e}_{i+r}, i, \overline{f}_{i+r}], [e_{i+r}, \overline{i}, f_{i+r}], [\overline{e}_{i+r}, i + \tfrac{v}{2}, \overline{f}_{i+r}], [e_{i+r}, \overline{i + \tfrac{v}{2}}, f_{i+r}] : i \in Z_{\frac{v}{2}}\}.$$

Then the following collections form an $LS_3(1, K_{1,2}, K_{v,v})$:

$\{\mathcal{A}_d^k : 1 \le k \le \frac{v-1}{2}, d \in Z_v\}$, for odd v;
$\{\mathcal{A}_d^k : 1 \le k \le \frac{v-2}{2}, d \in Z_v\} \cup \{\mathcal{B}_r : 0 \le r \le \frac{v-2}{2}\}$, for even v.

Proof. First, since each Ω_k is a Hamilton cycle, whereas Θ is an one-factor for even v, so each $\mathcal{A}_d^k, \mathcal{B}_r$ is just an $S_3(1, P_3, K_{v,v})$. The total number of these parallel classes is $\frac{v(v-1)}{2} = \frac{v(v-2)}{2} + \frac{v}{2}$, as expected. Next, we need to consider the P_3-blocks in the form $T = [\overline{y}, x, \overline{z}]$ (or $[y, \overline{x}, z]$) of $K_{v,v}$ on $Z_v \cup \overline{Z}_v$.

odd v: Let $\{\Omega_h : 1 \le h \le \frac{v-1}{2}\}$ be a Hamilton cycle decomposition on K_v. For each edge $\{y, z\}$, there exists an edge $\{a_{i+d}, a_{i+d+1}\}$ of Ω_k for certain $k \in [1, \frac{v-1}{2}]$, such that $\{a_{i+d}, a_{i+d+1}\}=\{y, z\}$. Furthermore, since $\{\overline{a}_i : i \in Z_v\}=\overline{Z}_v$, there exists some i, such that $\overline{x} = \overline{a}_i$. So, $T \in \mathcal{A}_d^k$.

even v: Let $\{\Omega_h : 1 \le h \le \frac{v-2}{2}\}$ be a Hamilton cycle decomposition on $K_v \backslash \Theta$. For each edge $\{y, z\}$, there exists some $k \in [1, \frac{v-2}{2}]$, such that $\{y, z\}$ appears in Ω_k or one-factor Θ. If the former is right, it is the same as case odd v. On the contrary, since $\{\{i\} \cup \{i + \frac{v}{2}\} : 0 \le i \le \frac{v-2}{2}\} = Z_v$, there exists some i, such that $i = x$ or $i + \frac{v}{2} = x$. So, $T \in \mathcal{B}_r$. ∎

Example 1 $LS_3(1, K_{1,2}, K_{4,4}) = \{(Z_4 \cup \overline{Z}_4, \mathcal{A}_d^1 \cup \mathcal{B}_r) : d \in Z_4, 0 \le r \le 1\}$.

First, let $\Theta = \{\{0,2\},\{1,3\}\}$ be an one-factor of K_4. Then a Hamilton cycle decomposition on $K_4 \backslash \Theta$ consists of one Hamilton cycle $\Omega_1 = \{(0,1,2,3)\}$. So, we can list the construction.

$\mathcal{A}_0^1 = \{[\overline{0},0,\overline{1}],[\overline{1},1,\overline{2}],[\overline{2},2,\overline{3}],[\overline{3},3,\overline{0}],[0,\overline{0},1],[1,\overline{1},2],[2,\overline{2},3],[3,\overline{3},0]\}$;
$\mathcal{A}_1^1 = \{[\overline{1},0,\overline{2}],[\overline{2},1,\overline{3}],[\overline{3},2,\overline{0}],[\overline{0},3,\overline{1}],[1,\overline{0},2],[2,\overline{1},3],[3,\overline{2},0],[0,\overline{3},1]\}$;
$\mathcal{A}_2^1 = \{[\overline{2},0,\overline{3}],[\overline{3},1,\overline{0}],[\overline{0},2,\overline{1}],[\overline{1},3,\overline{2}],[2,\overline{0},3],[3,\overline{1},0],[0,\overline{2},1],[1,\overline{3},2]\}$;
$\mathcal{A}_3^1 = \{[\overline{3},0,\overline{0}],[\overline{0},1,\overline{1}],[\overline{1},2,\overline{2}],[\overline{2},3,\overline{3}],[3,\overline{0},0],[0,\overline{1},1],[1,\overline{2},2],[2,\overline{3},3]\}$;
$\mathcal{B}_0 = \{[\overline{0},0,\overline{2}],[0,\overline{0},2],[\overline{1},1,\overline{3}],[1,\overline{1},3],[\overline{0},2,\overline{2}],[0,\overline{2},2],[\overline{1},3,\overline{3}],[1,\overline{3},3]\}$;
$\mathcal{B}_1 = \{[\overline{1},0,\overline{3}],[1,\overline{0},3],[\overline{0},1,\overline{2}],[0,\overline{1},2],[\overline{1},2,\overline{3}],[1,\overline{2},3],[\overline{0},3,\overline{2}],[0,\overline{3},2]\}$.

Example 2 $LS_3(1, K_{1,2}, K_{5,5}) = \{(Z_5 \cup \overline{Z}_5, \mathcal{A}_d^k) : 1 \le k \le 2, d \in Z_5\}$.

First, a Hamilton cycle decomposition on K_5 consists of two Hamilton cycles $\Omega_1 = \{(0,1,2,3,4)\}$ and $\Omega_2 = \{(0,2,4,1,3)\}$. So, we can list the construction.

$\mathcal{A}_0^1 = \{[\overline{0},0,\overline{1}],[\overline{1},1,\overline{2}],[\overline{2},2,\overline{3}],[\overline{3},3,\overline{4}],[\overline{4},4,\overline{0}],$
$[0,\overline{0},1],[1,\overline{1},2],[2,\overline{2},3],[3,\overline{3},4],[4,\overline{4},0]\}$;
$\mathcal{A}_1^1 = \{[\overline{1},0,\overline{2}],[\overline{2},1,\overline{3}],[\overline{3},2,\overline{4}],[\overline{4},3,\overline{0}],[\overline{0},4,\overline{1}],$
$[1,\overline{0},2],[2,\overline{1},3],[3,\overline{2},4],[4,\overline{3},0],[0,\overline{4},1]\}$;
$\mathcal{A}_2^1 = \{[\overline{2},0,\overline{3}],[\overline{3},1,\overline{4}],[\overline{4},2,\overline{0}],[\overline{0},3,\overline{1}],[\overline{1},4,\overline{2}],$
$[2,\overline{0},3],[3,\overline{1},4],[4,\overline{2},0],[0,\overline{3},1],[1,\overline{4},2]\}$;
$\mathcal{A}_3^1 = \{[\overline{3},0,\overline{4}],[\overline{4},1,\overline{0}],[\overline{0},2,\overline{1}],[\overline{1},3,\overline{2}],[\overline{2},4,\overline{3}],$
$[3,\overline{0},4],[4,\overline{1},0],[0,\overline{2},1],[1,\overline{3},2],[2,\overline{4},3]\}$;
$\mathcal{A}_4^1 = \{[\overline{4},0,\overline{0}],[\overline{0},1,\overline{1}],[\overline{1},2,\overline{2}],[\overline{2},3,\overline{3}],[\overline{3},4,\overline{4}],$
$[4,\overline{0},0],[0,\overline{1},1],[1,\overline{2},2],[2,\overline{3},3],[3,\overline{4},4]\}$;
$\mathcal{A}_0^2 = \{[\overline{0},0,\overline{2}],[\overline{1},1,\overline{3}],[\overline{2},2,\overline{4}],[\overline{3},3,\overline{0}],[\overline{4},4,\overline{1}],$

$$[0,\overline{0},2],[1,\overline{1},3],[2,\overline{2},4],[3,\overline{3},0],[4,\overline{4},1]\};$$

$$\mathcal{A}_1^2=\{[\overline{1},0,\overline{3}],[\overline{2},1,\overline{4}],[\overline{3},2,\overline{0}],[\overline{4},3,\overline{1}],[\overline{0},4,\overline{2}],$$
$$[1,\overline{0},3],[2,\overline{1},4],[3,\overline{2},0],[4,\overline{3},1],[0,\overline{4},2]\};$$

$$\mathcal{A}_2^2=\{[\overline{2},0,\overline{4}],[\overline{3},1,\overline{0}],[\overline{4},2,\overline{1}],[\overline{0},3,\overline{2}],[\overline{1},4,\overline{3}],$$
$$[2,\overline{0},4],[3,\overline{1},0],[4,\overline{2},1],[0,\overline{3},2],[1,\overline{4},3]\};$$

$$\mathcal{A}_3^2=\{[\overline{3},0,\overline{0}],[\overline{4},1,\overline{1}],[\overline{0},2,\overline{2}],[\overline{1},3,\overline{3}],[\overline{2},4,\overline{4}],$$
$$[3,\overline{0},0],[4,\overline{1},1],[0,\overline{2},2],[1,\overline{3},3],[2,\overline{4},4]\};$$

$$\mathcal{A}_4^2=\{[\overline{4},0,\overline{1}],[\overline{0},1,\overline{2}],[\overline{1},2,\overline{3}],[\overline{2},3,\overline{4}],[\overline{3},4,\overline{0}],$$
$$[4,\overline{0},1],[0,\overline{1},2],[1,\overline{2},3],[2,\overline{3},4],[3,\overline{4},0]\}.$$

3 Conclusion

Theorem 3.1. *There exists an $LS_\lambda(1,P_3,K_{v,v})$ if and only if $v\geq 2$, $3|\lambda v$ and $2\lambda|3v(v-1)$.*

Proof. By Theorem 2.4. and Theorem 2.5.,

when $3|v$, $LS(1,P_3,K_{v,v})=\{(Z_v\cup\overline{Z}_v,\mathcal{A}_i):1\leq i\leq\frac{3v(v-1)}{2}\}$ exists;

when $3\not|v$, $LS_3(1,P_3,K_{v,v})=\{(Z_v\cup\overline{Z}_v,\mathcal{B}_i):1\leq i\leq\frac{v(v-1)}{2}\}$ exists.

For any λ, by the necessary conditions, we need only to prove the sufficiency.

Case 1: $3\not|\lambda$. Then $3|v$ and $2\lambda|3v(v-1)$. Define

$$\mathcal{C}_k=\bigcup_{i=k\lambda+1}^{(k+1)\lambda}\mathcal{A}_i,\ 0\leq k\leq\tfrac{3v(v-1)}{2\lambda}-1,$$

then $\{(Z_v\cup\overline{Z}_v,\mathcal{C}_k):\ 0\leq k\leq\frac{3v(v-1)}{2\lambda}-1\}$ is just an $LS_\lambda(1,P_3,K_{v,v})$.

Case 2: $3|\lambda$. Then $\frac{2\lambda}{3}|v(v-1)$. For $3|v$, it is similar to above. For $3\not|v$, let $\lambda=3t$, define

$$\mathcal{C}_k=\bigcup_{i=tk+1}^{t(k+1)}\mathcal{B}_i,\ 0\leq k\leq\tfrac{v(v-1)}{2t}-1,$$

then $\{(Z_v\cup\overline{Z}_v,\mathcal{C}_k):\ 0\leq k\leq\frac{v(v-1)}{2t}-1\}$ is just an $LS_\lambda(1,P_3,K_{v,v})$. ∎

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