A Property of the Roots of *r*-D Rook Polynomials

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Abstract
In this paper, we prove an interesting property of rook polynomials for 2-D square boards and extend that for rook polynomials for 3-D cubic, and *r*-D "hypercubic", boards. In particularly, we prove that for *r*-D rook polynomials the modulus of the sum of their roots equals their degree. We end with some further questions, mainly for the 2-D and 3-D case, that could serve as future projects.

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1. Introduction

Rook polynomials, usually denoted by $R_{m,n}$, are a special class of polynomials which their study begun in the 1950's. During that time, Kaplansky and Riordan studied the concept as part of their studies in graph theory, the ménage problem, etc, and provided the first comprehensive analysis and main results (see [10],[13]). Nowadays, we know that rook polynomials have a close connection to graph theory (matching polynomials), special polynomials (Laguerre polynomials), and group representations (see [8],[4] and [9], respectively).

In general, a *rook polynomial* for a 2-D $m \times n$ chessboard is defined by:

$$R_{m,n}(x) = \sum_{k=0}^{\min\{m,n\}} r_k^{(m,n)} x^k \quad . \tag{1}$$

where $r_k^{(m,n)}$ is the number of ways one can place k non-attacking rooks on an $m \times n$ chessboard. It is not hard to see that $r_k^{(m,n)}$ is actually given by:

$$r_k^{(m,n)} = \binom{m}{k} P(n,k). \qquad (2)$$

For example, using formula (2), the number of ways one can place 2 rooks on a 2×2 board is $r_2^{(2,2)} = \binom{2}{2} 2! \binom{2}{2} = 2$ (see Fig.1). Similarly, $r_1^{(2,2)} = 4$ and $r_0^{(2,2)} = 1$.

Fig.1

Hence, the rook polynomial $R_{2,2}$ that corresponds to a 2×2 board is given by:

$$R_{2,2}(x) = 2x^2 + 4x + 1$$

In a similar way, one also computes $R_{3,3}$, $R_{4,4}$ and $R_{5,5}$. Namely:

$$R_{3,3}(x) = 6x^3 + 18x^2 + 9x + 1$$

$$R_{4,4}(x) = 24x^4 + 96x^3 + 72x^2 + 16x + 1$$

$$R_{5,5}(x) = 120x^5 + 600x^4 + 2400x^3 + 450x^2 + 36x + 1.$$

The polynomial $R_{1,1}$ is trivially computed to be $R_{1,1}(x) = x + 1$. Hereafter, we will focus on "square" boards. We examine 2-D square boards in Section 2, 3-D cubic boards in Section 3 and r-D hypercubic boards in Section 4.

2. Rook Polynomials in 2-D

By a simple counting argument, as we will see below, one can show that the maximum number of rooks could be placed on an $m \times m$ board is m. Most importantly, though, notice that the rook polynomials above ($R_{2,2}$, $R_{3,3}$, etc) have the following interesting property:

$$|S(R_{m,m})| = \deg(R_{m,m}) \quad (3)$$

where $S(R_{m,m})$ denotes the sum of the roots of $R_{m,m}$ and $\deg(R_{m,m})$ denotes the degree of $R_{m,m}$. Indeed, the roots of $R_{2,2}$ are $x_1 = \frac{-2+\sqrt{2}}{2}$, $x_2 = \frac{-2-\sqrt{2}}{2}$, which gives $|x_1 + x_2| = |-2| = 2 = \deg(R_{2,2})$. In Theorem 2.2 below, we prove that this is indeed true for any rook polynomial $R_{m,m}$. Furthermore, as we will see in the next section, the same fact is true for rook polynomials $R_{m,m,m}$ in 3-D (and R_m^r polynomials in r-D). But first, as we said in the beginning of the section, one can also show inductively that the maximum number of non-attacking rooks one can place on an $m \times m$ board B is m.

Proposition 2.1: Let B be an $m \times m$ board. Then, the maximum number of non-attacking rooks one can place on B is m.
Proof: Since there are m rows available on the board and no two rooks can lie in the same row, it is immediate that there can be at most m rooks on the board. □

Theorem 2.2: Let $R_{m,m}$ be any rook polynomial for an $m \times m$ board. Then, $|S(R_{m,m})| = \text{Degree}(R_{m,m})$. [1]

Proof: Let $R_{m,m}(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + 1$ be the rook polynomial for an $m \times m$ board. Then, by Vieta's Formulas we know that $S(R_{m,m}) = -\frac{a_{m-1}}{a_m}$. But, for the right-hand-side of the last equation, we know that:

a_{m-1} = the number of ways to place m - 1 rooks on the $m \times m$ board

and

a_m = the number of ways to place m rooks on the $m \times m$ board .

Hence, using equation (2), we find that:

$$\frac{a_{m-1}}{a_m} = \frac{(m-1)!\binom{m}{m-1}\binom{m}{m-1}}{m!\binom{m}{m}\binom{m}{m}} = \frac{m!m}{m!} = m$$

Therefore, $|S(R_{m,m})| = \left|-\frac{a_{m-1}}{a_m}\right| = |-m| = m = \deg(R_{m,m})$. □

Remark 2.3: Notice also that the real roots of $R_{m,m}(x)$ are always negative, in accordance to *Descartes' Rule of Signs.*

Motivated by the discussion above, we would like to extend the discussion in 3-D and see, most importantly, that Theorem 2.2 is true in 3-D as well.

3. Rook Polynomials in 3-D

In general, and in analogy[2] to the 2-D case, a rook polynomial in 3-D is defined by:

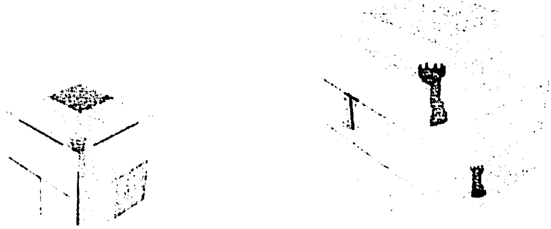
$$R_{m,n,d}(x) = \sum_{k=0}^{\min\{m,n,d\}} r_k^{(m,n,d)} x^k \tag{4}$$

where $r_k^{(m,n,d)}$ is the number of ways one can place k non-attacking rooks *in* an $m \times n \times d$ chessboard (see Fig.2). In [16], $r_k^{(m,n,d)}$ is actually shown to be:

$$r_k^{(m,n,d)} = \binom{m}{k} P(n,k)P(d,k). \tag{5}$$

For example, $R_{1,1,1}$ is trivially $R_{1,1,1}(x) = x+1$, and $R_{2,2,2}$ is given by (see Fig.2(a)):

$$R_{2,2,2}(x) = 4x^2 + 8x + 1$$



(a) Fig. 2 (b)

A few more examples are:

$$R_{3,3,3}(x) = 36x^3 + 108x^2 + 27x + 1$$
$$R_{4,4,4}(x) = 576x^4 + 2304x^3 + 864x^2 + 64x + 1$$
$$R_{5,5,5}(x) = 14400x^5 + 72000x^4 + 36000x^3 + 4000x^2 + 125x + 1.$$

Below, we can actually show that in the 3-D case the maximum number of rooks for an $m \times m \times m$ is m.

Proposition 3.1: Let *B* be an $m \times m \times m$ board. Then, the maximum number of non-attacking rooks one can place in *B* is m.
Proof: Since there are *m* level-planes available in the board and no two rooks can lie on the same level-plane (see Fig.2(b)), it is immediate that there can be at most *m* rooks in the board. □

Now, we prove the 3-D analogue of Theorem 2.2.

Theorem 3.2: Let $R_{m,m,m}$ be any rook polynomial for an $m \times m \times m$ board. Then, $|S(R_{m,m,m})| = \deg(R_{m,m,m})$.
Proof: Let $R_{m,m,m}(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + 1$ be the rook polynomial for an $m \times m \times m$ board. Then, by Vieta's Formulas we know that $S(R_{m,m,m}) = -\frac{a_{m-1}}{a_m}$. But, for the right-hand-side of the last equation, we know that:

a_{m-1} = the number of ways to place *m* - 1 rooks on the $m \times m \times m$ board and

a_m = the number of ways to place m rooks on the $m \times m \times m$ board .

Hence, using equation (5), we find that:

$$\frac{a_{m-1}}{a_m} = \frac{\binom{m}{m-1}[P(m,m-1)]^2}{\binom{m}{m}[P(m,m)]^2} = \frac{m \cdot \left(\frac{m!}{1}\right)^2}{1 \cdot (m!)^2} = m$$

Therefore, $|S(R_{m,m,m})| = |-\frac{a_{m-1}}{a_m}| = |-m| = m = \deg(R_{m,m,m})$. □

4. Rook Polynomials in *r*-D

The question now is whether the property of rook polynomials described in Theorems 2.2 and 3.2 is true in *r*-D. Consider an *r*-dimensional hypercubic board. As in the cases before (see footnote 2), a rook position is signified by the coordinate point $(i_1, i_2, \ldots, i_r)$, and forbidden positions would be determined by the $\binom{r}{r-1} = r$ orthogonal hyperplanes of dimension $r - 1$. [3] [For simplicity, hereafter, we denote the polynomials $R_{\underbrace{m,m,\ldots,m}_{r}}$ by R_m^r]. We recall that in the 2-D and 3-D cases the formulas for the rook polynomials are $R_m^2(x) = \sum_{k=0}^{m} \binom{m}{k} [P(m,k)] x^k$ and $R_m^3(x) = \sum_{k=0}^{m} \binom{m}{k} [P(m,k)]^2 x^k$, respectively, and notice that the powers of term $P(m,k)$ increase accordingly. Hence, we are able to prove [4] the following:

Theorem 4.1: Let R_m^r be any rook polynomial for an *r*-D hypercubic board. Then, R_m^r is given by:

$$R_m^r(x) = \sum_{k=0}^{m} \binom{m}{k} [P(m,k)]^{r-1} x^k .$$

Proof: By induction on the dimension r of the board. For $r = 2$, the claim is true since we have the formula for the 2-D rook polynomials. Suppose the claim is true for $r = s$. That is, we have $R_m^s(x) = \sum_{k=0}^{m} \binom{m}{k} [P(m,k)]^{s-1} x^k$. We show that is true for $r = s + 1$. In order to place k rooks in an (s+1)-D board, we utilize the fact that we know we have $\binom{m}{k} [P(m,k)]^{s-1}$ many ways to choose the positions for each rook in the hyperplane s-D board. Going one dimension higher now, we only need to select k tower positions from the m available, and permute them, in order to obtain all possible rook placements in the s-D board. Therefore, the number of ways to place k rooks in an (s+1)-D board is $\left(\binom{m}{k} [P(m,k)]^{s-1}\right) P(m,k) = \binom{m}{k} [P(m,k)]^{s}$. Hence, the rook polynomial for an r-D board is given by $R_m^r(x) = \sum_{k=0}^{m} \binom{m}{k} [P(m,k)]^{r-1} x^k$. □

Some examples of rook polynomials in r-D are:

$$R_1^r(x) = x + 1$$
$$R_2^r(x) = (2^{r-1})x^2 + (2 \cdot 2^{r-1})x + 1$$
$$R_3^r(x) = (6^{r-1})x^3 + (3 \cdot 6^{r-1})x^2 + (3 \cdot 3^{r-1})x + 1$$

$$R_4^r(x) = (24^{r-1})x^4 + (3 \cdot 24^{r-1})x^3 + (6 \cdot 12^{r-1})x^2 + (4 \cdot 4^{r-1})x + 1.$$

Finally, using Theorem 4.1 we can show that the connection of the sum of the roots with the degree generalizes in r-D:

Theorem 4.2: Let R_m^r be any rook polynomial for an r-D board. Then, $|S(R_m^r)| = \deg(R_m^r)$.

Proof: With the notation as in Theorem 2.2, we have: $|S(R_m^r)| = |-\frac{a_{m-1}}{a_m}| =$

$$\left| -\frac{\binom{m}{m-1}[P(m,m-1)]^{r-1}}{\binom{m}{m}[P(m,m)]^{r-1}} \right| = \left| -\frac{m\cdot\left(\frac{m!}{1}\right)^{r-1}}{1\cdot(m!)^{r-1}} \right| = \mid -m \mid = m = \deg(R_m^r). \ \square$$

4. Further Questions

Some further questions, mainly for the 2-D and 3-D case, that could serve as future projects are the following:

(a) *Connection between rook polynomials and rook matrices*: It well known that the non- attacking rook configurations in 2D could also be represented by matrices. For example,

R	
	R

$$\leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The question is whether any connections/relations exist between the rook matrices and rook polynomials (perhaps between eigenvalues, roots, etc), considering the fact that both represent configurations of non-attacking rooks. Consider also the fact that a connection exist between the eigenvalues of *circulant matrices* and the roots of polynomials (see [11]) and, as it seems, rook matrices are special cases of circulant matrices.

(b) *Connection to Laguerre Polynomials*: Rook polynomials relate to Laguerre polynomials as follows (see [3]):

$$R_{m,n}(x) = n!x^n L_n^{m-n}(-x^{-1})$$

In other words, the $L_n^{m-n}(-x^{-1})$ is generator for $R_{m,n}(x)$. Could some of the root properties of the rook polynomials shed more light to some of the root properties of the Laguerre polynomials, and vice versa?

(c) *Other "chess-based" polynomials*: A similar analysis and similar questions could be raised, as we did in this paper, perhaps for other similar type polynomials such as the Queen, or Bishop, polynomials, etc.

(d) *Rooks attacking in Lines*: Experimenting in 3-D with rooks that are prohibited to attack in the union of the orthogonal lines spanning out of (i,j,k), as in real chess, and *not* in the union of the three planes (see Fig.3(b)), we found that there are no known formulas for the rook polynomials yet, and generating the first few "by hand" is a cumbersome process. A program we built, that uses a "brute force" algorithm, produced for us the first four polynomials, $R_{1,1,1}$, $R_{2,2,2}$, $R_{3,3,3}$ and $R_{4,4,4}$, which are given by:

$$R_{1,1,1}(x) = x+1$$

$$R_{2,2,2}(x) = 2x^4 + 8x^3+16x^2 + 8x+1$$

$$R_{3,3,3}(x) = 12x^9 + 1086x^8+756x^7 + 2412x^6 + 3834x^5 + 3078x^4 + 1278x^3 + 270x^2 + 27x+1$$

$$R_{4,4,4}(x) = 576x^{16} + 9216x^{15}+110592x^{14} + 847872x^{13} + 4215744x^{12} + 13153536x^{11} + 25941504x^{10} + 32971008x^9 + 27534816x^8 + 15326208x^7 + 5728896x^6 + 143769x^5 + 239760x^4 + 25920x^3 + 1728x^2 + 64x+1$$

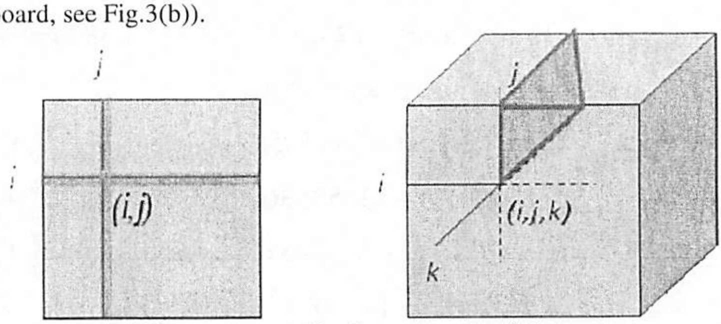
First, notice that the maximum number of rooks that could be placed on an $m \times m \times m$ board turns out to be m^2. Second, notice that for the first four polynomials above the modulus of the sum of the roots equals the degree. We believe that the program will not need too much time to find $R_{5,5,5}$, but for the cases of $R_{6,6,6}$ and $R_{7,7,7}$, we projected that the program will need several days and weeks respectively to compute them. Now, since the above polynomials were empirically produced, firstly, one needs to verify that the above polynomials are correct. Then, could a more efficient algorithm be built? Currently, we are looking at a way of representing the non-attacking configurations for the 3-D board with certain tri-partite graphs to achieve a better algorithm. (In 2-D there are several efficient algorithms, see [12]). And more importantly, could a general formula that generates all these polynomials be produced?

Notes:

1. Which, clearly, is also equal to the maximum number of rooks for the $m \times m \times m$ board.

2. The analogy here is as follows: In 2-D, placing a rook in the (i,j) position prohibits any further rook placement anywhere in the union of the ith row or jth

column (i.e., the union of the two lines (the $\binom{2}{1} = 2$ orthogonal hyperplanes) that span out of (i,j) of the 2-D board, see Fig.3(a)). In 3-D, placing a rook in the (i,j,k) position prohibits any further rook placement anywhere in the union of the three planes (the $\binom{3}{2} = 3$ orthogonal hyperplanes) that span out of (i,j,k) of the 3-D board, see Fig.3(b)).



(a) Fig.3 (b)

3. Conversely, we know that the r-many hyperplanes in r-D intersect at the (unique) point $(i_1, i_2, \ldots, i_r)$ by a standard Theorem from Linear Algebra, which says that the $r \times r$ system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1r}x_r &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2r}x_r &= b_2 \\ &\vdots \\ a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{rr}x_r &= b_r \end{aligned}$$

has a unique solution if and only if $\det A \neq 0$, where A is the coefficient matrix. (In essence, $\det A \neq 0$ means that the hyperplanes are not co-hyperplanar. In our case, they are orthogonal).

4. The proof here was based on the proof in [16, p.22], and extended his argument from 3-D to r-D inductively.

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