Homogeneous toroidal Latin bitrades

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Abstract

Let $\{T, T'\}$ be a Latin bitrade. Then T (and T') is said to be (r, c, e)-*homogeneous* if each row contains precisely r entries, each column contains precisely c entries, and each entry occurs precisely e times. An (r, c, e)-homogeneous Latin bitrade can be embedded on the torus only for three parameter sets, namely $(r, c, e) = (3,3,3), (4,4,2)$, or $(6,3,2)$. The first case has been completely classified by a number of authors. We present classifications for the other two cases.

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1 Introduction

Let R, C, and E be non-empty finite sets of cardinality n. A *partial Latin square* is an $n \times n$ array with rows indexed by R, columns indexed by C, and entries from E, in which each cell is either empty or contains precisely one entry, and no entry occurs more than once in any row or column. Let T be a partial Latin square. Then T is a *Latin trade* if there exist a partial Latin square T' with the properties that

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1. a cell is filled in T' if and only if it is filled in T,

2. no entry occurs in the same cell in T and T',

3. in any given row or column, T and T' contain precisely the same entries.

The partial Latin square T' is called a *trade mate* of T and the unordered pair $\{T, T'\}$ a *Latin bitrade.* The above definitions permit empty rows, empty columns, and non-occurring entries. However it is usual and convenient to ignore these and assume that every row and every column contains some entry, and each entry occurs in some cell of the partial Latin square.

A topological representation of a Latin bitrade $\{T, T'\}$ can be constructed as follows. For both T and T', re-define $R = \{r_1, r_2, \ldots, r_x\}$, $C = \{c_1, c_2, \ldots, c_y\}$, and $E = \{e_1, e_2, \ldots, e_z\}$ to be the sets of (non-empty) rows, (non-empty) columns, and (occurring) entries respectively. Let $m = x+y+z$. The Latin trade T can now be represented as a set S_T of triples $\{r_i, c_j, e_k\}$ where the entry e_k occurs in row r_i, column c_j of the trade and similarly for T'. The cardinality of S_T is called the *size* of the Latin trade; $n = |S_T| = |S_{T'}|$. Now take the sets of triples S_T and $S_{T'}$ as black and white triangular faces respectively and sew them together along common edges. This representation will not necessarily be connected but each component, which may be a surface or pseudosurface, will itself be the representation of a Latin bitrade. In this representation the *Euler characteristic* χ of a connected Latin bitrade is $F + V - E$ where F is the number of faces, V is the number of vertices, and E is the number of edges. Thus $V = m$, $F = 2n$, $E = 3n$ and therefore $\chi = m - n$. In this paper our interest is in the case where $m = n$ and the Latin bitrade forms a surface rather than a pseudosurface. Such bitrades are often called *separated* in the literature. Since the surface is necessarily orientable it is the torus.

A Latin trade T is *k-homogeneous* if each row and column contains precisely k entries and each entry occurs precisely k times in T. An example of a 2-homogeneous Latin trade is the Latin square of order 2 and it is easy to see that every 2-homogeneous Latin trade is the disjoint union of such squares. But the situation for $k = 3$ is more complex as the example in Figure 1 shows.

2	3	4	
3		1	4
	1	2	3
1	4		2

3	4	2	
1		4	3
	3	1	2
2	1		4

Figure 1: A 3-homogeneous Latin bitrade.

A construction for 3-homogeneous Latin trades, based on a hexagonal packing of circles in the plane, is given in [3]. Cavenagh [2], then classified all such trades, showing in fact that the construction in [3] gives <u>all</u> 3-homogeneous Latin trades.

Further independent proofs appear in [4] and [6]. An alternative solution to the classification, based on the work of Altshuler [1], and Negami [8], was then given in [5]. An essential feature of this latter classification is that the topological representation, as described above, of every 3-homogeneous Latin bitrade is necessarily the torus.

The concept of k-homogeneity of Latin trades may be generalized. A Latin trade T is (r,c,e)*-homogeneous* if each row contains precisely r entries, each column contains precisely c entries, and each entry occurs precisely e times in T. Clearly, if T' is any trade mate of T then T' itself is also (r,c,e)-homogeneous and the pair $\{T,T'\}$ is called a (r,c,e)*-homogeneous Latin bitrade.* For an (r,c,e)-homogeneous Latin trade, $n = xr = yc = ze$, so $\chi = x + y + z - xr = xr(1/r + 1/c + 1/e - 1)$. If $\chi = 0$ then $1/r + 1/c + 1/e = 1$. By considering the conjugates of a partial Latin square we may assume without loss of generality that $r \geq c \geq e$, and so there are just three solutions

1. $r = 3,\ c = 3,\ e = 3,$
2. $r = 4,\ c = 4,\ e = 2,$
3. $r = 6,\ c = 3,\ e = 2.$

The first of these is the case considered in [5]. In the next two sections we consider the remaining two cases and obtain a complete classification of separated Latin bitrades of these types. The situation is therefore slightly different from case 1. since for 3-homogeneous Latin bitrades, every such bitrade is separated.

2 Separated (4,4,2)-homogeneous Latin bitrades

Let $\{T,T'\}$ be a separated $(4,4,2)$-homogeneous Latin bitrade, and consider its topological representation on the torus. Then every vertex $r_i \in R$ and $c_j \in C$ has valency 8 and every vertex $e_k \in E$ has valency 4. Now remove all the vertices e_k, together with their incident edges. The graph which remains has $x + y = 2x$ vertices, all of which have valency 4. Hence the graph has $4x$ edges and so its embedding on the torus has $2x$ faces. Since the graph is also bipartite with the sets R and C forming the vertex bipartition, it follows that every face is a 4-cycle, i.e. what remains is a quadrangulation of the torus by a 4-regular graph. Such quadrangulations have been completely classified by Altshuler [1], see also [7]. First consider the quadrangulation Q, shown in Figure 2, of the domain

$$\{(x,y) \in R^2 :\ 0 \leq x \leq s,\ \ 0 \leq y \leq p\},$$

where p and s are positive integers.

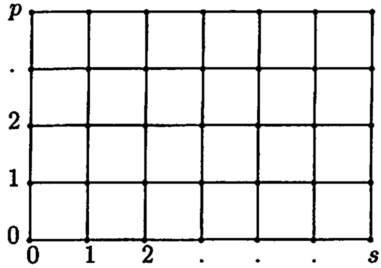

Figure 2: Quadrangulation of $\{(x,y) \in R^2 : \ 0 \leq x \leq s, \ \ 0 \leq y \leq p\}$.

In order to convert this into a quadrangulation of the torus, first identify the upper and lower sides of the rectangle in the usual way to form an open-ended cylinder. Now glue one of the boundaries of the cylinder to the other so that the point $(0,y)$, $0 \leq y \leq p$ coincides with the point (s,y'), $0 \leq y' \leq p$ if $y - y' \equiv q$ (mod p), where q is an integer satisfying $0 \leq q < p$. Informally we make a "twist" in the cylinder before gluing the two boundaries. This procedure defines *the standard 4-regular quadrangulation* $Q(p,q,s)$. For our purposes, the main result in both [1] and [7] is the following theorem.

Theorem 2.1 *Every quadrangular embedding of a 4-regular graph in the torus, is isomorphic to some standard quadrangulation* $Q(p,q,s)$ *for some integers* $s \geq 1$, $p \geq 3$, *and* $q \geq 0$.

However, not every quadrangulation $Q(p,q,s)$ can arise from the process of removing the vertices $e_k \in E$, together with their incident edges, from the toroidal embedding of a separated $(4,4,2)$-homogeneous Latin bitrade. As stated above, the 4-regular graph which remains is bipartite. So we must determine which of the quadrangulations $Q(p,q,s)$ are of bipartite graphs. Since the upper and lower sides of Q are identified, it follows immediately that p must be even. Further, the vertices $(0,y)$ and (s,y) will be in the same vertex partition if s is even and different vertex partitions if s is odd. Hence, in gluing the boundaries of the cylinder so that the points $(0,y)$ and (s,y') coincide, in order for the graph to be bipartite, $y - y'$ must have the same parity as s. Since $y - y' \equiv q$ (mod p), and p is even it follows that q and s are either both even or both odd. Finally we need to recover the Latin bitrade from the toroidal quadrangulation of a 4-regular bipartite graph. This is easy. Simply place a new vertex into each face of the quadrangulation and insert edges connecting it to each of the four vertices which determine the face. We therefore have the following theorem.

Theorem 2.2 *There is a one-one correspondence between separated* $(4,4,2)$*-homogeneous Latin bitrades and quadrangulations* $Q(p,q,s)$ *of the torus with* $p \equiv 0$ *(mod 2),* $p \geq 4$, *and* $q \equiv s$ *(mod 2),* $s \geq 1$, $q \geq 0$.

It is also worth remarking that if T is a separated $(4,4,2)$-homogeneous Latin trade, then any trade mate T' will be unique. Let $e_k \in E$. Then there exist $r_i,\ r_{i'} \in R$ and $c_j,\ c_{j'} \in C$ such that $\{r_i, c_j, e_k\},\ \{r_{i'}, c_{j'}, e_k\} \in S_T$. Moreover these are the only triples containing e_k. Thus $\{r_i, c_{j'}, e_k\},\ \{r_{i'}, c_j, e_k\} \in S_{T'}$, i.e. the trade mate is uniquely determined. (The same argument applies to any homogeneous Latin trade with r, c, or $e = 2$.) We conclude this section with an example. Figure 3 below shows a separated $(4,4,2)$-homogeneous Latin trade together with its trade mate and Figure 4 shows the quadrangulation $Q(4,3,5)$ corresponding to this Latin bitrade.

	0	1	2	3	4	5	6	7	8	9
0	r		q				a			w
1	b	t		s				c		
2	s		r	m	k					
3		v		t		n				p
4			k		f		e		j	
5				n	m	h		g		
6	a				g		f	b		
7		c				i		h	d	
8		d					w		e	v
9			j			p			i	q

	0	1	2	3	4	5	6	7	8	9
0	a		r				w			q
1	s	c		t				b		
2	r		k	s	m					
3		t		n		p				v
4			j		k		f		e	
5				m	g	n		h		
6	b				f		a	g		
7		d				h		c	i	
8		v					e		d	w
9			q			i			j	p

Figure 3: A $(4,4,2)$-homogeneous Latin bitrade.

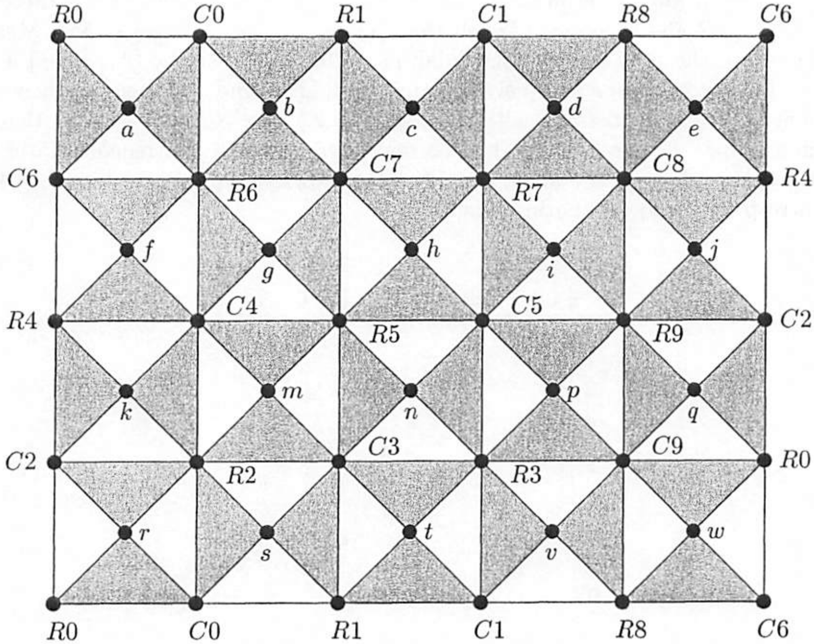

Figure 4: The quadrangulation $Q(4, 3, 5)$.

3 Separated (6,3,2)-homogeneous Latin bitrades

The approach to the classification of separated $(6, 3, 2)$-homogeneous Latin bitrades is similar to that in the previous section. Let $\{T, T'\}$ be a separated $(6, 3, 2)$-homogeneous Latin bitrade, and consider its topological representation on the torus. Then every vertex $r_i \in R$ has valency 12, every vertex $c_j \in C$ has valency 6, and every vertex $e_k \in E$ has valency 4. Remove all the edges which join vertices in the set R to vertices in the set E. Now every vertex r_i has valency 6 and every vertex e_k has valency 2, whilst the valency of every vertex c_j remains unchanged. Then suppress all the vertices e_k. The graph which remains is 6-regular, with each vertex r_i incident with 6 vertices from the set C and each vertex c_j incident with 3 other vertices from the set C and 3 vertices from the set R, occurring alternately around the vertex c_j. There are $x + y = 3x$ vertices and $9x$ edges and so its embedding on the torus has $6x$ faces. It follows that every face is a 3-cycle, i.e. what remains is a triangulation of the torus by a 6-regular graph. Such triangulations have also been completely classified first by Altshuler [1], and later by Negami [8]. Our approach, which follows that of Negami, is similar to the previous section. First consider the triangulation $\mathcal{T}$, shown in Figure 5, of the domain

$$\{(x,y) \in R^2 : \ 0 \leq x \leq s, \ \ 0 \leq y \leq p\},$$

where p and s are positive integers.

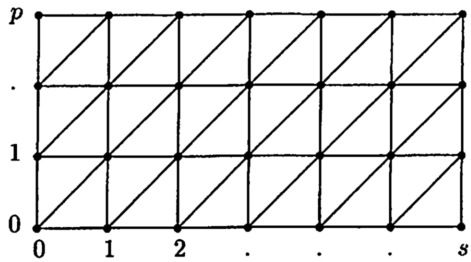

Figure 5: Triangulation of $\{(x,y) \in R^2 : \ 0 \leq x \leq s, \ \ 0 \leq y \leq p\}$.

Again, in order to convert this into a triangulation of the torus, first identify the upper and lower sides of the rectangle in the usual way to form an open-ended cylinder. Then glue one of the boundaries of the cylinder to the other so that the point $(0,y)$, $0 \leq y \leq p$ coincides with the point (s,y'), $0 \leq y' \leq p$ if $y - y' \equiv q \pmod{p}$, where q is an integer satisfying $0 \leq q < p$. This defines *the standard 6-regular triangulation* $T(p,q,s)$. For our purposes, the main result in both [1] and [7] is the following theorem.

Theorem 3.1 *Every triangular embedding of a 6-regular graph in the torus, is isomorphic to some standard triangulation $T(p,q,s)$ for some integers $s \geq 1$, $p \geq 3$, and $q \geq 0$..*

As in the previous section, not every triangulation $T(p,q,s)$ can arise from the above process and again we must determine those which do. So suppose that the vertex $(x,y) \in R$. Then the vertices $(x,y+1)$, $(x-1,y)$, $(x-1,y-1)$, $(x,y-1)$, $(x+1,y)$, $(x+1,y+1) \in C$. Moving to the vertex $(x+1,y) \in C$, it then follows that $(x+1,y-1) \in R$, $(x+2,y) \in C$, and $(x+2,y+1) \in R$, and further by moving to the vertex $(x+2,y) \in C$, that $(x+2,y-1) \in C$, $(x+3,y) \in R$, and $(x+3,y+1) \in C$. In particular if $(x,y) \in R$ then $(x+1,y)$, $(x+2,y) \in C$ and $(x+3,y) \in R$. Similarly $(x,y+1)$, $(x,y+2) \in C$ and $(x,y+3) \in R$. Also $(x+1,y-1) \in R$. Therefore the toroidal embedding is labelled as shown in Figure 6 below.

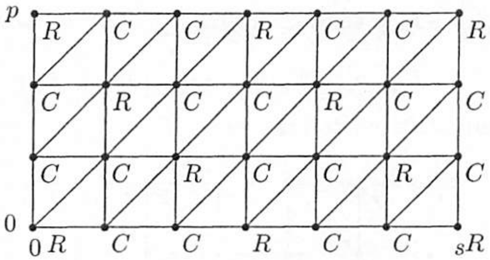

Figure 6: Labelled triangulation of $T(p, q, s)$.

Since the points $(x, 0)$ and (x, p), $0 \le x \le s$ are identified, it follows immediately that $p \equiv 0 \pmod 3$. Further, the vertex $(0, y)$, $0 \le y \le p$ coincides with the vertex (s, y'), $0 \le y' \le p$, if $y \equiv y' + q \pmod p$ and, since $p \equiv 0 \pmod 3$, this implies that $y \equiv y' + q \pmod 3$. But $y \equiv y' + s \pmod 3$ so $s \equiv q$ (mod 3). Finally we need to recover the Latin bitrade from the toroidal triangulation. First reinstate the vertices $e_k \in E$ by placing one on every edge which connects two vertices from the set C. Now as stated above, each vertex $r_i \in R$ is incident with 6 vertices all belonging to the set C. In cyclic order around r_i let these be $c_{i(0)}$, $c_{i(1)}$, $c_{i(2)}$, $c_{i(3)}$, $c_{i(4)}$, $c_{i(5)}$. Moreover each vertex $c_{i(j)}$ was, (before the reinstatement of the vertices e_k), incident with $c_{i(j+1)}$, arithmetic modulo 6. Insert edges connecting each vertex r_i to these reinstated vertices e_k. This re-establishes the separated $(6, 3, 2)$-homogemeous bitrade and the situation is illustrated in Figure 7 below.

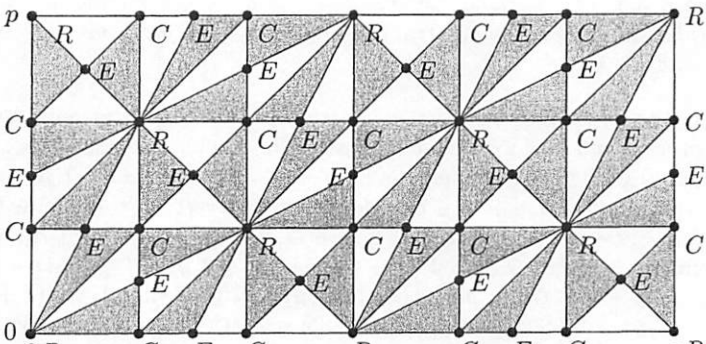

Figure 7: Labelled triangulation of $T(p, q, s)$.

We have the following theorem. As in the previous section, $r = 2$ and so by the argument given there a trade mate T' of a separated $(6, 3, 2)$-homogeneous Latin trade T is unique.

Theorem 3.2 *There is a one-one correspondence between separated* $(6,3,2)$*-homogeneous Latin bitrades and triangulations* $T(p,q,s)$ *of the torus with* $p \equiv 0$ *(mod 3),* $p \geq 3$*, and* $q \equiv s$ *(mod 3),* $s \geq 1$*,* $q \geq 0$.

References

[1] A. Altshuler, Construction and enumeration of regular maps on the torus, *Discrete Math.* **115** (1973), 201-217.

[2] N. J. Cavenagh, A uniqueness result for 3-homogeneous Latin trades, *Comment. Math. Univ. Carolin.* **47** (2006), 337-358.

[3] N. J. Cavenagh, D. M. Donovan and A. Drápal, 3-homogeneous Latin trades, *Discrete Math.* **300** (2005), 57-70.

[4] D. M. Donovan, A. Drápal and J. G. Lefevre, Permutation representation of 3 and 4-homogeneous Latin bitrades, *Fundamenta Informaticae* **20** (2007), 1-13.

[5] M. J. Grannell, T. S. Griggs and M. Knor, Biembeddings of symmetric configurations and 3-homogeneous Latin trades, *Comment. Math. Univ. Carolin.* **49** (2008), 411-420.

[6] C. Hämäläinen, Partitioning 3-homogeneous latin bitrades, *Geom. Dedicata* **133** (2008), 181-193.

[7] A. Nakamoto, Four-regular graphs that quadrangulate both the torus and the Klein bottle, *Australas. J. Combin.* **29** (2004), 249-257.

[8] S. Negami, Uniqueness and faithfulness of embedding of toroidal graphs, *Discrete Math.* **44** (1983), 161-180.