A Study on Radial Graphs

KM.Kathiresan and G.Marimuthu*
Centre for Research and Post Graduate Studies in Mathematics,
Ayya Nadar Janaki Ammal College,[Autonomous],
Sivakasi- 626 124,Tamil Nadu, India.
E-Mail : kathir2esan@yahoo.com and yellowmuthu@yahoo.com

ABSTRACT

In a graph G, the *distance* $d(u,v)$ between a pair of vertices u and v is the length of a shortest path joining them. The *eccentricity* $e(u)$ of a vertex u is the distance to a vertex farthest from u. The minimum eccentricity is called the *radius* of the graph and the maximum eccentricity is called the *diameter* of the graph. The *radial graph* $R(G)$ based on G has the vertex set as in G. Two vertices u and v are adjacent in $R(G)$ if the distance between them in G is equal to the radius of G. If G is disconnected, then two vertices are adjacent in $R(G)$ if they belong to different components. The main objective of this paper is to find a necessary and sufficient condition for a graph to be a radial graph.

Key words. Radius , Diameter , Radial Graph.

AMS Subject Classification: 05C12

1. Introduction

The graphs considered here are non-trivial and simple. For other graph theoretic notation and terminology, we follow [4,6]. For a graph G, the *distance* $d(u,v)$ between a pair of vertices u and v is the length of a shortest path joining them. The *eccentricity* $e(u)$ of a vertex u is the distance to a vertex farthest from u. The *radius* $r(G)$ of G is defined by $r(G)=\min\{e(u):u\in V(G)\}$ and the *diameter* $d(G)$ of G is defined by $d(G)=\max\{e(u):u\in V(G)\}$. A graph G for which $r(G)=d(G)$ is called a *self-centered graph* of radius $r(G)$. A vertex v is called an *eccentric vertex* of a vertex u if $d(u,v)=e(u)$. A vertex v of G is called an *eccentric vertex* of G if it is the eccentric vertex of some vertex of G. Let S_i be denote a subset of the vertex set of G such that $e(u) = i$ for all $u\in S_i$. The concept of antipodal graph was initially introduced by[5] and was further expanded by [2,3]. The antipodal graph of a graph G, denoted by $A(G)$, is the graph on the same vertices as of G, two vertices being adjacent if the distance between them is equal to the diameter of G. A graph is said to be antipodal if it is the antipodal $A(H)$ of some graph H. The concept of eccentric graph was introduced by[1].

* Research Supported by Government of India, CSIR, New Delhi.

The *eccentric graph* based on G is denoted by G_e, whose vertex set is $V(G)$ and two vertices u and v are adjacent in G_e if and only if $d(u,v) = min\{e(u), e(v)\}$. We introduce a new type of graph called *radial graph*. Two vertices of a graph are said to be *radial* to each other if the distance between them is equal to the radius of the graph. The radial graph of a graph G, denoted by $R(G)$, has the vertex set as in G and two vertices are adjacent in $R(G)$ if and only if they are radial in G. If G is disconnected, then two vertices are adjacent in $R(G)$ if they belong to different components of G. A graph G is called a *radial graph* if $R(H) = G$ for some graph H.

2. Radial Graph of Some Classes of Graphs

Result 2.1. *Let P_n be any path on $n \geq 5$ vertices, then*

$$R(P_n) = \begin{cases} (n/2)\,K_2 & \text{if } n \text{ is even} \\ P_3 \cup ((n-3)/2)\,K_2 & \text{if } n \text{ is odd} \end{cases}$$

Proof. Let $v_1, v_2, \ldots v_n$ be the vertices of the path P_n, ≥ 5. $r(P_n) = n / 2$ if n is even and $r(P_n) = (n - 1)/2$ if n is odd. The radial pairs are (v_1, v_{r+1}), (v_2, v_{r+2}), . . ., (v_{n-r}, v_n). The result follows from the definition.

Result 2.2. *Let C_n be any cycle on $n \geq 4$ vertices, then*

$$R(C_n) = \begin{cases} (n/2)\,K_2 & \text{if } n \text{ is even} \\ \cong C_n & \text{if } n \text{ is odd} \end{cases}$$

Proof. Let $v_1, v_2, \ldots v_n$ be the vertices of the cycle C_n. When n is even, a vertex and its eccentric vertex are radial to each other and hence $R(C_n) = (n/2)\,K_2$, $n/2$ disjoint copies of K_2. When n is odd, $r(C_n) = (n - 1)/2 = m$. $R(G)$ is the cycle $v_1 v_{r+1} v_{2r+1}\ v_r v_{2r}\ v_{r-1} \ldots v_{2r+2} v_1$, which is isomorphic to C_n

Result 2.3. $R(K_{m,n}) = K_m \cup K_n$

Proof. Let X, Y be the bipartition of the vertex set of $K_{m,n}$. The eccentricity of each vertex is two. Any two distinct vertices within a partition are eccentric to each other and hence the result follows.

Corollary 2.4. *For a graph G, $R(G) = K_{1,m}$ if and only if $G = K_{1,m}$ or G is disconnected with exactly two components out of which one is an isolated vertex.*

Result 2.5. *For a graph G of order n, $R(G) = K_n$ if and only if either G or $\bar{G}$ is K_n.*

Proof. If $R(G) = K_n$, then either $G \cong \overline{K}_n$ or G contains a vertex adjacent to all other vertices. In the second case if the vertices u and v are non adjacent in G, then they are not radial to each other. Thus the result follows.

3. Some Propositions on Radial Graphs

Proposition 3.1. *If $r(G) > 1$, then $R(G) \subseteq \overline{G}$*

Propositions 3.2. *Let G be a graph of order n. Then $r(R(G)) = 1$ if and only if either $\Delta(G) = n-1$ or G is disconnected with at least one isolated vertex.*

Proposition 3.3. *Every graph G of order n with $\Delta(G) = n-1$ is a radial graph of itself*

.

Proposition 3.4. *Every path P_n, $n \neq 4$ is a radial graph.*

Proof. When $n = 1,2,3$, P_n is the radial graph of itself. We can easily verify that P_4 is not a radial graph of any graph on four vertices. Let $v_1, v_2, \ldots v_n$ be the vertices of P_n, $n \geq 5$ *where* v_1 *and* v_n are the end vertices of P_n. For each i, $2 \leq i \leq n-1$, v_i is non-adjacent to v_{i-1} and v_{i+1} in $\overline{P}_n$ and it is adjacent to all other vertices of $\overline{P}_n$. Hence eccentric vertices of v_i in $\overline{P}_n$ are v_{i-1} and v_{i+1}. The eccentric vertex of v_1 in $\overline{P}_n$ is v_2 and the eccentric vertex of v_n in $\overline{P}_n$ is v_{n-1}. Therefore $S_2(\overline{P}_n) = V(\overline{P}_n)$ and hence $R(\overline{P}_n) = P_n$.

Proposition 3.5. *A cycle C_n on n vertices is a radial graph.*

Proof. C_3 is a radial graph of itself and $R(\overline{C}_n) = C_n$, $n \geq 4$.

Proposition 3.6. *Every complete n-partite graph is a radial graph.*

Proof. Let $K_{m1,m2,\ldots mn}$ be a n-partite graph. Then $R(\overline{K}_{m1,m2,\ldots mn}) = K_{m1,m2,\ldots mn}$

4. A Necessary and Sufficient Condition for a Graph to be a Radial Graph.

Let F_1, F_{22}, F_{23}, F_{24}, F_3 denote the set of all connected graphs G for which $r(G) = 1$, $r(G) = d(G) = 2$, $r(G) = 2$ and $d(G) = 3$, $r(G) = 2$ and $d(G) = 4$, $r(G) > 2$ respectively and F_4 denote the set of all disconnected graphs.

Theorem 4.1 *Let G be graph of order n, then $R(G) = G$ if and only if $G \in F_1$.*

Theorem A[6] If *G is a simple graph with diameter at least 3 ,then* $\bar{G}$ *has diameter at most 3.*

Theorem B[6] *If G is a simple graph with diameter at least 4,then* $\bar{G}$ *has diameter at most 2.*

Theorem C[6] *If G is a simple graph with radius at least 3,then* $\bar{G}$ *has radius at most 2.*

Theorem D[4] *If G is a self centered graph with* $r(G) \geq 3$*, then* $\bar{G}$ *is a self centered graph of radius 2*

Lemma 4.2. *Let G be a graph of order n. Then* $R(G) = \bar{G}$ *if and only if either* $S_2(G)=V(G)$ *or G is disconnected in which each component is complete.*

Proof. If $S_2(G) = V(G)$, then in $R(G)$, two vertices are adjacent if and only if they are nonadjacent in G. Also there are no vertices u and v in G such that $d(u,v) > 2$. Therefore $R(G) = \bar{G}$.

Assume that G is disconnected and each component of G is complete. Then $r(G)>1$. By proposition 3.1, $R(G) \subseteq \bar{G}$. Let u and v be any two adjacent vertices in $\bar{G}$. Then u and v are nonadjacent in G. This implies that u and v are in different components of G. Therefore $\bar{G} \subseteq R(G)$ and hence $R(G) = \bar{G}$.

Conversely let $R(G)= \bar{G}$ and let r be the radius of G. Since $R(G)= \bar{G}$, the vertices which are nonadjacent in G are adjacent in $R(G)$. Hence if u and v are any two vertices in G, then $d(u,v)=1$ or r. We claim that $S_2(G) = V(G)$ *or* $r = \infty$. If $r(G)=1$, then $R(G)=G$, a contradiction. Assume that $2<r<\infty$. Then there exists at least two vertices say x and y in G such that $d(x,y)=2$. Since x and y are nonadjacent in G, they should be adjacent in $\bar{G}$. But x and y are nonadjacent in $R(G)$ as $d(x,y)=2<r$. This is a contradiction to $R(G)= \bar{G}$. Next we claim that G is a self-centered graph of radius 2 if R(G)= $\bar{G}$. It is well known that $r(G) \leq d(G)$. Suppose that $r(G) < d(G)$. Then there exists at least two vertices say x and y such that $d(x,y) = d(G)$. x and y are non adjacent in G and are adjacent in $\bar{G}$. But they are non adjacent in $R(G)$,a contradiction to $R(G)= \bar{G}$ .Therefore $r(G) = d(G) = 2$.

Now let us show that each component of G is complete if G is disconnected. Let G_i be a component of G which is not complete and let z and w be two nonadjacent vertices in G_i. The distance between z and w in G_i is a finite number which is not equal to the radius of G and hence z and w are nonadjacent in $R(G)$. But $zw \in E(\bar{G})$ which is a contradiction to $R(G) = \bar{G}$.

Corollary 4.3. *If both G and $\bar{G}$ are of self centered graphs of radius 2, then so is R(G).*

Proof. Let both G and $\bar{G}$ be selfcentered graphs of radius 2. Then by lemma 4.2, R(G)= $\bar{G}$ and hence R(G) is a self centered graph of radius 2.

Lemma 4.4. *If G is disconnected, then each component of $\overline{R(G)}$ is complete.*

Proof. Follows from the definition.

Theorem 4.5 *Let G be a connected graph with r(G)>1. If $\bar{G}$ is disconnected with at least one non complete component, then G is not a radial graph.*

Proof. Suppose there exists a graph H such that $R(H) = G$. If $r(H) = 1$, then $R(H) = H = G$, a contradiction since $r(G)>1$ and so $r(H)>1$. By proposition 3.1, $R(H) \subseteq \bar{H}$. This implies that $H \subseteq \bar{G}$. As $\bar{G}$ is disconnected, H can not be connected. By lemma 4.4, each component of $\overline{R(H)}$ is complete. But $\overline{R(H)} = \bar{G}$. This is a contradiction to the fact that $\bar{G}$ is disconnected with at least one non complete component.

Corollary 4.6. *Let G be a connected graph with r(G)>1. If $\bar{G}$ is disconnected with each component complete, then G is a radial graph.*

Proof. Follows from lemma 4.2.

Corollary 4.7 *If $G \in F_{22}$ and $\bar{G} \in F_{22}$, then G is a radial graph.*

Proof. By lemma 4.2, R($\bar{G}$) = G.

Theorem 4.8. *If $G \in F_{22}$ and $\bar{G} \in F_{23}$, then G is not a radial graph.*

Proof. Suppose G is radial graph. Then there exists a graph H such that $R(H) = G$. Also $\overline{R(H)} = \bar{G}$. If H is disconnected, then by lemma 4.4, $\overline{R(H)}$ is disconnected and each component of $\overline{R(H)}$ is complete, a contradiction to the fact that $\bar{G}$ is connected. Hence H must be connected. $R(H) = G$ implies that $R(H) \subseteq \bar{H}$ and hence $H \subseteq \bar{G}$. Since $r(\bar{G})=2$ and $d(\bar{G})=3$, $r(H) \geq 2$ and $d(H) \geq 3$. Let $u \in V(\bar{G})$. Then u is adjacent to all the vertices v such that $d_{\bar{G}}(u,v) \geq 2$ in G. But u is adjacent to the vertices w such that $d(u,w)=r(H)$ in $R(H)$. From this we conclude that $R(H)$ is not equal *to G, a* contradiction.

Using the same proof technique used in Theorem 4.8, we prove the following corollaries.

Corollary 4.9. *If* $G \in F_{22}$ *and* $\bar{G} \in F_{24}$, *then G is not a radial graph.*

Corollary4.10 *If* $G \in F_{22}$ *and* $\bar{G} \in F_3$, *then G is not a radial graph.*

Corollary 4.11. *If* $G \in F_{23}$ *and* $\bar{G} \in F_{23}$, *then G is not a radial graph.*

Lemma 4.12. *If* $G \in F_{23}$ *and* $\bar{G} \in F_{22}$ *then G is a radial graph.*

Proof. By Lemma *4.2,* $R(\bar{G})=G$.

Lemma 4.13. *If* $G \in F_{24}$, *then G is a radial graph.*

Proof. By Theorem B, $\bar{G} \in F_{22}$. By Lemma 4.2, $R(\bar{G})=G$.

Lemma 4.14. *If* $G \in F_3$, *then G is a radial graph.*

Proof. Since $G \in F_3$, $r(G) \geq 3$ and hence $d(G) \geq 3$. If $r(G) = d(G) = 3$ then by Theorem *D*, $\bar{G} \in F_{22}$. By Theorem *C*, $\bar{G} \in F_{22}$ if $d(G) > 3$. *By Lemma 4.2,* $R(\bar{G})=G$.

Lemma 4.15. *If* $G \in F_4$ *without isolated vertices, then G is a radial graph.*

Proof. Since *G* has no isolated vertices, $\bar{G} \in F_{22}$ and hence by lemma 4.2, $R(\bar{G})=G$.

Lemma 4.16. *If* $G \in F_4$ *has at least one isolated vertex, then G is not a radial graph.*

Proof. Suppose there exists a graph *H* such that $R(H) = G$. *If H* is disconnected, then *R(H)* is connected and hence *G* is connected, a contradiction. This implies that *H* must be connected. Since *H* is connected, for each vertex *u* of *H*, there exists at least one vertex *v* in *H* such that $d(u,v)=r(H)$ and hence $uv \in E(R(H))$. Thus *R(H)* contains no isolated vertex.

Based on the above discussion, we conclude the following characterization Theorem for the radial graphs.

Theorem 4.17. *A graph G is a radial graph if and only if either G is a radial graph of itself or the radial graph of its complement.*

Proof. The proof follows from the following table.

<table>
<tr><td colspan="3">Case 1: G is connected.</td></tr>
<tr><td>Case 1.1
$r(G) = 1$</td><td>Case 1.2
$r(G) = 2$</td><td>Case 1.3
$r(G)>2$</td></tr>
<tr><td rowspan="4">Theorem 4.1</td><td>Case 1.2.1. $\bar{G}$ is connected.
$r(G) = d(G) = 2$
a) $r(\bar{G}) = d(\bar{G}) = 2$. Corollary 4.7
b) $r(\bar{G}) = 2$, $d(\bar{G}) = 3$. Theorem 4.8
c) $r(\bar{G}) = 2$, $d(\bar{G}) = 4$. Corollary 4.9
d) $r(\bar{G})>2$. Corollary 4.10</td><td rowspan="4">Theorem B and Lemma 4.14</td></tr>
<tr><td>Case 1.2.2 $\bar{G}$ is connected.
$r(G)=2,d(G)=3$
a) $r(\bar{G}) = d(\bar{G}) = 2$. Lemma 4.12
b) $r(\bar{G}) = 2$, $d(\bar{G}) =3$. Corollary 4.11
c) $r(\bar{G}) = 2$, $d(\bar{G}) = 4$.
(ruled out from Theorem B)
d) $r(\bar{G})\geq 3$. (ruled out from Theorem C)</td></tr>
<tr><td>Case 1.2.3. $\bar{G}$ is connected.
$r(G)=2, d(G)=4$ Lemma 4.13</td></tr>
<tr><td>Case 1.2.4. $\bar{G}$ is disconnected. Theorem 4.5 and Corollary 4.6</td></tr>
<tr><td colspan="3">Case 2. G is disconnected. Lemma 4.15 and Lemma 4.16</td></tr>
</table>

Acknowledgement

The authors wish to thank the anonymous referee for various suggestions for improving the paper. They also wish to thank the Management and the Principal of **Ayya Nadar Janaki Ammal College, Sivakasi** for providing facilities to do this work. The second author wishes to thank the Government of India, CSIR, New Delhi for providing financial support.

References

[1] Akiyama, J., Ando,K., Avis,D., Eccentric graphs, Discrete Math., **16**, 187-195(1976).

[2] Aravamuthan,R.,Rajendran,B.,Graph equations involving antipodal graphs, presented at the seminar on combinatorics and applications held at ISI, Culcutta during 14-17 December, 40-43(1982).

[3] Aravamuthan,R.,Rajendran,B., On antipodal graphs, Discrete Math., *49*, 193-195(1984).

[4] Buckley, F., and Harary, F., Distance in graphs, Addison-wesley, Reading (1990).

[5] Singleton,R.R., There is no irregular moore graph, Amer. Math. Monthly, *75*, 42-43(1968).

[6] West ,D.B., Introduction to graph theory, Prentice – Hall of India, New Delhi (2003).