$(d,1)$-Total labellings of regular nonbipartite graphs and an application to flower snarks *

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Abstract

A $(d,1)$-*total labelling* of a graph G is an assignment of integers to $V(G) \cup E(G)$ such that: (i) any two adjacent vertices of G receive distinct integers, (ii) any two adjacent edges of G receive distinct integers, and (iii) a vertex and its incident edge receive integers that differ by at least d in absolute value. The *span* of a $(d,1)$-total labelling is the maximum difference between two labels. The minimum span of labels required for such a $(d,1)$-total labelling of G is called the $(d,1)$-*total number* and is denoted by $\lambda_d^T(G)$. In this paper, we prove that $\lambda_d^T(G) \geq d+r+1$ for r-regular nonbipartite graphs with $d \geq r \geq 3$ and determine the $(d,1)$-*total numbers* of flower snarks and of quasi flower snarks.
Keywords: *(d,1)-Total labelling; Minimum span; Flower snark*

1 Introduction

We consider only finite undirected graphs without loops or multiple edges. Let $G=(V(G),E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$.

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A $(d,1)$-*total labelling* of a graph G, introduced by Havet and Yu [3, 4], is an assignment f of integers to $V(G) \cup E(G)$ such that: (i) any two adjacent vertices of G receive distinct integers, (ii) any two adjacent edges of G receive distinct integers, and (iii) a vertex and its incident edge receive integers that differ by at least d in absolute value. The *span* of a $(d,1)$-total labelling is the maximum difference between two labels. The minimum span of labels required for such a $(d,1)$-total labelling of G is called the $(d,1)$-*total number* and is denoted by $\lambda_d^T(G)$. The $(1,1)$-total labelling is the traditional total coloring.

Bazzaro et al [1] proved that $\lambda_d^T(G) \leq \Delta + 2d - 2$ for planar graphs with sufficiently large girth and high maximum degree Δ. Chen and Wang [2] proved that $\lambda_2^T(G) \leq \Delta + 2$ for outer planar graph with $\Delta \geq 5$, or $\Delta = 3$ and G being a 2-connected graph, or $\Delta = 4$ and G containing no intersecting triangles. Havet and Yu [5] provided lower and upper bounds of $(d,1)$-total number and determined the exact value of $\lambda_d^T(K_n)$ except for even n in the interval $[d+5, 6d^2 - 10d + 4]$. Montassier and Raspaud [9] proved that $\lambda_d^T(G) \leq \Delta(G) + 2d - 2$ for connected graphs with a given maximum average degree.

Let G_n be the simple nontrivial connected cubic graph with vertex set $V(G_n) = \{a_i, b_i, c_i,\ d_i\ :\ 0 \leq i \leq n-1\}$ and edge set $E(G_n) = \{a_i a_{i+1}, b_i b_{i+1}, c_i c_{i+1}, a_i d_i, b_i d_i, c_i d_i\ :\ 0 \leq i \leq n-1\}$, where the vertex labels are read modulo n. Let H_n be a graph obtained from G_n by replacing the edges $b_{n-1}b_0$ and $c_{n-1}c_0$ with $b_{n-1}c_0$ and $c_{n-1}b_0$ respectively. For odd $n \geq 5$, H_n is called a snark, namely *flower snark*. G_n and H_n ($n = 3$ or even $n \geq 4$) are *quasi flower snarks* (or *related graphs of the flower snarks*).

Figure 1.1 shows the flower snark H_5 and quasi flower snark G_5.

The flower snark, defined by Isaacs [6], is certainly one of the most famous cubic graphs that theorists have come across. Mohammad et al [7] determined the circular chromatic index of flower snarks. Zheng Wenping et al [11] studied the crossing number of flower snarks and of their related graphs. Mominul et al [8] investigated the prime cordial labelling of flower snarks and of their related graphs. Xi Yue et al [10] proved that flower snarks and of their related graphs are super vertex-magic.

In Section 2, we will prove $\lambda_d^T(G) \geq d + r + 1$ for r-regular nonbipartite graphs with $d \geq r \geq 3$. This provids a lower bound of $\lambda_d^T(H_{2m+1})$ and $\lambda_d^T(G_{2m+1})$ $(m \geq 1)$. In Section 3, we will exhibit a (d+4)-total labellings of H_{2m+1} and G_{2m+1}, providing the matching upper bound.

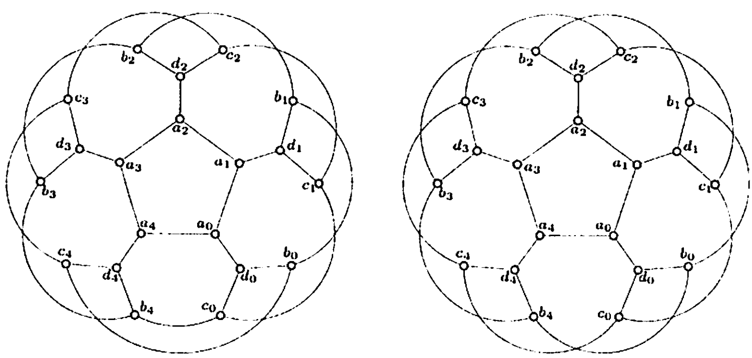


Figure 1.1. The flower snark H_5 and quasi flower snark G_5

2 Basic lemmas

Let f be a $(d,1)$-total labelling of G in $[0, \lambda_d^T(G)]$. For $v \in V(G)$, let $N_v(v) = \{u \in V(G) | uv \in E(G)\}$, $N_e(v) = \{(v,w) \in E(G) | w \in V(G)\}$. For $S_e \subseteq E(G)$, let $f(S_e) = \{f(e) | e \in S_e\}$. And for $S_v \subseteq V(G)$, let $f(S_v) = \{f(v) | v \in S_v\}$.

By [3, 4], we have Lemma 2.1.

Lemma 2.1. If G is an r-regular graph, then $\lambda_d^T(G) \geq d + r$.

Lemma 2.2. If G is an r-regular graph with $\lambda_d^T(G) = d+r$ and $d \geq r \geq 3$, then $f(v) \in \{0, 1, d+r-1, d+r\}$.

Proof. If $f(v) \in \{2, 3, ..., d-1\}$, then $f(N_e(v)) \subseteq \{d + f(v), d + f(v) + 1, ..., d+r\} \subseteq \{d+2, d+3, ..., d+r\}$. If $f(v) = d$ and $d = r$, then $f(N_e(v)) \subseteq \{0, d+r\}$. If $f(v) = d$ and $d > r$, then $f(N_e(v)) \subseteq \{0\}$. If $f(v) \in \{d+1, d+2, ..., d+r-2\}$, then $f(N_e(v)) \subseteq \{0, 1, ..., f(v) - d)\} \subseteq \{0, 1, ..., r-2)\}$. For all these cases, we have $|f(N_e(v))| < r$, a contradiction to the fact that $|f(N_e(v))| = r$. Hence, $f(v) \in \{0, 1, d+r-1, d+r\}$. □

Lemma 2.3. If G is an r-regular graph with $\lambda_d^T(G) = d+r$ and $d \geq r \geq 3$, then G is bipartite.

Proof. Suppose to the contrary that G contains an odd cycle, say $C = v_0 v_1 ... v_{2k} v_0$.

By Lemma 2.2, we have $f(v) \in \{0, 1, d+r-1, d+r\}$. Let $B(i) = \bigcup_{f(v)=i} f(N_e(v))$, $A(i) = \bigcup_{f(v)=i} f(N_v(v))$. Then $B(0) \subseteq \{d, d+1, ..., d+r\}$, $B(1) = \{d+1, d+2, ..., d+r\}$, $B(d+r-1) = \{0, 1, ..., r-1\}$, $B(d+r) \subseteq \{0, 1, ..., r\}$.

Case 1. $d = r$.

Since $B(0) \bigcap B(1) \neq \phi$, $B(0) \bigcap B(d+r-1) = \phi$, $B(0) \bigcap B(d+r) \neq \phi$, $B(1) \bigcap B(d+r-1) = \phi$, $B(1) \bigcap B(d+r) = \phi$ and $B(d+r-1) \bigcap B(d+r) \neq \phi$, we have $A(0) \subseteq \{1, d+r\}$, $A(1) = \{0\}$, $A(d+r-1) = \{d+r\}$ and $A(d+r) \subseteq \{0, d+r-1\}$.

If $0 \notin f(V(C))$, then $f(V(C)) = \{d+r-1, d+r\}$, a contradiction to G is nonbipartite. If $0 \in f(V(C))$, without loss of generality , we may assume that $f(v_0) = 0$. Then we have $f(v_1) \in \{1, d+r\}$. It follows $f(v_2) \in \{0, d+r-1\}$, $f(v_3) \in \{1, d+r\}$. By induction, we can get $f(v_{2j-1}) \in \{1, d+r\}$, $f(v_{2j}) \in \{0, d+r-1\}$ for $1 \leq j \leq k$. Hence, we have $f(v_{2k}) \in \{0, d+r-1\}$, it follows $f(v_0) \in \{1, d+r\}$, a contradiction to $f(v_0) = 0$.

Case 2. $d > r$.

Since $B(0) \bigcap B(d+r-1) = \phi$, $B(0) \bigcap B(d+r) = \phi$, $B(1) \bigcap B(d+r-1) = \phi$ and $B(1) \bigcap B(d+r) = \phi$, then $f(V(C)) \subseteq \{0,1\}$ or $f(V(C)) \subseteq \{d+r-1, d+r\}$. A contradiction to G is nonbipartite. □

From Lemma 2.3, we have the following theorem.

Theorem 2.4. If G is an r-regular nonbipartite graph with $d \geq r \geq 3$, then $\lambda_d^T(G) \geq d+r+1$.

Since H_{2m+1} and G_{2m+1} $(m \geq 1)$ are 3-regular nonbipartite graphs, by Theorem 2.4, we have the following corollaries.

Corollary 2.5. $\lambda_d^T(H_{2m+1}) \geq d+4$ for $m \geq 1$ and $d \geq 3$.

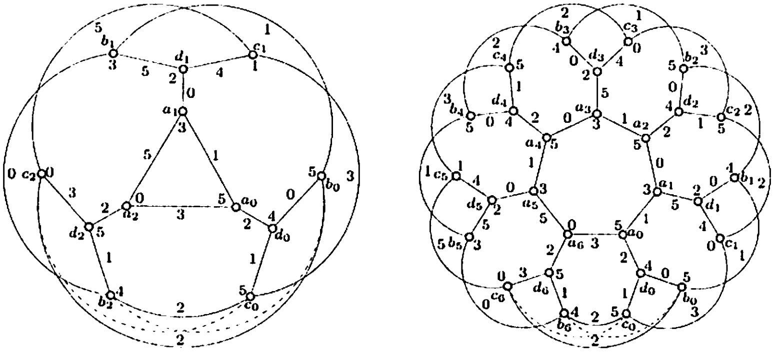
Corollary 2.6. $\lambda_d^T(G_{2m+1}) \geq d+4$ for $m \geq 1$ and $d \geq 3$.

3 $(d,1)$-total labelling of flower snarks and of quasi flower snarks

Lemma 3.1. $\lambda_2^T(H_3) = \lambda_2^T(G_3) = 5$.

Proof. Figure 3.1(a) shows a (2,1)-total labelling for H_3 and G_3. We then have $\lambda_2^T(H_3) \leq 5$ and $\lambda_2^T(G_3) \leq 5$. By Lemma 2.1, $\lambda_2^T(H_3) \geq 5$ and $\lambda_2^T(G_3) \geq 5$. Hence we have $\lambda_2^T(H_3) = \lambda_2^T(G_3) = 5$. □

Lemma 3.2. $\lambda_2^T(H_{2m+1}) = \lambda_2^T(G_{2m+1}) = 5$ for $m \geq 2$.



(a) $H_3(G_3)$ (b) $H_7(G_7)$

Figure 3.1. A (2, 1)-total labelling of $H_n(G_n)$ for $n = 3, 7$.

Proof. For $0 \le i \le 2m$, we construct a function f as follows:

$$f(a_i) = \begin{cases} 5, & i \bmod 2 = 0 \text{ and } i \le 2m-2, \\ 3, & i \bmod 2 = 1 \text{ and } i \le 2m-1, \\ 0, & i = 2m, \end{cases}$$

$$f(b_i) = \begin{cases} 5, & i \bmod 2 = 0 \text{ and } i \le 2m-2, \\ 4, & i \bmod 2 = 1 \text{ and } i \le 2m-3, i = 2m, \\ 3 & i = 2m-1, \end{cases}$$

$$f(c_i) = \begin{cases} 5, & i \bmod 2 = 0 \text{ and } i \le 2m-2, \\ 0, & i \bmod 2 = 1 \text{ and } i \le 2m-3, i = 2m, \\ 1, & i = 2m-1, \end{cases}$$

$$f(d_i) = \begin{cases} 4, & i \bmod 2 = 0 \text{ and } i \le 2m-2, \\ 2, & i \bmod 2 = 1 \text{ and } i \le 2m-1, \\ 5, & i = 2m, \end{cases}$$

$$f(a_ia_{i+1}) = \begin{cases} 1, & i \bmod 2 = 0 \text{ and } i \le 2m-2, \\ 0, & i \bmod 2 = 1 \text{ and } i \le 2m-3, \\ 5, & i = 2m-1, \\ 3, & i = 2m, \end{cases}$$

$$f(b_ib_{i+1}) = \begin{cases} 1, & i \bmod 2 = 0 \text{ and } i \le 2m-2, \\ 2, & i \bmod 2 = 1 \text{ and } i \le 2m-3, i = 2m, \\ 0, & i = 2m-1, \end{cases}$$

$$(f(b_{2m}c_0) = 2 \text{ for } H_{2m+1},)$$

$$f(c_ic_{i+1}) = \begin{cases} 3, & i \bmod 2 = 0 \text{ and } i \le 2m-2, \\ 2, & i \bmod 2 = 1 \text{ and } i \le 2m-3, i = 2m, \\ 5, & 2m-1, \end{cases}$$

$$(f(c_{2m}b_0) = 2 \text{ for } H_{2m+1},)$$

$$f(a_id_i) = \begin{cases} 2, & i \bmod 2 = 0 \text{ and } i \le 2m, \\ 5, & i \bmod 2 = 1 \text{ and } i \le 2m-3, \\ 0, & i = 2m-1, \end{cases}$$

$$f(b_id_i) = \begin{cases} 0, & i \le 2m-2, \\ 5, & i = 2m-1, \\ 1, & i = 2m, \end{cases}$$

$$f(c_i d_i) = \begin{cases} 1, & i \bmod 2 = 0 \text{ and } i \le 2m-2, \\ 4, & i \bmod 2 = 1 \text{ and } i \le 2m-1, \\ 3, & i = 2m. \end{cases}$$

Clearly, the total labelling f has the required properties of a (2,1)-total labelling for $m \ge 2$. We then have $\lambda_2^T(H_{2m+1}) \le 5$ and $\lambda_2^T(G_{2m+1}) \le 5$. By Lemma 2.1, $\lambda_2^T(H_{2m+1}) \ge 5$ and $\lambda_2^T(G_{2m+1}) \ge 5$. This concludes the proof. □

Figure 3.1(b) shows a $(2,1)$-total labelling of $H_7(G_7)$.

Theorem 3.3. $\lambda_d^T(H_{2m+1}) = \lambda_d^T(G_{2m+1}) = d+4$ for $m \ge 1$ and $d \ge 3$.
Proof. For $0 \le i \le 2m$, we construct a function f as follows:

$$\begin{aligned}
f(a_i) = f(b_i) = f(c_i) &= \begin{cases} d+3, & i \bmod 2 = 0 \text{ and } i \le 2m-2, \\ d+2, & i \bmod 2 = 1 \text{ and } i \le 2m-1, \\ d+4, & i = 2m, \end{cases} \\
f(d_i) &= \begin{cases} d+2, & i \bmod 2 = 0 \text{ and } i \le 2m, \\ d+3, & i \bmod 2 = 1 \text{ and } i \le 2m-1, \end{cases} \\
f(a_i a_{i+1}) &= \begin{cases} 0, & i \bmod 2 = 0 \text{ and } i \le 2m-2, \\ 1, & i \bmod 2 = 1 \text{ and } i \le 2m-1, \\ 3, & i = 2m, \end{cases} \\
f(b_i b_{i+1}) &= \begin{cases} 0, & i \bmod 2 = 0 \text{ and } i \le 2m-2, \\ 2, & i \bmod 2 = 1 \text{ and } i \le 2m-1, \\ 3, & i = 2m, \end{cases} \\
(f(b_{2m} c_0) &= 3 \text{ for } H_{2m+1},) \\
f(c_i c_{i+1}) &= \begin{cases} 1, & i \bmod 2 = 0 \text{ and } i \le 2m-2, \\ 2, & i \bmod 2 = 1 \text{ and } i \le 2m-1, \\ 3, & i = 2m, \end{cases} \\
(f(c_{2m} b_0) &= 3 \text{ for } H_{2m+1},) \\
f(a_i d_i) &= 2, \\
f(b_i d_i) &= 1, \\
f(c_i d_i) &= 0.
\end{aligned}$$

Clearly, the total labelling f has the required properties of a $(d,1)$-total labelling for $m \ge 1$ and $d \ge 3$. We then have $\lambda_d^T(H_{2m+1}) \le d+4$ and $\lambda_d^T(G_{2m+1}) \le d+4$. By Corollary 2.5 and Corollary 2.6, $\lambda_d^T(H_{2m+1}) \ge d+4$ and $\lambda_d^T(G_{2m+1}) \ge d+4$. This concludes the proof. □

Figure 3.2(a) shows a $(3,1)$-total labelling of $H_5(G_5)$.

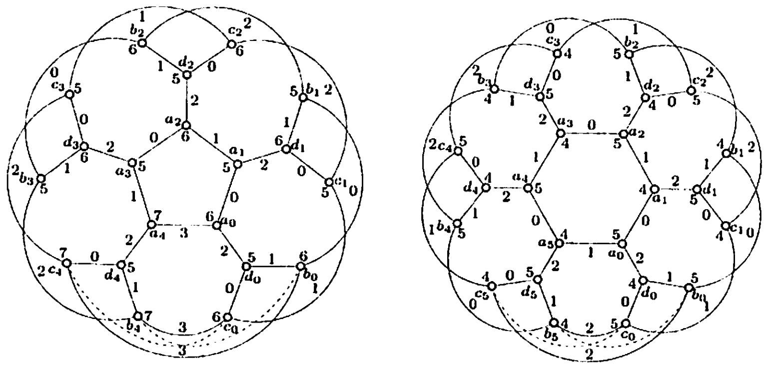
By [5], we have Theorem 3.4.

Theorem 3.4. If G is an r-regular bipartite graph, then $\lambda_d^T(G) = d + r$.

Since H_{2m} and G_{2m} are 3-regular bipartite graphs, By Theorem 3.4, we have Corollary 3.5.

Corollary 3.5. $\lambda_d^T(H_{2m}) = \lambda_d^T(G_{2m}) = d+3$ for $m \ge 2$ and $d \ge 2$.

For the sake of completeness, we show a $(d+3)$-total labelling f $(0 \le$



(a) $H_5(G_5)$ (b) $H_6(G_6)$

Figure 3.2. A (3,1)-total labelling of $H_5(G_5)$ and a (2,1)-total labelling of $H_6(G_6)$.

$i \le 2m-1$) as follows:

$$\begin{aligned}
f(a_i)=f(b_i)=f(c_i) &= \begin{cases} d+3, & i \bmod 2 = 0 \text{ and } i \le 2m-2, \\ d+2, & i \bmod 2 = 1 \text{ and } i \le 2m-1, \end{cases} \\
f(d_i) &= \begin{cases} d+2, & i \bmod 2 = 0 \text{ and } i \le 2m-2, \\ d+3, & i \bmod 2 = 1 \text{ and } i \le 2m-1, \end{cases} \\
f(a_i a_{i+1}) &= \begin{cases} 0, & i \bmod 2 = 0 \text{ and } i \le 2m-2, \\ 1, & i \bmod 2 = 1 \text{ and } i \le 2m-1, \end{cases} \\
f(b_i b_{i+1}) &= \begin{cases} 0, & i \bmod 2 = 0 \text{ and } i \le 2m-2, \\ 2, & i \bmod 2 = 1 \text{ and } i \le 2m-1, \end{cases} \\
(f(b_{2m} c_0) &= 2 \text{ for } H_{2m},) \\
f(c_i c_{i+1}) &= \begin{cases} 1, & i \bmod 2 = 0 \text{ and } i \le 2m-2, \\ 2, & i \bmod 2 = 1 \text{ and } i \le 2m-1, \end{cases} \\
(f(c_{2m} b_0) &= 2 \text{ for } H_{2m},) \\
f(a_i d_i) &= 2, \\
f(b_i d_i) &= 1, \\
f(c_i d_i) &= 0.
\end{aligned}$$

Figure 3.2(b) shows a $(2,1)$-total labelling of $H_6(G_6)$.

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