Equality in Vizing's Conjecture Fixing One Factor of the Cartesian Product

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Abstract.

In this paper, we investigate the existence of nontrivial solutions for the equation $\gamma(G\square H) = \gamma(G)\,\gamma(H)$ fixing one factor. For the complete bipartite graphs $K_{m,n}$; we characterize all nontrivial solutions when $m = 2$, $n \geq 3$ and prove the nonexistence of solutions when $m, n \geq 3$. In addition, it is proved that the above equation has no nontrivial solution if H is one of the graphs obtained from C_n, the cycle of length n, either by adding a vertex and one pendant edge joining this vertex to any $v \in V(C_n)$, or by adding one chord joining two alternating vertices of C_n.

Keywords: Domination number, Cartesian product, Vizing's conjecture.

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1. Introduction.

All graphs considered in this paper are simple and finite. Let G be a graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. The *open neighborhood* of $v \in V(G)$ is $N(v) = \{u : uv \in E(G)\}$ and the *open neighborhood of a subset* X of vertices is $N(X) = \cup_{v \in X} N(v)$. Similarly, we define the *closed neighborhood* $N[v] = N(v) \cup \{v\}$ and $N[X] = N(X) \cup X$. A subset D of $V(G)$ is called a *dominating set* of G if for each $x \in V(G) - D$, there is $y \in D$ such that $xy \in E(G)$. The *domination number*, $\gamma(G) = \min\{|D| : D$ is a dominating set of $G\}$, where $|D|$ denotes the number of elements of D. A dominating set with smallest cardinality will be called a $\gamma(G)$-set or simply, a γ-set. The *cartesian product* $G \Box H$ of two graphs G and H is the graph with vertex set $V(G \Box H) = V(G) \times V(H)$ and two vertices in $V(G \Box H)$ are adjacent if and only if they are equal in one coordinate and adjacent in the other. The two graphs G and H are called the *factors* of the graph $G \Box H$. We think of the vertices of $G \Box H$ as being laid out in a matrix form where for $u \in V(G)$, the row $\{(u, v) : v \in V(H)\}$ induces a subgraph of $G \Box H$, which is isomorphic to H. This graph will be denoted by H_u. Similarly, for $v \in V(H)$, the column $\{(u, v) : u \in V(G)\}$ induces the subgraph G_v of $G \Box H$. Clearly, $G_v \cong G$.

The interest in dominating the cartesian product of two graphs stems from a conjecture suggested by V.G. Vizing in 1963 [10], which states that *for any two graphs G and H, $\gamma(G \Box H)$ is not less than* $\gamma(G)\,\gamma(H)$. Most of the progress to resolve this conjecture has been to show that the conjectured inequality holds when some structural properties are imposed on one or both graphs. While, for the general case, this conjecture is still open.

Several authors considered the problem of determining pairs of graphs for which the conjectured lower bound is attained. Jacobson and Kinch [4] studied the case when both factors are trees. Fink et. al [3] proved that equality holds when both factors have domination number half their order. On the other hand, Hartnell and Rall [5] gave five instances of infinite families of graphs for which Vizing's conjecture holds with equality. For more about equality, the interested reader may refer to the survey article by Hartnell and Rall [6] and to the more recent articles [2] and [8]. In [6], the authors posed the following problem: *can we characterize the graphs H that satisfies the equation.*

$$\gamma(G \Box H) = \gamma(G)\,\gamma(H), \qquad (1)$$

when G is some fixed graph?. Later on, the same authors answered this question in the affirmative for $G = K_2$ [8]. They further proved that Vizing's conjecture holds strictly for the star graph $K_{1,m}$; $m \geq 2$, [7]. Moreover, they pointed out that for any generalized comb H, $\gamma(K_{2,m} \Box H) = 2\gamma(H)$; $m \geq 2$, [6]. El-Zahar, Khamis, and Nazzal [2] gave a characterization of graphs H when $G = C_4 \cong K_{2,2}$. They also considered equation (1) when one factor of the cartesian product is a cycle. This motivates the investigation of nontrivial solutions for equation (1) when G is either the complete bipartite graph $K_{m,n}$, where $m \geq 2$ and $n \geq 3$, or G is the graph obtained from the cycle of any length as described below.

The main results of the present paper are given as follows: in section 3, a characterization of all nontrivial solutions for equation (1) in case of $G = K_{2,n}$; $n \geq 3$, is given. On the other hand, it is shown that Vizing's conjecture holds strictly if $G = K_{m,n}$ for $m, n \geq 3$. In section 4, it is proved that equation (1) has no nontrivial solution if G is the graph obtained from C_n either by adding one vertex and a pendant edge joining this vertex to any $v \in V(C_n)$, or by adding one chord joining two alternating vertices of C_n. For ease of reference, those graphs will be called C''_n and C'_n, respectively. Section 5 is dedicated to the study of more graphs with domination number 2, where we either give solutions for equation (1), or else, prove the nonexistence of nontrivial solutions.

2. Preliminaries.

Before proceeding, some previous results and some related ideas are presented.

Theorem 2.1 [2]. *Let D be a γ-set for G. Then there is a vertex $v \in V(G) - D$ such that v is adjacent to at most two vertices of D.* □

Theorem 8 of [1] states that *cycles, C_n; $n \geq 3$, satisfy Vizing's conjecture.* The proof of this theorem made use of the fact that if D is a γ-set for $C_n \square H$, $n \geq 6$, then the graph $C_{n-3} \square H$ and a corresponding $\gamma(C_{n-3} \square H)$-set, D'', may be constructed from $C_n \square H$ and D, respectively. This is simply done if two successive rows in $V(C_n \square H)$ are deleted and then the two rows adjacent to the deleted ones are identified. Here the rows corresponding to the vertices of H_{n-1} and H_n are deleted and then the vertices corresponding to H_1 and H_{n-2} are dentified. According to this construction, the following corollary is obtained.

Corollary 2.2 [1]. *For any connected graph H and $n \geq 6$,*

$$\gamma(C_n \square H) \geq \gamma(C_{n-3} \square H) + \gamma(H). \qquad \square$$

Obviously, if C_n and C_{n-3} in corollary 2.2 are replaced by C''_n and C''_{n-3}, respectively, the resulting inequality is valid, as long as we keep away from the vertex adjacent to the newly added vertex either in the deletion or in the identification process. An analogous result holds for the graph C'_n described above. This proves the following corollary:

Corollary 2.3. *For any connected graph H and $n \geq 6$, the following inequalities hold:*

$$(i)\ \gamma(C'_n \square H) \geq \gamma(C'_{n-3} \square H) + \gamma(H), \text{ and}$$

$$(ii)\ \gamma(C''_n \square H) \geq \gamma(C''_{n-3} \square H) + \gamma(H). \qquad \square$$

Let G be a fixed connected graph with domination number 2. To gain some insight in the case when the lower bound of Vizing's conjecture can actually be achieved, we are going to recall and extend the proof of El-Zahar and Pareek [1] that graphs with domination number 2 satisfy Vizing's conjecture.
Let G be a connected graph for which equation (1) is satisfied. Assume that A is a minimum dominating set for the product $G\square H$.

Define:

$$B_0 = \{y \in V(H): |V(G_y) \cap A| = 0\},$$

$$B_1 = \{y \in V(H): |V(G_y) \cap A| = 1\}, \text{ and}$$

$$B_2 = \{y \in V(H): |V(G_y) \cap A| \geq 2\}.$$

Evidently, $B_0 \cup B_1 \cup B_2$ is a partition of $V(H)$. Since $\gamma(G) = 2$, $V(G)$ can be partitioned into V_1 and V_2 such that each of the sets V_1 and V_2 is a dominating set of $\overline{G}$; the complementary graph of G, [1]. In fact, for the graphs which are under consideration in this paper, several such partitions exist; those different partitions are employed to investigate the existence of solutions for equation (1). For our purposes, assume $V(G)$ has the following two different partitions:

$$V(G) = V_1 \cup V_2 \quad \text{and} \quad V(G) = V_1' \cup V_2'.$$

For $i = 1, 2$; let

$$B_{1i} = \{y \in B_1 : V(G_y) \cap A = \{(x,y)\}, \text{with } x \in V_i\}, \text{ and}$$

$$B_{1i}' = \{y \in B_1 : V(G_y) \cap A = \{(x,y)\}, \text{with } x \in V_i'\}.$$

It can be shown that each one of the sets $B_2 \cup B_{1i}$ and $B_2 \cup B_{1i}'$; $i = 1,2$, is a dominating set of H, and thus, it has cardinality greater than or equal to $\gamma(H)$. In particular, $|B_2 \cup B_{11}| \geq \gamma(H)$ and $|B_2 \cup B_{12}| \geq \gamma(H)$, [1]. This implies that $2\gamma(H) = |A| \geq 2|B_2| + |B_1| \geq 2\gamma(H)$. Therefore, $|B_2 \cup B_{11}| = |B_2 \cup B_{12}| = \gamma(H)$, and

$$B_2 = \{y \in V(H): |V(G_y) \cap A| = 2\}.$$

Hence, $|B_{11}| = |B_{12}|$. Considering the second partition and applying a similar argument, one can get $|B_{11}'| = |B_{12}'| = |B_{11}| = |B_{12}|$.

For each $v \in V(G)$, let

$$F_v = \{y \in V(H): V(G_y) \cap A = \{(v,y)\}\}.$$

Then, for $i = 1, 2$;

$$B_{1i} = \bigcup_{v \in V_i} F_v \quad \text{and} \quad B_{1i}' = \bigcup_{v \in V_i'} F_v.$$

Thus, the following equality holds:

$$\sum_{v\in V_1}|F_v| = \sum_{v\in V_2}|F_v| = \sum_{v\in V_1'}|F_v| = \sum_{v\in V_2'}|F_v| \quad (2)$$

3. The Graphs $K_{m,n} \Box H$; $m \geq 2$ and $n \geq 3$.

We are now ready to investigate the existence of solutions for equation (1) for some fixed graphs G with domination number 2.

If one factor of the cartesian product is $K_{2,n}$, then the results of [6] and [2] imply that both K_2 and C_4, respectively, are solutions for equation (1). For any graph H having at least 4 vertices, a characterization of H for which Vizing's conjecture holds with equality is given as follows:

Theorem 3.1. *Let H be a connected graph of order at least four. Then H satisfies $\gamma(K_{2,n} \Box H) = 2\gamma(H)$, $n \geq 3$, if and only if H is either C_4 or a generalized comb.*

Proof. Assume H is a connected graph with order at least four and let A be a minimum dominating set for $K_{2,n}\Box H$ with cardinality $2\gamma(H)$.

Since $\gamma(K_{2,n}) = 2$, there is a partition of $V(K_{2,n})$ into V_1 and V_2 such that each of the sets V_1 and V_2 is a dominating set of $\overline{K_{2,n}}$. In fact, $V(K_{2,n})$ has several such partitions each of which satisfies this property. Note that $\overline{K_{2,n}} = K_2 \cup K_n$, and label the vertices of K_2 by u_1, u_2, and those of K_n by v_1, v_2, ..., v_n. Consider the following partitions of $V(K_{2,n})$:

$V_1 = \{u_1, v_1\}$, $V_2 = \{u_2, v_2, \ldots, v_n\}$, and

$V_1' = \{u_1, v_1, v_2\}$, $V_2' = \{u_2, v_3, \ldots, v_n\}$.

As a result of equality (2), one can conclude that

$\left|F_{u_1} \cup F_{v_1}\right| = \left|F_{u_1} \cup F_{v_1} \cup F_{v_2}\right|$, and hence, $\left|F_{v_2}\right| = 0$.

Considering other different partitions gives:

$\left|F_{v_i}\right| = 0$, for each i; $i = 1, 2, \ldots, n$, and $\left|F_{u_1}\right| = \left|F_{u_2}\right|$. Now, the following two cases are studied.

Case 1: $F_{u_1} = F_{u_2} = \varnothing$. This means that B_2 is a $\gamma(H)$-set. Note that $B_0 \neq \varnothing$. If not, then H is a null graph which contradicts the hypothesis of the theorem. So, let $x_0 \in B_0$. Then, in order for G_{x_0} to be dominated by A, x_0 should be adjacent to at least 3 distinct vertices in B_2, since $n \geq 3$, which contradicts theorem 2.1.

Case 2: Both sets F_{u_1} and F_{u_2} are nonempty. Suppose $B_0 \neq \varnothing$, and consider the vertex $x_0 \in B_0$. Note that x_0 is adjacent to exactly two vertices in B_2. Thus x_0 is adjacent to at least one vertex in one of the sets F_{u_1} and F_{u_2}, as well as the two vertices say $y_1, y_2 \in B_2$. Without loss of generality, assume x_0 is adjacent to $x_1 \in F_{u_i}$. But then, the set $(B_2 \cup F_{u_1} - \{y_1, y_2\}) \cup \{x_0\}$ would be a $\gamma(H)$-set with smaller cardinality. So, $B_0 = \varnothing$. Therefore, $|V(H)| = |B_1| + |B_2|$ but, $2\gamma(H) = |B_1| + 2|B_2|$ and so $\gamma(H) > \frac{1}{2}|V(H)|$, which is a contradiction. This implies that $B_2 = \varnothing$. Consequently, $V(H) = B_1$ and $2\gamma(H) = |B_1| = |V(H)|$, that is, H is either C_4 or a generalized comb. Conversely, if H is a generalized comb, denote the set of end vertices of H by U and let $W = V(H) - U$. Clearly, the set $(\{u_1\} \times U) \cup (\{u_2\} \times W)$ is a γ-set for $K_{2,n} \Box H$ with cardinality $2\gamma(H)$. Also, if $H = C_4$, then the set $(\{u_1\} \times \{1,3\}) \cup (\{u_2\} \times \{2,4\})$ is a γ-set for $K_{2,n} \Box C_4$ with cardinality $2\gamma(C_4)$. □

The above theorem implies that a sharp lower bound is attained infinitely often for the graph $K_{2,n}$. However, the next theorem shows that this is not the case for the complete bipartite graph, $K_{m,n}$ where $m, n \geq 3$.

Theorem 3.2. *For any connected graph H of order at least four;* $\gamma(K_{m,n} \Box H) > 2\gamma(H)$; $m, n \geq 3$.

Proof. The graph $K_{m,n}$ has domination number 2 and thus it satisfies Vizing's conjecture, so it remains to prove that equality does not hold.

Suppose H is a graph for which $\gamma(K_{m,n} \Box H) = 2\gamma(H)$. Let A be a minimum dominating set for $K_{m,n} \Box H$ such that $|A| = 2\gamma(H)$. Observe that $\overline{K_{m,n}} = K_m \cup K_n$, let $V(K_m) = \{u_1, u_2, \ldots, u_m\}$ and $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Consider the following partitions of $V(K_{m,n})$, where each of the sets V_k and V_k'; $k = 1,2$, is a dominating set for $\overline{K_{m,n}}$:

$V_1 = \{u_1, v_1\}$, $\quad V_2 = \{u_2, \ldots, u_m, v_2, \ldots, v_n\}$, and

$V_1' = \{u_1, u_2, v_1, v_2\}$, $\quad V_2' = \{u_3, \ldots, u_m, v_3, \ldots, v_n\}$.

This implies that $|F_{u_2}| = |F_{v_2}| = 0$. Considering other different partitions, it can easily be realized that $|F_{u_i}| = 0$ for all i; $1 \leq i \leq m$, and $|F_{v_j}| = 0$ for all j, $1 \leq j \leq n$. It follows that $B_1 = \varnothing$, and hence, B_2 is a $\gamma(H)$-set. Now, any $y \in V(H) - B_2$ must be adjacent to at least 3 distinct vertices in B_2, otherwise; not all vertices of the column G_y would be dominated by A. This contradicts Theorem 2.1, and thus the result follows. □

4. The Graphs $C_n' \Box H$ and $C_n'' \Box H$.

Now, the effect of adding one chord joining two alternating vertices of C_n is studied. Assume that $V(C_n) = \{1, 2, \ldots, n\}$. For the case $n = 4$; $\gamma(C_4') = 1$, therefore, equation (1) has no solution [7]. A similar result holds for C_3''. For the graph C_5' (see Fig.1.) the result is addressed in the following lemma.

Lemma 4.1. *For any connected graph H of order at least four,* $\gamma(C_5' \square H) > 2\gamma(H)$.

Proof. Assume that there exists a graph H such that $\gamma(C_5' \square H) = 2\gamma(H)$ and let A be a minimum dominating set for $C_5' \square H$ with cardinality $2\gamma(H)$. Consider the following partitions of $V(C_5')$:

1) $V_1 = \{1, 5\}$, $V_2 = \{2, 3, 4\}$,

2) $V_1' = \{1, 2, 5\}$, $V_2' = \{3, 4\}$, and

3) $V_1'' = \{1, 2, 3\}$, $V_2'' = \{4, 5\}$.

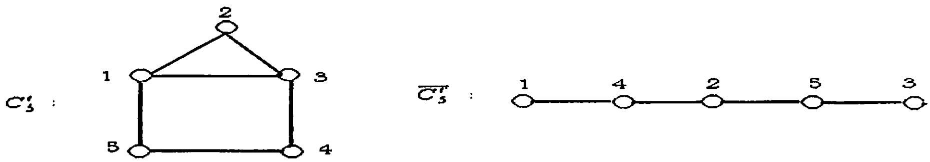

Fig. 1.

This implies that $F_2 = \varnothing$, $|F_1| = |F_4|$ and $|F_3| = |F_5|$. Now the following two cases are tackled.

Case 1: $F_5 \neq \varnothing$, then $F_3 \neq \varnothing$. For each i such that $1 \le i \le 5$, let $A_i = \{x \in V(H) : (i, x) \in A\}$. Note that $F_5 \subseteq N(A_2)$. If not, then for some $y \in F_5$, the vertex $(2, y)$ would not be dominated by A. Since $F_2 = \varnothing$ and $F_5 \subseteq N(A_2)$, then y is adjacent to some vertex $z \in B_2$, this implies that the set $(B_2 \cup F_1 \cup F_5) - \{z\}$ dominates H and has cardinality $\gamma(H)-1$ which is a contradiction.

Case 2: $F_5 = \varnothing$, then $F_3 = \varnothing$. Furthermore, if $F_4 = \varnothing$, then $F_1 = \varnothing$, consequently, B_2 is $\gamma(H)$-set, which leads to a contradiction. So, $F_4 \neq \varnothing$ and $F_4 \subseteq N(A_2)$ which again leads to a contradiction. Therefore, for the graph C_5', there exists no graph H for which Vizing's lower bound is sharp. □

Note that the graph C_4'' is a spanning subgraph of C_5' with the same domination number. This implies that

Corollary 4.2. *For any connected graph H of order at least four;* $\gamma(C_4'' \square H) > 2\gamma(H)$. □

Lemma 4.3. *For any connected graph H of order at least 4,* $\gamma(C_6' \square H) > 2\gamma(H)$.

Proof. The graph C_6' and its complement are shown in Fig.2.

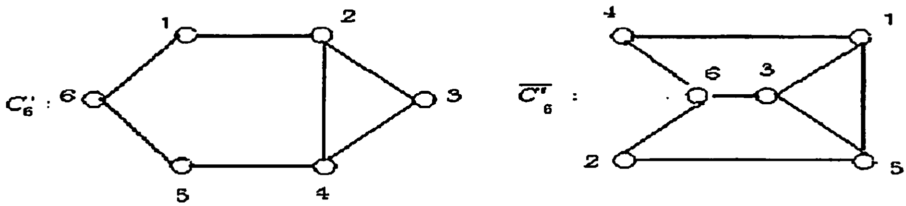

Fig. 2.

Considering different partitions of $V(C'_6)$ into $V_1 \cup V_2$ where each of the sets V_1 and V_2 is a dominating set of $\overline{C'_6}$ yields $B_1 = \varnothing$, and thus B_2 is a $\gamma(H)$-set, which leads to a contradiction. So, the result follows. □

Let us remark here that a similar result of the above could be obtained if another chord joining vertex 1 to vertex 5 is added to the graph C'_6 of lemma 4.3.

The following corollary is an immediate result of lemma 4.3.

Corollary 4.4. *For any connected graph H of order at least 4,* $\gamma(C''_5 \square H) > 2\gamma(H)$. □

To this point, it has been shown that for $n = 4, 5, 6$; $\gamma(C'_n \square H) > \gamma(C'_n)\gamma(H)$, and for $n = 3, 4, 5$; $\gamma(C''_n \square H) > \gamma(C''_n)\gamma(H)$. The general cases are obtained using corollary 2.3. This can be stated as follows.

Theorem 4.5. *For any connected graph H of order at least 4,*

$$\gamma(C'_n \square H) > \gamma(C'_n)\gamma(H);\ n \geq 4, \text{ and}$$

$$\gamma(C''_n \square H) > \gamma(C''_n)\gamma(H);\ n \geq 3.$$ □

5. More Graphs with Domination Number 2.

In this section, two results which are immediate consequences of the results of section 3 are demonstrated. Let G_1 be the graph obtained from C_5 by adding the chords $\{1,3\}$ and $\{2,5\}$, while G_2 is the graph obtained from C_6 by adding the chords $\{1,3\}$, $\{2,6\}$, $\{3,5\}$, and $\{4,6\}$, which are shown in Fig.3. Then some solutions for equation (1) are given in the following lemma.

Lemma 5.1. *Let G be one of graphs G_1 or G_2 and let H be either C_4 or a generalized comb. Then* $\gamma(G \square H) = \gamma(G)\gamma(H)$.

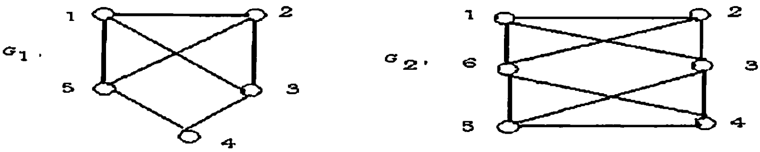

Fig . 3.

Proof. The two mentioned graphs G_1 and G_2 are supergraphs of $K_{2,3}$ and $K_{2,4}$, respectively, with the same order and domination number. So, if $n=3$ or 4, then $\gamma(G\Box H)\leq\gamma(K_{2,n}\Box H)=2\gamma(H)$, where H is either C_4 or a generalized comb.

□

Corollary 5.2. *Let G be the graph obtained from C_6 by adding at least one of the chords $\{i,i+3\}$ where $i=1, 2, 3$. Then, for any connected graph H;* $\gamma(G\Box H)>2\gamma(H)$.

Proof. Note that if all three mentioned chords are added to C_6 then G is isomorphic to $K_{3,3}$. Thus the result follows from theorem 3.2. On the other hand, if not all three chords are added, then G is a spanning subgraph of $K_{3,3}$ with the same domination number and hence

$\gamma(G\Box H)\geq\gamma(K_{3,3}\Box H)>2\gamma(H)$. □

We end this section with the following result concerning the graph $Q_3=C_4\Box K_2$, since $\gamma(Q_3)=2$.

Theorem 5.3. *For any connected graph H of order at least 4,* $\gamma(Q_3\Box H)>2\gamma(H)$.

Proof. The graph Q_3 and its complement are shown in Fig. 4.

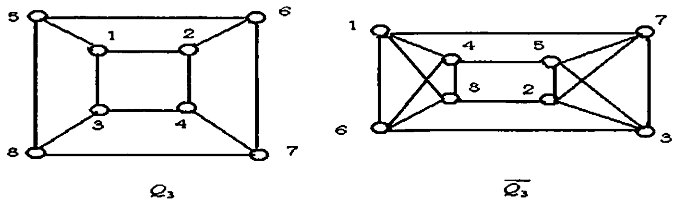

Fig. 4

Consider the following partitions of $V(Q_3)$:

1) $V_1=\{1, 4\}$, $V_2=\{2, 3, 5, 6, 7, 8\}$,

2) $V_1'=\{2, 3\}$, $V_2'=\{1, 4, 5, 6, 7, 8\}$, and

3) $V_1''=\{3,5\}$, $V_2''=\{1, 2, 4, 6, 7, 8,\}$.

Which implies that for each i, $i=1, 2, \ldots, 8$, F_i is empty. So, B_1 is empty, and thus B_2 is a γ-set for H which leads to a contradiction. □

This shows that if one factor of the cartesian product is Q_3 then the lower bound of Vizing's conjecture is not attained. However, considering the graph $Q_4=Q_3\Box K_2$ proves that the upper bound, given in [10], is actually achieved, since $4=|V(K_2)|\gamma(Q_3)\geq\gamma(Q_3\Box K_2)=\gamma(Q_4)=4$.

References.

[1] M. H. El-Zahar, C. M. Pareek, *Domination number of products of graphs*, Ars Combin. 31, (1991), 223-227.

[2] M. H. El-Zahar, S. M. Khamis, Kh. M. Nazzal, *On the domination number of the Cartesian product of the cycle of length n and any graph*, Discrete Applied Math, (submitted).

[3] J. F. Fink, M.S. Jacobson, *n-domination in graphs*, In Y.Alavi and A. J. Schwenk, editors, Graph Theory with Applications to Algorithms and Computer Science, (Kalamazoo, MI 1984), Wiley, (1985), 283-300.

[4] M.S. Jacobson, L.F. Kinch, *On the domination of the products of graphs II: trees*, J.Graph Theory 10 (1986), 97-106.

[5] B. L. Hartnell, D. F. Rall, *On Vizing's conjecture*, Congr. Numer. 82, (1991), 87-96.

[6] B. L. Hartnell, D. F. Rall, *Domination in Cartesian products*, *Vizing's conjecture*, In T. W. Haynes, S. T. Hedetniemi , P. J. Slater, editors, Domination in Graphs: Advanced Topics, Marcel Dekker, New York, (1998), 163-189.

[7] B. L. Hartnell, D.F. Rall, *Lower bounds for dominating Cartesian products*, J. Comb. Math. Comb. Comp. 31, (1999), 219-226.

[8] B. L. Hartnell, D. F. Rall, *On dominating the Cartesian product of a graph and K_2*, Discussiones Mathematicae - Graph Theory, 24(3), (2004), 389-402.

[9] T. W. Haynes, S. T. Hedetniemi, P. S. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York, (1998).

[10] V. G. Vizing, *The Cartesian product of graphs*, Vyčisl. Sistemy 9, (1963), 30-43.