A new product construction for large sets of resolvable Mendelsohn triple systems *

Hongtao Zhao
School of Mathematics and Physics,
North China Electric Power University,
Beijing 102206, China
ht_zhao@163.com

Abstract
A large set of resolvable Mendelsohn triple systems of order v, denoted by LRMTS(v), is a collection of $v-2$ RMTS(v)s based on v-set X, such that every Mendelsohn triple of X occurs as a block in exactly one of the $v-2$ RMTS(v)s. In this paper, we use TRIQ and LR-design to present a new product construction for LRMTS(v)s. This provides some new infinite families of LRMTS(v)s.
Keywords: Large set; Resolvable Mendelsohn triple system; Transitive resolvable idempotent quasigroup; LR-design

1 Introduction

Let X be a v-set. A *Mendelsohn triple* is a cyclic tripe $\langle x,y,z\rangle$ (or $\langle y,z,x\rangle$, or $\langle z,x,y\rangle$) based on X which consists of three order pairs: (x,y), (y,z) and (z,x). A *Mendelsohn triple system* of order v, denoted by MTS(v), is a pair $(X,\mathcal{B})$ where $\mathcal{B}$ is a collection of cyclic triples on X, called *blocks*, such that each ordered pair of X occurs in exactly one block of $\mathcal{B}$.

An MTS(v) is called *resolvable* if its blocks can be partitioned into subsets (called *parallel classes*), each containing every element of X exactly once. A resolvable MTS(v) is denoted by RMTS(v).

A *large set* of Mendelsohn triple systems of order v, denoted by LMTS(v), is a collection of $(v-2)$ MTS(v)s based on X such that every Mendelsohn triple from X occurs as a block in exactly one of the $(v-2)$ MTS(v)s. Existence results for LMTSs and RMTSs are well known from [1, 10].

*Research supported by NSFC Grant 10901051, NSFC Grant 10971051 and Doctoral Grant of North China Electric Power University.

Theorem 1.1 (1) *There exists an* LMTS(v) *if and only if* $v \equiv 0, 1 \pmod 3$ *and* $v \neq 6$.

(2) *There exists an* RMTS(v) *if and only if* $v \equiv 0 \pmod 3$ *and* $v \neq 6$.

A large set of disjoint RMTS(v)s is denoted by LRMTS(v). The existence of LRMTS(v)s has been investigated by Chang [2], Kang [9], Kang and Lei [11], Kang and Tian [12], Kang and Xu [13], Kang and Zhao [14], Xu and Kang [18] and Zhou and Chang [23]. We can list the known conclusions as follows.

Theorem 1.2 *There exists an* LRMTS(v) *for the following orders* v :

(1) $v = 3^k m$, *where* $k \geq 1$ *and* $m \in \{1, 4, 5, 7, 11, 13, 17, 23, 25, 35, 37, 41, 43, 47, 53, 55, 57, 61, 65, 67, 91, 123\}$.

(2) $v = 7^k + 2,\ 13^k + 2,\ 25^k + 2,\ 2^{4k} + 2$ *and* $2^{6k} + 2$, *where* $k \geq 0$.

(3) $v = 12(t + 1)$, *where* $t \in \{0, 1, 2, 3, 4, 6, 7, 8, 9, 14, 16, 18, 20, 22, 24\}$.

(4) $v = 6t + 3$, *where* $t \in \{35, 38, 46, 47, 48, 51, 56, 60\}$.

Also, if there exists an LRMTS(v), *then there exists an* LRMTS($(2 \cdot s^k + 1)v$) *for any* $k \geq 0$, $s = 7, 13$ *and* $v \equiv 0, 3, 9 \pmod{12}$.

Some orders in this theorem come from the existence of LKTS(v)s, which are defined below.

A *group-divisible design* (briefly GDD) is a triple $(X, \mathcal{G}, \mathcal{B})$ with the following properties: (i) X is a finite set of points; (ii) $\mathcal{G}$ is a partition of X into subsets called groups; (iii) $\mathcal{B}$ is a set of subsets of X (called blocks) such that a group and a block contain at most one common point, and any pair of points from distinct groups occur in exactly one block of $\mathcal{B}$. A GDD $(X, \mathcal{G}, \mathcal{B})$ is called *resolvable*, denoted by RGDD, if there exists a partition $\Gamma = \{P_1, P_2, \cdots, P_r\}$ of $\mathcal{B}$ such that each part P_i (called *parallel classes*) is a partition of X.

A GDD is called a *transversal design* if it has exactly k groups of size n and every block has size k. We denoted such a GDD by TD(k, n). A TD is called *resolvable* (denoted by RTD) if it is a RGDD.

A GDD $(X, \mathcal{G}, \mathcal{B})$ is called a *Steiner triple system* if $|X| = v$ and it has v groups of size 1 and every block has size 3. Such a GDD is denoted briefly by STS(v) $(X, \mathcal{B})$. A resolvable STS(v) is called a *Kirkman triple system* and denoted by KTS(v).

A large set of Kirkman triple system of order v, denoted by LKTS(v), is a collection of $v - 2$ KTS(v)s based on a v-set X, such that each triple from X occurs in exactly one of the $v - 2$ KTS(v)s. In a KTS(v), if we replace any triple $\{x, y, z\}$ by two Mendelsohn triples $\langle x, y, z \rangle$ and $\langle z, y, x \rangle$, then we obtain an RMTS(v). It is obvious that the existence of an LKTS(v) implies the existence of an LRMTS(v). However, this approach can provide only odd orders of v since the existence of a KTS(v) implies $v \equiv 3 \pmod 6$. The existence of LKTS(v)s, known as the general Sylvester's problem of the 15 schoolgirls, has a long history [4]. The recent investigation was started in 1974 by Denniston [4, 5, 6, 7]. Some recursive constructions were given by Ji and Lei [8], Lei [15, 16], Yuan and Kang [19, 20] and Zhang and Zhu [21, 22]. We summarize the known results on LKTS(v)s as follows.

Theorem 1.3 (1) *There exists an* LKTS($3^n m(2\cdot 13^{n_1}+1)(2\cdot 13^{n_2}+1)\cdots(2\cdot 13^{n_t}+1)$) *for* $n \geq 1, m \in M, t \geq 1$ *and* $n_i \geq 1$ $(i = 1, 2, \cdots, t)$, *where* $M = \{1, 5, 11, 17, 25, 35, 43, 67, 91, 123\} \cup \{2^{2r+1}25^s + 1 : r \geq 0, s \geq 0\}$.

(2) *There exists an* LKTS($3\prod_{i=1}^{p}(2q_i^{r_i}+1)\prod_{j=1}^{q}(4^{s_j}-1)$) *for* $p + q \geq 1, r_i, s_j \geq 1$ *and prime power* $q_i \equiv 7$ (*mod* 12).

The main result of this paper is to give a new product construction for LRMTSs. This provides some new infinite families of LRMTS(v)s. In Section 2, we give some concepts such as transitive resolvable idempotent quasigroup (TRIQ(v)), LR-design (LR(u)), etc. In Section 3, we make use of TRIQ(v) and LR(u) to present a new product construction. In Section 4, we give new orders for LRMTS(v)s.

2 Definitions

A quasigroup is a pair $(X, \circ)$, where X is a set and $(\circ)$ is a binary operation on X such that the equation $a \circ x = b$ and $y \circ a = b$ are uniquely solvable for every pair of elements a, b in X. The order of a quasigroup $(X, \circ)$ is the size of X.

A quasigroup of order v is called *idempotent* if the identity $x \circ x = x$ holds for all x in X. An idempotent quasigroup of order v is denoted by IQ(v). A quasigroup of order v is called *symmetric* if the identity $x \circ y = y \circ x$ holds for every pair of elements x, y in X. A symmetric quasigroup of order v is denoted by SQ(v).

A quasigroup $(X, \circ)$ is called *resolvable* if all $v(v-1)$ pairs of distinct elements can be partitioned into subsets $T_i, 1 \leq i \leq 3(v-1)$, such that every $\{(x, y, x \circ y) : (x, y) \in T_i\}$ is a partition of X. An idempotent quasigroup IQ(v) is called (*sharply*) *transitive* if there exists a group of order v acting transitively on X which forms an automorphism group of the IQ(v). A transitive resolvable IQ(v) is denoted by TRIQ(v). A transitive resolvable symmetric IQ(v) is denoted by TRISQ(v). In [2], Chang gave the following existence result.

Lemma 2.1 *[2] There exists a* TRIQ(v) *for* $v \equiv 0, 3, 9$ (*mod* 12).

Transitive IQ has been used to give a tripling construction for large sets of STSs in Teirlinck [17]. To consider the similar problem for large sets of KTSs and large sets of RMTSs, we demand that the transitive IQ must have certain property of resolvability. TRISQ(v) was used to construct LKTSs [21]. TRIQ(v) was used to construct LRMTSs [2, 23].

In [15], Lei introduced a kind of combinatorial design named LR-design, denoted by LR(u). An LR(u) is a collection $\{(X, \mathcal{A}_k^j) : 1 \le k \le \frac{u-1}{2}, j = 0, 1\}$, where each $(X, \mathcal{A}_k^j)$ is a KTS(u) based on u-set X and $\{A_k^j(h); 1 \le h \le \frac{u-1}{2}\}$ is a *resolution* (collection of parallel classes) of $\mathcal{A}_k^j$ with the properties.

i) $\bigcup_{k=1}^{\frac{u-1}{2}} A_k^0(1) = \bigcup_{k=1}^{\frac{u-1}{2}} A_k^1(1) = \mathcal{A}$ forms a KTS(u) over X too;

ii) Any triple from X is contained in $\bigcup_{k=1}^{\frac{u-1}{2}} \bigcup_{j=0}^{1} \mathcal{A}_k^j$.

Lei [15] and Ji and Lei [8] obtained some existence results for LR(u).

Lemma 2.2 *[15, 8] There exists an* $\mathrm{LR}(3^a 5^b \prod_{i=1}^{r}(2 \cdot 13^{n_i} + 1) \prod_{j=1}^{p}(2 \cdot 7^{m_j} + 1))$ *for any integer* $n_i, m_j \ge 1$ $(1 \le i \le r, 1 \le j \le p)$, $a, b, r, p \ge 0$ *with* $a + r + p \ge 1$.

Recently, using these auxiliary designs and their existence, Chang et al. [2, 23] proved the following conclusions.

Lemma 2.3 *[2] If there exist both a* TRIQ(v) *and an* LRMTS(v), *then there exists an* LRMTS($3v$).

Lemma 2.4 *[23] If there exist an* LRMTS(v), *a* TRIQ(v) *and an* LR(u), *then there exists an* LRMTS(uv).

Next, we introduce the concept of complete mapping in a finite group. We follow the definition in Denes and Keedwell [3].

A complete mapping of a group $(G, \cdot)$, is a bijection mapping $x \to \theta(x)$ of G upon G, such that the mapping $\eta(x) = x \cdot \theta(x)$ is also a bijection mapping of G upon G. The following existence results were stated in [3].

Lemma 2.5 *[3] If G is an arbitrary group of order $n = 4k+2$, then G has no complete mapping. If G is an abelian group of order $n \ne 4k+2$, then G does have a complete mapping.*

Let $X = \{0, 1, \cdots, v-1\}$ and $(X, \circ)$ be an idempotent quasigroup with a sharply transitive automorphism group G written multiplicatively. It is easy to see that there is a unique $g \in G$ such that $g(x) = y$ for every pair of elements x, y in X. Let the first row of $(X, \circ)$ be of the following ordered triples:

$$(0, h(0), h^*(0)), \ h \in G.$$

Then $h \mapsto h^*$ is a bijection between G, denoted by Φ. Hence, $(g(0), gh(0), gh^*(0)), g, h \in G$ forms the quasigroup $(X, \circ)$.

Then $(g, gh, gh^*), g, h \in G$ is a latin square on G, which implies that

$$\{(gh, gh^*) : g, h \in G\} = G \times G.$$

So, we have

$$\{h(h^*)^{-1} : h \in G\} = G \tag{1}$$

Note that the mapping $\overline{\Phi} : h \mapsto (h^*)^{-1}$ is also a bijection between G. By the definition of complete mapping and formula (1), $\overline{\Phi}$ is a complete mapping of G. Next we record the result as follows.

Lemma 2.6 *If there exists a transitive* IQ *with G as a sharply transitive automorphism group, then G has a complete mapping.*

3 A new product construction for LRMTS

Let $X = \{0, 1, \cdots, v-1\}$ and $(X, \circ)$ be an idempotent quasigroup with a sharply transitive automorphism group $G = \{\sigma_0, \sigma_1, \cdots, \sigma_{v-1}\}$. By Lemma 2.6, G has a complete mapping, say, Φ^{-1}. Let $\sigma^* = \Phi(\sigma)$ for $\sigma \in G$. Then, by the definition of complete mapping, we have

$$\{\sigma(\sigma^*)^{-1} : \sigma \in G\} = G \tag{2}$$

Theorem 3.1 *If there exist an* LRMTS($3v$), *a* TRIQ(v) *and an* LR(u), *then there exists an* LRMTS(uv).

Proof. Suppose that X is a u-set with a linear order " $<$ " (i.e. for any $x \neq y$, $x, y \in X$, either $x < y$ or $y < x$). We have an LR(u) over X with the following collection of $u-1$ KTS(u)s

$$\{(X, \mathcal{A}_k^l) : \ 1 \leq k \leq \frac{u-1}{2}, \ l = 0, 1\}$$

which with following properties:

(i) Let the resolution of $\mathcal{A}_k^l$ be $\Gamma_k^l = \{A_k^l(h) : 1 \leq h \leq \frac{u-1}{2}\}$, and

$$\bigcup_{k=1}^{\frac{u-1}{2}} A_k^0(1) = \bigcup_{k=1}^{\frac{u-1}{2}} A_k^1(1) = \mathcal{A},$$

$(X, \mathcal{A})$ is a KTS(u).

(ii) For any triple $T = \{x, y, z\} \subset X$, $x \neq y \neq z \neq x$, there exist k, l such that $T \in \mathcal{A}_k^l$.

Furthermore, suppose that Y is a set of size v. So we have a TRIQ(v) over Y. Let $(Y, \circ)$ be a TRIQ(v), $G = \{\sigma_0, \sigma_1, \cdots, \sigma_{v-1}\}$ be the transitive automorphism group of $(Y, \circ)$. We will construct an LRMTS(uv) on the point set $X \times Y$. The construction proceeds in 2 steps.

Step 1: For any $\{x,y,z\} \subseteq X$, $\{x,y,z\} \in \mathcal{A} = \bigcup_{k=1}^{\frac{u-1}{2}} A_k^0(1)$.

(1) If $\{x,y,z\} \in A_1^0(1)$, we have an LRMTS($3v$) on the point set $\{x,y,z\} \times Y$. Let its block set be $\{\mathcal{B}_i^{\{x,y,z\}} : 1 \le i \le v-2\} \bigcup \{\mathcal{B}_j^l(\{x,y,z\}) : 0 \le j \le v-1, l = 0,1\}$, and each $\mathcal{B}_i^{\{x,y,z\}}$ can be partitioned into parallel classes $B_i^{\{x,y,z\}}(n)$, $1 \le n \le 3v-1$, each $\mathcal{B}_j^l(\{x,y,z\})$ can be partitioned into parallel classes $B_j^l(\{x,y,z\},n)$, $1 \le n \le 3v-1$.

(2) If $\{x,y,z\} \notin A_1^0(1)$, i.e. $\{x,y,z\} \in A_k^0(1)$ for some k, $2 \le k \le \frac{u-1}{2}$, $x < y < z$, let

$$P_{j,s}^{\{x,y,z\}} = \{\langle (x,a),(y,\sigma_s(a)),(z,\sigma_j\sigma_s^*(a))\rangle : a \in Y\},$$

$$\overline{P}_{j,s}^{\{x,y,z\}} = \{\langle w,v,u\rangle : \langle u,v,w\rangle \in P_{j,s}^{\{x,y,z\}}\},$$

where $\sigma_s, \sigma_j \in G$ and let

$$\mathcal{A}_j^{\{x,y,z\}} = \bigcup_{\sigma_s \in G} P_{j,s}^{\{x,y,z\}},$$

$$\overline{\mathcal{A}}_j^{\{x,y,z\}} = \bigcup_{\sigma_s \in G} (P_{j,s}^{\{x,y,z\}} \bigcup \overline{P}_{j,s}^{\{x,y,z\}}).$$

Then by formula (2), it is easy to show that each $(\{x,y,z\} \times Y, \mathcal{A}_j^{\{x,y,z\}})$ ($0 \le j \le v-1$, in $\mathcal{A}_j^{\{x,y,z\}}$, $\langle u,v,w\rangle$ is replaced by $\{u,v,w\}$.) is a resolvable TD($3,v$) with the parallel classes $P_{j,s}^{\{x,y,z\}}$, $\sigma_s \in G$, and these v TDs form a large set of disjoint RTDs.

Since $(Y,\circ)$ is a TRIQ(v), for any ordered pair $(a,b) \in Y \times Y$ $(a \ne b)$ and any $\sigma \in G$, we get an element $a \circ b$ in Y such that $\sigma(a) \circ \sigma(b) = \sigma(a \circ b)$. Define

$$B_{a,b,x,j}^{(0)} = \langle (x,a),(x,b),(y,\sigma_j(a\circ b))\rangle,$$

$$B_{a,b,y,j}^{(0)} = \langle (y,\sigma_j(a)),(y,\sigma_j(b)),(z,\sigma_0\sigma_j^*(a\circ b))\rangle,$$

$$B_{a,b,z,j}^{(0)} = \langle (z,\sigma_0\sigma_j^*(a)),(z,\sigma_0\sigma_j^*(b)),(x,a\circ b)\rangle,$$

$$B_{a,b,x,j}^{(1)} = \langle (x,a),(x,b),(z,\sigma_{v-1}\sigma_j^*(a\circ b))\rangle,$$

$$B_{a,b,y,j}^{(1)} = \langle (y,\sigma_j(a)),(y,\sigma_j(b)),(x,a\circ b)\rangle,$$

$$B_{a,b,z,j}^{(1)} = \langle (z,\sigma_{v-1}\sigma_j^*(a)),(z,\sigma_{v-1}\sigma_j^*(b)),(y,\sigma_j(a\circ b))\rangle$$

and

$$\mathcal{B}_j^l(\{x,y,z\}) = (P_{v-l,j}^{\{x,y,z\}} \bigcup \overline{P}_{v-l,j}^{\{x,y,z\}}) \bigcup (\bigcup_{\substack{(a,b)\in Y\times Y \\ a \ne b}} \{B_{a,b,x,j}^{(l)}, B_{a,b,y,j}^{(l)}, B_{a,b,z,j}^{(l)}\}),$$

where $0 \le j \le v-1, l = 0, 1$ and $v - l = 0, v-1$.

Note that the blocks $\langle (x,a),(y,\sigma_j(a)),(z,\sigma_0\sigma_j^*(a))\rangle$, $\langle (x,b),(y,\sigma_j(b)),(z,\sigma_0\sigma_j^*(b))\rangle$ and $\langle (x,a\circ b),(y,\sigma_j(a\circ b)),(z,\sigma_0\sigma_j^*(a\circ b))\rangle$ are the three blocks of $P_{0,j}^{\{x,y,z\}}$, and the blocks $\langle (x,a),(y,\sigma_j(a)),(z,\sigma_{v-1}\sigma_j^*(a))\rangle$, $\langle (x,b),(y,\sigma_j(b)),(z,\sigma_{v-1}\sigma_j^*(b))\rangle$ and $\langle (x,a\circ b),(y,\sigma_j(a\circ b)),(z,\sigma_{v-1}\sigma_j^*(a\circ b))\rangle$ are the three blocks of $P_{v-1,j}^{\{x,y,z\}}$. Furthermore, $(\{x,y,z\} \times Y, \mathcal{B}_j^l(\{x,y,z\}))$, $0 \le j \le v-1, l = 0, 1$, is an RMTS($3v$). Let each $\mathcal{B}_j^l(\{x,y,z\})$ can be partitioned into parallel classes $B_j^l(\{x,y,z\},n)$, $1 \le n \le 3v-1$.

(For any triple T of $X \times Y$, T is form as $\langle (x,a),(x,b),(x,c)\rangle$ or $\langle (x,a),(x,b),(y,c)\rangle$ or $\langle (x,a),(y,b),(z,c)\rangle$ with $\{x,y,z\} \in \mathcal{A}$, then T appears in *Step* 1.)

Step 2: For any $\{x,y,z\} \subseteq X$, $x < y < z$, $\{x,y,z\} \notin \mathcal{A}$, (i.e. there exist k, l such that $\{x,y,z\} \in \mathcal{A}_k^l \setminus A_k^l(1)$) define

$$P_{j,s}^{\{x,y,z\}} = \{\langle (x,a),(y,\sigma_s(a)),(z,\sigma_j\sigma_s^*(a))\rangle : a \in Y\},$$

$$\overline{P}_{j,s}^{\{x,y,z\}} = \{\langle w,v,u\rangle : \langle u,v,w\rangle \in P_{j,s}^{\{x,y,z\}}\},$$

where $\sigma_s, \sigma_j \in G$ and let

$$\mathcal{A}_j^{\{x,y,z\}} = \bigcup_{\sigma_s \in G} P_{j,s}^{\{x,y,z\}},$$

$$\overline{\mathcal{A}}_j^{\{x,y,z\}} = \bigcup_{\sigma_s \in G} (P_{j,s}^{\{x,y,z\}} \bigcup \overline{P}_{j,s}^{\{x,y,z\}}).$$

Then by formula (2), it is easy to show that each $(\{x,y,z\} \times Y, \mathcal{A}_j^{\{x,y,z\}})$ ($0 \le j \le v-1$, in $\mathcal{A}_j^{\{x,y,z\}}$, $\langle u,v,w\rangle$ is replaced by $\{u,v,w\}$.) is a resolvable TD($3,v$) with the parallel classes $P_{j,s}^{\{x,y,z\}}$, $\sigma_s \in G$, and these v TDs form a large set of disjoint RTDs.

Define

$$C_i = (\bigcup_{\{x,y,z\} \in \mathcal{A} \setminus A_1^0(1)} \overline{\mathcal{A}}_i^{\{x,y,z\}}) \bigcup (\bigcup_{\{x,y,z\} \in A_1^0(1)} \mathcal{B}_i^{\{x,y,z\}}).$$

It is not difficult to check that each $(X \times Y, C_i)$, $1 \le i \le v-2$, is an RMTS(uv) with the following parallel classes:

$$C_i(n) = \bigcup_{\{x,y,z\} \in A_1^0(1)} B_i^{\{x,y,z\}}(n), \ 1 \le n \le 3v-1;$$

$$C_i(k,s) = \bigcup_{\{x,y,z\} \in A_k^0(1)} P_{i,s}^{\{x,y,z\}}, 2 \le k \le \frac{u-1}{2}, \ 0 \le s \le v-1.$$

$$\overline{C}_i(k,s) = \bigcup_{\{x,y,z\}\in A_k^0(1)} \overline{P}_{i,s}^{\{x,y,z\}},\ 2 \le k \le \frac{u-1}{2},\ 0 \le s \le v-1.$$

Furthermore, these $v-2$ RMTSs are obviously disjoint.
Define

$$\mathcal{D}_{k,j}^l = (\bigcup_{\{x,y,z\}\in A_k^l(1)} \mathcal{B}_j^l(\{x,y,z\}))\bigcup(\bigcup_{\{x,y,z\}\in A_k^l\setminus A_k^l(1)} \overline{\mathcal{A}}_j^{\{x,y,z\}}),$$

where $1 \le k \le \frac{u-1}{2}, 0 \le j \le v-1$, $l = 0, 1$. It is not difficult to check that each $(X \times Y, \mathcal{D}_{k,j}^l)$ is an RMTS(uv) with the following parallel classes:

$$\mathcal{D}_{k,j}^l(n) = \bigcup_{\{x,y,z\}\in A_k^l(1)} \mathcal{B}_j^l(\{x,y,z\},n),\ 1 \le n \le 3v-1,$$

$$\mathcal{D}_{k,j}^l(h,s) = \bigcup_{\{x,y,z\}\in A_k^l(h)} P_{j,s}^{\{x,y,z\}},\ 2 \le h \le \frac{u-1}{2},\ 0 \le s \le v-1.$$

$$\overline{\mathcal{D}}_{k,j}^l(h,s) = \bigcup_{\{x,y,z\}\in A_k^l(h)} \overline{P}_{j,s}^{\{x,y,z\}},\ 2 \le h \le \frac{u-1}{2},\ 0 \le s \le v-1.$$

and these $(u-1)v$ RMTSs are disjoint. We obtain a total of $uv-2$ disjoint RMTS(uv), a large set. This completes the proof. □

4 New orders

From Lemmas 2.1, 2.2 and Theorem 3.1, we can obtain the following conclusion.

Theorem 4.1 *For $v \equiv 0, 3, 9$ (mod 12), if there exists an* LRMTS$(3v)$, *then there exists an* LRMTS$(v \cdot 3^a 5^b \prod_{i=1}^r (2\cdot 13^{n_i}+1) \prod_{j=1}^p (2\cdot 7^{m_j}+1))$, *where integers $n_i, m_j \ge 1$ $(1 \le i \le r, 1 \le j \le p)$, $a, b, r, p \ge 0$ with $a+r+p \ge 1$.*

For example, from Theorem 1.2, for $s \in \{57, 93, 132, 240, 255\}$, the existence of LRMTS$(s)$ is unknown. But the existence of LRMTS$(3s)$ is known. And from Lemma 2.1, there exist a TRIQ(s). Thus, from Theorem 4.1, we get the following result.

Theorem 4.2 *There exists an* LRMTS$(s \cdot 3^a 5^b \prod_{i=1}^r (2\cdot 13^{n_i}+1) \prod_{j=1}^p (2\cdot 7^{m_j}+1))$, *where $s \in \{57, 93, 132, 240, 255\}$, integers $n_i, m_j \ge 1$ $(1 \le i \le r, 1 \le j \le p)$, $a, b, r, p \ge 0$ with $a+r+p \ge 1$.*

Remark: The smallest order of v (unknown before this paper) obtained from Theorem4.2 is $1395, 1980, 3600, 3825, \cdots$ in turn.

References

[1] J.C. Bermond, A. Germa, D. Sotteau, Resolvable decomposition of K_n^*, J. Combin. Theory Ser. A 26(1979) 179–185.

[2] Y. Chang, Transitively resolvable idempotent quasigroup and large sets of resolvable Mendelsohn triple systems, Discrete Math.(2009), dio:10.1016/j.disc.2009.04.012.

[3] J. Denes and A.D. Keedwell, Latin Square and their Applications, Academic Press, New York, 1974.

[4] R.H.F. Denniston, Sylvester's problem of the 15 schoolgirls, Discrete Math. 9(1974) 229-238.

[5] R.H.F. Denniston, Double resolvability of some complete 3-design, Manuscripta Math. 12(1974) 105-112.

[6] R.H.F. Denniston, Four doubly resolvable complete 3-design, Ars Combin. 7(1979) 265-272.

[7] R.H.F. Denniston, Further cases of double resolvability, J. Combin. Theory Ser. A 26(1979) 298-303.

[8] L. Ji, J. Lei, Further results on large sets of Kirkman triple systems, Discrete Math. 308(2008) 4643-4652.

[9] Q. Kang, Large sets of resolvable MTS and DTS of order $p^n + 2$, J. Combin. Des. 4(1996) 301-321.

[10] Q. Kang, J. Lei, A completion of the spectrum for large sets of disjoint Mendelsohn triple systems, Bull. Inst. Combin. Appl. 9(1993) 14-26.

[11] Q. Kang, J. Lei, On large sets of resolvable and almost resolvable oriented triple systems, J. Combin. Des. 4(1996) 95-104.

[12] Q. Kang, Z. Tian, Large sets of oriented triple systems with resolvability, Discrete Math. 212(2000) 199-221.

[13] Q. Kang, R. Xu, More large sets of resolvable MTS and resolvable DTS with even orders, Acta Math. Appl. Sin. Engl. Ser. 24(2008) 233-252.

[14] Q. Kang, H. Zhao, More large sets of resolvable MTS and resolvable DTS with odd orders, Discrete Math. 308(2008) 886-895.

[15] J. Lei, On the large sets of Kirkman systems with holes, Discrete Math. 254(2002) 259-274.

[16] J. Lei, On the large sets of Kirkman triple systems, Discrete Math. 257(2002) 63-81.

[17] L. Teirlinck, On the maximum number of disjoint Steiner triple systems, Discrete Math. 6(1973) 299-300.

[18] R. Xu, Q. Kang, A construction for LRMTS($3v$) and LRDTS($3v$), Journal of Hebei Normal College 2(1997) 8-13. (in Chinese)

[19] L. Yuan, Q. Kang, some infinite families of large sets of Kirkman triple systems, J. Combin. Des. 16(2008) 202-212.

[20] L. Yuan, Q. Kang, Another construction for large sets of Kirkman triple systems, Des. Codes Cryptogr. 48(2008) 35-42.

[21] S. Zhang, L. Zhu, Transitive resolvable idempotent symmetric quasi-groups and large sets of Kirkman triple systems, Discrete Math. 247(2002) 215-223.

[22] S. Zhang, L. Zhu, An improved product construction for large sets of Kirkman triple systems, Discrete Math. 260(2003) 307-313.

[23] J. Zhou, Y. Chang, Product constructions for large sets of resolvable MTSs and DTSs, Australas. J. Combin. 33(2005) 47-56.