On questions on (total) domination vertex critical graphs

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Abstract

Let $G = (V, E)$ be a graph. Let $\gamma(G)$ and $\gamma_t(G)$ be the domination and total domination number of a graph G respectively. The γ-criticality and γ_t-criticality of Harary graphs are studied. The Question 2 of the paper; [W. Goddard et al., The Diameter of total domination vertex critical graphs, Discrete Math. 286 (2004), 255–261] is fully answered with the family of Harary graphs. It is answered to the second part of Question 1 of that paper with some Harary graphs.

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1 Introduction

Let $G = (V, E)$ be a graph. A vertex $v \in V(G)$ *dominates* itself and its neighbors. A subset S of $V(G)$ is called a *dominating set* if it dominates every vertex of G, i.e., every vertex $v \in V(G)$ is either an element of S or is adjacent to an element of S [3]. The *domination number* $\gamma(G)$ of G is the minimum cardinality among the dominating sets of G. A $\gamma(G)$-set

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is a dominating set of cardinality $\gamma(G)$. The open neighborhood of vertex $v \in V$ is denoted by $N(v) = \{u \in V \mid uv \in E\}$. For a set $S \subseteq V$, $N(S) = \bigcup_{s \in S} N(s)$. A set S is called a *total dominating set* if its open neighborhood is $V(G)$, i.e, $N(S) = V(G)$. The *total domination number* $\gamma_t(G)$ is the minimum cardinality among the total dominating sets of G. A $\gamma_t(G)$-set is a total dominating set of cardinality $\gamma_t(G)$ (see [1-3]).

For a set $S \subseteq V$, denote the subgraph of G induced by S by $G[S]$. The minimum and maximum degrees of the graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. An *end-vertex* is a vertex of degree one and a *support vertex* is one that is adjacent to an end-vertex. Let $S(G)$ be the set of support vertices of G. We say that a vertex $v \in V(G)$ is *critical* if $\gamma(G-v) < \gamma(G)$. A graph G is called domination vertex critical if $\gamma(G-v) < \gamma(G)$, for every vertex v in $V(G)$. We say that a vertex $v \in V(G)$ is total critical if $\gamma_t(G-v) < \gamma_t(G)$. Since total domination is undefined for a graph with isolated vertices, we say that a graph G is *total domination vertex critical*, or just γ_t-critical, if every vertex of $V - S(G)$ is total critical. If G is γ_t-critical, and $\gamma_t(G) = k$, then we say that G is *k-γ_t-critical* (see [1,5]).

Given $k \leq n$, place n vertices around a circle, equally spaced. If k is even, form $H_{k,n}$ by making each vertex adjacent to the nearest $k/2$ vertices in each direction around the circle. If k is odd and n is even, form $H_{k,n}$ by making each vertex adjacent to the nearest $(k-1)/2$ vertices in each direction and to the diametrically opposite vertex. In each case, $H_{k,n}$ is k-regular. When k and n are both odd, index the vertices by the integers modulo n. Construct $H_{k,n}$ from $H_{k-1,n}$ by adding the edges between vertices i and $i+(n-1)/2$ for each $1 \leq i \leq (n+1)/2$. In this case the vertex with index $n+1/2$ has degree $k+1$ and the others have degree k. The graph $H_{k,n}$ in each case is known as Harary graph (see [6]). Note that for any Harary graph H, $S(H) = \emptyset$.

In [4], A. Khodkar et al. investigated the domination numbers, the forcing domination numbers, the independent domination numbers and the total domination numbers of Harary graphs. In this paper, we shall study the γ-criticality and γ_t-criticality of Harary graphs and it is answered to the Question 2 of [1] with the family of Harary graphs and is partially answered to the Question 1 of [1] with some Harary graphs.

The following results (see [4]) are useful for providing the main results.

Theorem A. $\gamma(H_{2m,n}) = \lceil \frac{n}{2m+1} \rceil$.

Theorem B. Let $n-(m+1)=(2m+2)t+r$, where $0\le r\le 2m+1$. Then

$$\gamma(H_{2m+1,2n})=\begin{cases}\lceil\frac{n}{m+1}\rceil+1 & \text{if } 2\le r\le m+1 \text{ and } t+r\ge m+1\\ \lceil\frac{n}{m+1}\rceil & \text{otherwise.}\end{cases}$$

Theorem C. Let $n-(m+1)=(2m+2)t+r$, where $0\le r\le 2m+1$. Then

$$\gamma(H_{2m+1,2n+1})=\begin{cases}\lceil\frac{n}{m+1}\rceil+1 & \text{if } 2\le r\le m+1 \text{ and } t+r\ge m\\ \lceil\frac{n}{m+1}\rceil & \text{otherwise.}\end{cases}$$

Theorem D. Let $n=(3m+1)l+r$, where $0\le r\le 3m$. Then

$$\gamma_t(H_{2m,n})=\begin{cases}2l & \text{if } r=0\\ 2l+1 & \text{if } 1\le r\le m\\ 2l+2 & \text{if } m+1\le r\le 3m.\end{cases}$$

Theorem E. Let $2n=(4m+2)l+r$, where $0\le r\le 4m$ and $2n\ge 2m+2$. Then

$$\gamma_t(H_{2m+1,2n})=\begin{cases}2l & \text{if } r=0\\ 2l+2 & \text{otherwise.}\end{cases}$$

Theorem F. Let $2n+1=(4m+2)l+(2r+1)$, where $0\le r\le 2m$. Then

$$\gamma_t(H_{2m+1,2n+1})=\begin{cases}2l+1 & \text{if } r=0\\ 2l+2 & \text{if } 1\le r\le 2m.\end{cases}$$

Criticality and total criticality of all types of Harary graphs are fully verified and we study the diameter of those graphs and the following questions of [1].

1. Does there exist a 4-γ_t-critical graph with diameter 2?

2. Which graphs are domination vertex critical and total domination vertex critical or one but not the other?

2 Elementary results

In this section some elementary results are studied for providing the main results.

Lemma 1 *In a Harary graph $H_{2m,n}$ we have.*
1. Any k adjacent vertices dominate at most $(k+1)m+1$ vertices.
2. Any $2k$ vertices totally dominate at most $k(3m+1)$ vertices and any $2k+1$ vertices totally dominate at most $k(3m+1)+m$ vertices.

Proof.
1. In $H_{2m,n}$, the vertex $i+jm$ dominates itself and $2m$ vertices $\{i+(j-1)m, i+(j-1)m+1, \cdots, i+(j-1)m+m-1, i+jm+1, \cdots, i+(j+1)m$. Every two adjacent vertices have at least m common vertices in their dominated set and two adjacent vertices such as $i+jm$ and $i+(j+1)m \pmod n$ exactly dominate $3m+1$ vertices $\{i+(j-1)m, i+(j-1)m+1, \cdots, i+(j-1)m+m-1, i+jm, i+jm+1, \cdots, i+(j+1)m, i+(j+1)m+1, \cdots, i+(j+2)m$. Thus every k adjacent vertices dominate the most vertices of G if these vertices are such as $S=\{i+m, i+2m, \cdots, i+km\} \pmod n$. The set S dominates at most $(2m+1)+(k-1)m=(k+1)m+1$ vertices.

2. Any $2k$ vertices ($2k+1$ vertices) in Harary graph $H_{2m,n}$ totally dominate the most vertices if these $2k$ vertices ($2k+1$ vertices) are separated to k subsets of 2-set such as $\{i+jm, i+(j+1)m\}$ ($k-1$ subsets of 2-set such as $\{i+jm, i+(j+1)m\}$ and 1 subset of 3-set such as $\{i+jm, i+(j+1)m, i+(j+2)m\}$) and the totally dominated vertices of any of these 2-sets (and the totally dominated vertices of any of these 2-sets and 3-set) do not have common vertices. The part 1 implies that $2k$ vertices ($2k+1$ vertices) totally dominate at most $k(3m+1)$ vertices ($k(3m+1)+m$ vertices). □

Lemma 2 *In the Harary graphs of types $H_{2m+1,2n}$ and $H_{2m+1,2n+1}$ we have.*
1. Any k vertices dominate at most $k(2m+2)$ and $k(2m+2)+1$ vertices respectively.
2. Any $2k+1$ vertices totally dominate at most $2k(2m+1)+m$ vertices and any $2k$ vertices totally dominate at most $2k(2m+1)$ vertices

Proof. 1. Each vertex in $H_{2m+1,2n}$ dominates $2m+2$ vertices. So k vertices dominates at most $k(2m+2)$ vertices of $H_{2m+1,2n}$. The graph $H_{2m+1,2n+1}$, has only one vertex of degree $2m+2$ and the others are of degree $2m+1$. So in a graph $H_{2m+1,2n+1}$ any k vertices dominate at most $k(2m+2)+1$.

2. $2k$ vertices in Harary graph $H_{2m+1,2n}$ and $H_{2m+1,2n+1}$ totally dominate the most vertices if these $2k$ vertices are separated to k subsets of 2-set such that each 2-set is such as $\{v_i, v_{i+n}\}$ (mod n) (is diametrically opposite vertices in $H_{2m+1,2n}$) and the totally dominated vertices of any of these 2-sets do not have common vertices. Now it is obvious that $2k$ vertices totally dominates $k(4m+2) = 2k(2m+1)$, because each of these 2-sets totally dominates $4m+2$.
$2k+1$ vertices in Harary graph $H_{2m+1,2n}$ and $H_{2m+1,2n+1}$ totally dominate the most vertices if $2k+1$ vertices are separated to $k-1$ subsets of 2-set and 1 subset of 3-set such that each 2-set is such as $\{v_i, v_{i+n}\}$ (mod n) and 3-set is such as $\{v_i, v_{i+n}\}$ (mod n) and one of the vertices of the set $\{v_{i-m}, v_{i+m}, v_{i+n-m}, v_{i+n+m}\}$ and the totally dominated vertices of any of these 2-sets and 3-set do not have common vertices. Now it is obvious that $2k+1$ vertices totally dominates $(k-1)(4m+2)+(4m+2)+m = 2k(2m+1)+m$. □

The following lemmas are useful.

Lemma 3 *Let $G = H_{2m+1,2n}$ where $2n = (2t+1)(2m+2)+2r$, $3 \le r \le m+1$ and $t+r \ge m+1$.*
1. There are at most $t+k = \lfloor \frac{n+m}{2m+1} \rfloor$ vertices so that dominate $(t+k)(2m+2)$ vertices and $2n-(t+k)(2m+2)$, other vertices are dominated by at least $t+3-k$ vertices so that each of $t+1-k$ vertices dominate exactly $2m$ vertices and each of two others dominate at least 2 vertices.
2. There are at most $2t+1$ vertices that dominate $(2t+1)(2m+2)$ vertices and $2r$ other vertices of G are dominated by at least 2 vertices.

Proof. 1. Let k be a positive integer and $t+k = \lfloor \frac{n+m}{2m+1} \rfloor$. Let $S_1 = \{m+1, 3m+2, 5m+3, \cdots, (2(t+k)-1)m+(t+k)\}$. Since $m+1+n > n+m \ge (t+k)(2m+1) = (2(t+k)-1)m+(t+k)+m$ and $(2(t+k)-1)m+(t+k)+n = (t+k)(2m+1)-m+n \le 2n$, hence S_1 exactly dominate $(t+k)(2m+2)$ vertices, named $D_1 = \{1, 2, \cdots, (t+k)(2m+1), m+1+n, 3m+2+n, \cdots, (t+k)(2m+1)-m+n\}$. Let $\{x = (2l-1)m+l+n, y = (2l+1)m+(l+1)+n\} \subset n+S_1 = \{n+u \mid u \in S_1\}$, then $y-x = 2m+1$ and the set $\{x+1, x+2, \cdots, x+2m\}$ has not dominated yet for $x \in n+S_1$. Let $v \notin D_1$ be any vertex. Then $(t+k)(2m+1)+1 \le v \le 2n$

and $v \notin n + S_1$. So v dominates $2m$ vertices, because $v + n \in D_1$. There are $t+k-1$ vertices such as v and so a set of $2(t+k-1)m$ vertices named D_2 is dominated by them.
If $k = 1$, then $|D_1 \cup D_2| = (t+1)(2m+2) + 2tm = 2n - 2(2t+1) - 2r + 2(t+1)$ and $2(t+r) \geq 2m+2$. Since these $2(t+1) \geq 2m+2$ vertices are dominated by at least two vertices so that each of both dominates at least 2 vertices, thus $2n - (t+k)(2m+2) = 2n - |D_1|$ vertices are dominated by $t+2 = t+3-1$ vertices.
If $k \geq 2$, then $2n - |D_1| = 2n - (t+k)(2m+2) \geq 2n - (n+m) - (t+k) = n - m - (t+k) \geq (t+k-1)2m$. This relation shows that we need at least $t+k-1 \geq t+3-k$ vertices to dominate $2n - |D_1|$ vertices so that each vertex must dominate at least $2m$ vertices.

2. Let $S_2 = \{(2i+1)(m+1) | \ 0 \leq i \leq t)\} \cup \{n + j(2m+2) | \ 1 \leq j \leq t\}$ (mod $2n$). The set S exactly dominates $(2t+1)(2m+2)$ vertices of the $V(H_{2m+1,2n}) \backslash S_1' \cup S_2'$ where $S_1' = \{(2t+1)(m+1)+m+1, \cdots, (2t+1)(m+1)+m+r\}$ and $S_2' = \{n+t(2m+2)+m+2, \cdots, n+t(2m+2)+m+r+1\}$. Each vertex of the set S_1' or S_2' cannot dominate $S_1' \cup S_2'$, because $r \geq 3$. Thus we need at least 2 vertices for dominating of $S_1' \cup S_2'$ so that each vertex must dominate at least $r-1$ vertices. □

Lemma 4 *Let $G = H_{2m+1,2n+1}$ where $2n = (2t+1)(2m+2) + 2r$, $2 \leq r \leq m+1$ and $t + r \geq m+1$.*
1. There are at most $t+k = \lfloor \frac{n+m}{2m+1} \rfloor$ vertices so that dominate $(t+k)(2m+2)$ or $(t+k)(2m+2)+1$ vertices and $2n-(t+k)(2m+2)+1$ or $2n-(t+k)(2m+2)$ other vertices are dominated by at least $t+3-k$ vertices.
2. There are at most $2t+1$ vertices that dominate $(2t+1)(2m+2)$ and $2r+1$ other vertices are dominated by at least 2 vertices.
3. There are at most $2t$ vertices that dominate $2t(2m+2)$, 1 vertex (the vertex with maximum degree) dominates $2m+3$ vertices, and $2r$ other vertices are dominated by at least 2 vertices.

Proof. Let the vertex $n+1$ have maximum degree $2m+2$.
1. The proof is the same as the proof of part 1 of Lemma 3 depending on $n+1 \in S_1$ or $n+1 \notin S_1$.
2. Let $S_2 = \{(2i+1)(m+1) | \ 0 \leq i \leq t)\} \cup \{(2m+2)j + n | \ 1 \leq j \leq t\}$. S_2 does not dominate $2r+1$ vertices of the set $\{v | \ (t+1)(2m+2) \leq v \leq (t+1)(2m+2) + r - 1$ and $(2t+1)(2m+2) + r + 1 \leq v \leq 2n+1\}$. It easy to see that, this set is dominated by 2 vertices.

3. There are several ways for choosing a set of $2t+1$ vertices containing the vertex $n+1$. We can give two ways. If S_3 is a set of $2t+1$ vertices so that $n+1 \in S_3$ such as $\{(2i+1)(m+1)+r+1 | \, 0 \leq i \leq t\} \cup \{(2m+2)j+r+1+n | \, 1 \leq j \leq t\}$ then S_2 dominates $(2t+1)(2m+2)+1$ vertices, and $2r$ vertices $\{2, \cdots, r+1, n+m+2, \cdots n+m+1+r\}$ are not dominated by S_3. These $2r$ vertices are dominated by 2 vertices and as well, if one of the vertices $r+1$ or $n+m+2$ is deleted, the resulted set is also dominated by 2 vertices. Let $v_i = (m+2)+(2m+2)i$, $w_j = (n+1)+(2m+2)j$ and $S_4 = \{v_i \mid 0 \leq i \leq t-1\} \cup \{w_j \mid 1 \leq j \leq t\} \cup \{n+1\}$. Then S_4 dominates $2t(2m+2)+(2m+3)$. The $2r$ vertices of the set $\{v_{t-1}+m+2, v_{t-1}+m+3, \ldots, n-m\} \cup \{w_t+m+1, w_t+m+2, \ldots, 2n\}$ has not been dominated yet. This set is dominated by 2 vertices and as well if one of the vertices $n-m$ or w_t+m+1 is deleted, the resulted set is also dominated by 2 vertices. □

In order to clear the Lemmas some examples are given.

Example 1. $H_{5,48}$ is not γ-critical. Theorem B implies that the set $\gamma(H_{5,48}) = \lceil \frac{24}{3} \rceil + 1 = 9$ and $\lfloor \frac{n+m}{2m+1} \rfloor = \lfloor \frac{24+2}{5} \rfloor = 5$. There are 2 cases. If $S_1 = \{3, 8, 13, 18, 23\}$, then the set $D_1 = \{1, 2, \cdots, 25, 27, 32, 37, 42, 47\}$ is dominate by S. Each vertex of $V(H_{5,48}) \setminus D_1$ dominates at most 4 vertices. So 5 vertices dominate $V(H_{5,48}) \setminus D_1$. Further, removing any vertex does not decrease the domination number.
If $S_2 = \{3, 9, 15, 21, 28, 36, 42\}$, then S_2 dominates 42 vertices. The non-dominated vertices $S_3 = \{24, 25, 26, 46, 47, 48\}$ are dominated by at least 2 vertices and $\gamma(H_{5,48} - v) = \gamma(H_{5,48})$ for every vertex v in $H_{5,48}$.

Example 2. $H_{5,35}$ is not γ-critical. By Theorem C, $\gamma(H_{5,35}) = 7$. Lemma 4 implies that, there are three cases.
1. There are $3 = \lfloor \frac{19}{5} \rfloor$ vertices such as $S_1 = \{3, 8, 13\}$ so that dominate 18 vertices. The other vertices $\{16, \cdots, 35\} \setminus \{20, 25, 30\}$ are dominated by at least 4 vertices and each of them cannot dominates more than 4 vertices except for the vertex 18. Now, removing any vertex does not decrease the domination number.
2. There are 5 vertices such as $\{3, 9, 15, 23, 29\}$ so that dominate 30 vertices. This yields 5 other vertices $\{18, 19, 33, 34, 35\}$ are dominated by 2 vertices. It is easy to see that $\gamma(H_{5,35} - v) = \gamma(H_{5,35})$.
3. There are $2t+1$ vertices such as $\{4, 10, 18, 24, 30\}$ that dominate $24+7 = 31$ vertices and the vertex set $\{14, 15, 33, 34\}$ is dominate by 2 vertices and as well $\{14, 33, 34\}$ or $\{14, 15, 34\}$ is dominated by 2 vertices. Thus $H_{5,35}$ is not γ-critical.

3 Main results

We verify the vertex criticality of dominating and total dominating sets of Harary graphs. A sample of any types of the Harary graphs is shown in Figure 1.

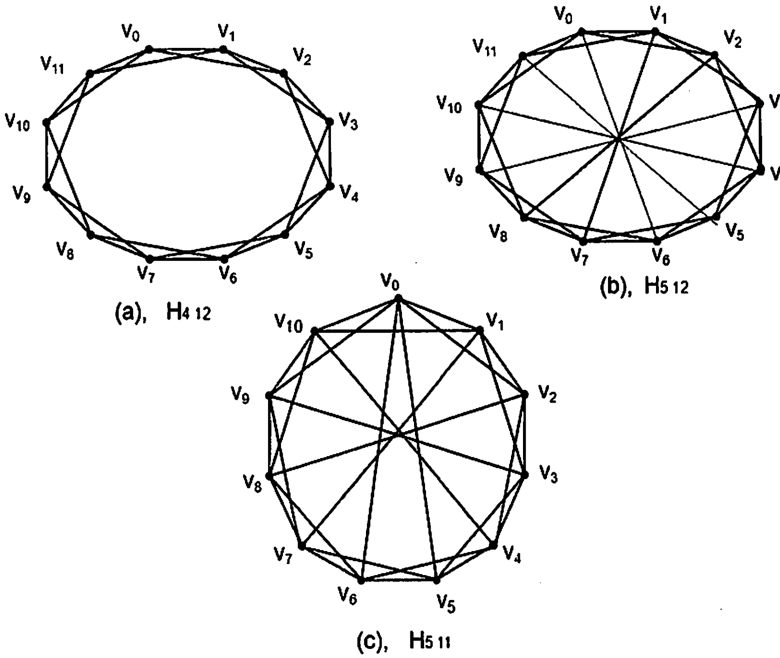

Figure 1.

3.1 $H_{2m,n}$

The criticality and total criticality of the Harary graphs of type $H_{2m,n}$ are determined.

Theorem 5 *The graph $H_{2m,n}$ where $n = (2m+1)t + r$ and $0 \leq r \leq 2m$ is γ-critical for $r = 1$ and is not γ-critical for $2 \leq r \leq 2m$ and $r = 0$.*

Proof. By Theorem A, $\gamma(H_{2m,n}) = \lceil \frac{n}{2m+1} \rceil$.
Let $r = 1$. Then $\gamma(H_{2m,n}) = t+1$. If the vertex v is deleted of $H_{2m,n}$, then the set $S = \{v+m+1+i(2m+1) \mid 0 \leq i \leq t-1\}$ (mod n) is a dominating set for $H_{2m,n} - v$ for the given n.
Let $r = 0$. Then $\gamma(H_{2m,n}) = \lceil \frac{n}{2m+1} \rceil = t$. Each vertex dominate $2m+1$ vertices and since $n = (2m+1)t$, it is clear that $\gamma(H_{2m,n} - v) = t$.
Let $2 \leq r \leq 2m$. Then $\gamma(H_{2m,n}) = t+1$ and $n = (2m+1)t+r$ where $r \geq 2$. Since t vertices dominate at most $(2m+1)t$ vertices, and $H_{2m,n} - v$ has at least $n-1 = (2m+1)t+1$ vertices. So, the dominating set of $H_{2m,n} - v$ has at least $t+1$ vertices. Hence, $H_{2m,n}$ is not γ-critical for the given n. □

Theorem 6 *The graph $H_{2m,n}$ where $n = (3m+1)l+r$ and $0 \leq r \leq 3m$ is γ_t-critical for $r \in \{1, m+1\}$ and is not γ_t-critical for another r.*

Proof. Let $r = 1$. Theorem D says $\gamma_t(H_{2m,n}) = 2l+1$. If the vertex v of $H_{2m,n}$ for the given n is deleted, then the set $\{v+m+1+i(3m+1), v+2m+1+i(3m+1) \mid 0 \leq i \leq l-1\}$ (mod n) is a totally dominating set for $H_{2m,n} - v$ and this shows $\gamma_t(H_{2m,n}) = 2l$.
Let $r = m+1$, Theorem D says $\gamma_t(H_{2m,n}) = 2l+2$. If the vertex v is deleted of $H_{2m,n}$, then the set $\{v+m+1+i(3m+1), v+2m+1+i(3m+1) \mid 0 \leq i \leq l-1\} \cup \{v+(3m+1)l\}$ (mod n) is a total dominating set of $H_{2m,n} - v$ and this shows $\gamma_t(H_{2m,n} - v) = 2l+1$. Therefore, $H_{2m,n}$ for $n = (3m+1)l+1$ and $n = (3m+1)l+m+1$ is γ_t-critical.
Let $r = 0$, $\gamma_t(H_{2m,n}) = 2l$ and $n = (3m+1)l$. Since $n-1 = (3m+1)(l-1)+3m$ and by Lemma 1, two (three) adjacent vertices dominate at most $3m+1$ ($4m+1$) vertices, and so to totally dominate of $n-1 = (3m+1)(l-1)+3m$ vertices, it needs at least $2l$ vertices. Thus for $r = 0$, $H_{2m,n}$ is not γ_t-critical.
Let $2 \leq r \leq m$, $\gamma_t(H_{2m,n}) = 2l+1$ and $n = (3m+1)l+r$ and $2l$ vertices totally dominate at most $l(3m+1)$ vertices. Since $n-1 \geq (3m+1)l+1$, so at least $2l+1$ vertices dominate $H_{2m,n} - v$ for any vertex $v \in H_{2m,n}$. Thus, $H_{2m,n}$ for the given n is not γ_t-critical. Let $m+2 \leq r \leq 3m$, $\gamma_t(H_{2m,n}) = 2l+2$ and $n = (3m+1)l+r$. Since $n-1 \geq (3m+1)l+m+1$ and three adjacent vertices totally dominate at most $4m+1$ vertices and $n-1 \geq (3m+1)(l-1)+4m+2$, and so to totally dominate of $n-1$ vertices we need at least $2l+2 = 2(l-1)+4$ vertices. Thus, $H_{2m,n}$ for this n is not total domination vertex critical too. □

By using Theorems 5 and 6, we have:

Corollary 7 *Let k be any non negative integer. The graphs $H_{2m,n}$ for*
(i) $n = (2m+1)(3m+1)k+1 = (3m+1)(2m+1)k+1$ is γ-critical and

γ_t*-critical.*
(ii) $n = (3m+1)((2m+1)k+1)+m+1 = (2m+1)((3m+1)k+2)$ *is* γ_t*-critical but is not* γ*-critical.*
(iii) $n = ((3m-2)+(3m+1)k)(2m+1)+1 = ((2m-1)+(2m+1)k)(3m+1)$ *is* γ*-critical but not* γ_t*-critical.*
(iv) $n = (3m+1)(2m+1)k$ *is not* γ*-critical and* γ_t*-critical.*

3.2 $H_{2m+1,2n}$

The criticality and total criticality of the Harary graphs of type $H_{2m+1,2n}$ are determined.

Theorem 8 *The Harary graph* $H_{2m+1,2n}$ *where* $2n = (2t+1)(2m+2)+2r$ *and* $0 \le r \le 2m+1$ *is not* γ*-critical for* $r = 0$, $m+2 \le r \le 2m+1$, $2 \le r \le m+1$ *and* $1 \le t+r \le m$, *or* $3 \le r \le m+1$ *and* $t+r \ge m+1$ *and it is* γ*-critical for* $r = 2$ *and* $t+r \ge m+1$.

Proof. Let $r = 0$. Then $\gamma(H_{2m+1,2n}) = 2t+1$ and $2n = (2t+1)(2m+2)$, $2n-1 = 2t(2m+2)+2m+1$. Since each vertex dominates at most $2m+2$ vertices, $\gamma(H_{2m+1,2n}-v) = 2t+1$ for any vertex v.
Let $m+2 \le r \le 2m+1$. Then $2n = (2t+1)(2m+2)+2r \ge (2t+1)(2m+2)+2m+4 = (2t+2)(2m+2)+2$. By Theorem B, $\gamma(H_{2m+1,2n}) = 2t+3$. Since $2n-1 \ge (2t+2)(2m+2)+1$ and each vertex dominates at most $(2m+2)$ vertices. So we need at least $2t+3$ vertices to dominate $H_{2m+1,2n}-v$ for any vertex v.
Let $2 \le r \le m+1$ and $1 \le t+r \le m$. Then $\gamma = 2t+2$ and $2n-1 = (2t+1)(2m+2)+2r-1 \ge (2t+1)(2m+2)+3$. As above, at least $2t+2$ vertices can dominate $H_{2m+1,2n}-v$ for any vertex v.
Let $3 \le r \le m+1$ and $t+r \ge m+1$. By Theorem B, $\gamma(H_{2m+1,2n}) = 2t+3$. Now Lemma 3 and its proof show that $H_{2m+1,2n}$ cannot be γ-critical for the given n.
Let $r = 2$ and $r+t \ge m+1$. Then $\gamma(H_{2m+1,2n}) = 2t+3$ for $2n = (2t+1)(2m+2)+4$. If the vertex v is deleted, then the set $S = \{v+m+2+i(2m+2) \mid 0 \le i \le t\} \cup \{v+m+4+j(2m+2) \mid t+1 \le j \le 2t\} \cup \{v+n+1\} \pmod{2n}$ dominates the $H_{2m+1,2n}-v$ for any vertex v. So, $\gamma(H_{2m+1,2n}-v) = 2t+2$. It shows that $H_{2m+1,2n}$ is γ-critical for this n. □

Theorem 9 *The Harary graph $H_{2m+1,2n}$ where $2n = 2l(2m+1) + 2r$ and $0 \leq r \leq 2m$ is γ_t-critical for $r = 1$, and is not γ_t-critical for $r \neq 1$.*

Proof. Let $r = 1$ then $\gamma_t(H_{2m+1,2n}) = 2l + 2$. The set $S = \{v + m + 1 + i(2m+1) | \ 0 \leq i \leq l-1\} \cup \{v + 3m + 3 + j(2m+1) | \ l-1 \leq j \leq 2(l-1)\} \cup \{v + n + 1\}$ (mod $2n$) is the set with $2l+1$ vertices that totally dominates $H_{2m+1,2n} - v$ for any vertex v, so $H_{2m+1,2n}$ is γ_t-critical for the given n and $r = 1$.
Let $r = 0$. By Theorem E, $\gamma_t(H_{2m+1,2n}) = 2l$ for $2n = 2l(2m+1)$. It is obviously $2n-1 = 2(l-1)(2m+1)+2m$ vertices of this $H_{2m+1,2n}$ cannot be dominated by $2l-1$ vertices. So $H_{2m+1,2n}$ for the given n is not γ_t-critical.
Let $2r \geq m+2$. Then $\gamma_t(H_{2m+1,2n}) = 2l+2$ and $2n = 2l(2m+1) + 2r \geq 2l(2m+1) + m + 2$. Since $2n - 1 \geq 2l(2m+1) + m + 1 = 2(l-1)(2m+1) + 5m + 3$. So by Lemma 2, $H_{2m+1,2n} - v$ cannot be totally dominated by $2(l-1) + 3 = 2l + 1$ vertices for any vertex v.
Let $4 \leq 2r \leq m+1$. Then $\gamma_t(H_{2m+1,2n}) = 2l+2$. If we say the set $S_1 = \{(2i+1)m + i + 1, n + (2i+1)m + i + 1 | \ 0 \leq i \leq l-1\}$ $(mod\ 2n)$ and the set $S_2 = \{l(2m+1) + k, l(2m+1) + k + n\}$ for a $k \in \{1, 2, \cdots, r\}$. Then by Lemma 2, S_1 totally dominates $2l(2m+1)$ distinct vertices and S_2 totally dominates $2r$ distinct vertices that have not been dominated by S_1. Also each vertex adjacent to a vertex of S_1 dominates at most r vertices of $2r$ vertices dominated by S_2. Hence if we delete a vertex v of $H_{2m+1,2n}$, then it has $2n-1 = 2l(2m+1)+2r-1 \geq 2l(2m+1)+r+1$ vertices, and $2l+1$ vertices totally dominate at most $(2l-2)(2m+1) + 2(2m+1) + r = 2l(2m+1) + r$ vertices. So $\gamma_t(H_{2m+1,2n} - v) = 2l + 2$ for any vertex v and given n. □

By using Theorems 8 and 9, we have:

Corollary 10 *Let k be any non negative integer. The graph $H_{2m+1,2n}$ for*
(i) $2n = (2t+1)(2m+2)+4 = 2l(2m+1)+2$, where $t = (m-1)+(2m+1)k$ and $l = m + (2m+2)k$ is γ_t-critical and γ-critical.
(ii) $2n = (2t+1)(2m+2) + 2 = 2l(2m+1) + 2$, where $t = m + (2m+1)k$ and $l = (m+1) + (2m+2)k$ is γ_t-critical and is not γ-critical.
(iii) $2n = (2t+1)(2m+2)+4 = 2l(2m+1)+2m+4$, where $t = 2m+(2m+1)k$ and $l = (2m+1) + (2m+2)k$ is γ-critical and is not γ_t critical.
(iv) $2n = (2t+1)(2m+2)+2 = 2l(2m+1)+(2m+4)$, where $t = (2m+1)k$ and $l = (2m+2)k$ is not γ_t-critical and γ-critical.

3.3 $H_{2m+1,2n+1}$

The criticality and total criticality of the Harary graphs of type $H_{2m+1,2n+1}$ are determined.

Theorem 11 *The Harary graph $H_{2m+1,2n+1}$ where $2n+1 = (2t+1)(2m+2)+2r+1$ and $0 \le r \le 2m+1$ is γ-critical for ($2 \le r \le m+1$ and $t+r = m$) and it is not γ-critical for $r \in \{0, 1, m+2, \cdots, 2m+1\}$ or $t+r \neq m$.*

Proof. Let $2 \le r \le m+1$ and $t+r = m$. Then $\gamma(H_{2m+1,2n+1}) = 2t+3$. If we delete the vertex v of $H_{2m+1,2n+1}$, then the set $S = \{v+m+1+i(2m+1), v+m+1+i(2m+1)+n \mid 0 \le i \le t\}$ dominates $H_{2m+1,2n+1} - v$ for any vertex v and the given n. So $\gamma(H_{2m+1,2n+1} - v) = 2t+2$, and $H_{2m+1,2n+1}$ is γ-critical for this n.
Let $r = 0$ and $2n+1 = (2t+1)(2m+2)$. Then $\gamma(H_{2m+1,2n+1}) = 2t+1$ and each vertex except $n+1$ dominates $2m+2$ vertices. If $n+1$ is deleted, it needs at least $2t+1$ vertices to dominate $H_{2m+1,2n+1} - (n+1)$. So $H_{2m+1,2n+1}$ for this n is not γ-critical.
Let $r = 1$, $2n+1 = (2t+1)(2m+2)+3$ and $\gamma(H_{2m+1,2n+1}) = 2t+2$. The graph $H_{2m+1,2n+1} - v$ has $2n = (2t+1)(2m+2)+2$ vertices. It is clear that $\gamma(H_{2m+1,2n+1} - v) = 2t+2$ for the given n.
Let $m+2 \le r \le 2m+1$ and $2n+1 = (2t+1)(2m+2)+2r+1$. Then $\gamma(H_{2m+1,2n+1}) = 2t+3$ and $2n+1 \ge (2t+2)(2m+2)+3$. Since $H_{2m+1,2n+1} - v$ has $(2t+2)(2m+2)+2$ and $(2t+2)$ vertices dominate at most $(2t+2)(2m+2)+1$ vertices so $\gamma(H_{2m+1,2n+1} - v) = 2t+3$.
Let $2 \le r \le m+1$, $1 \le t+r \le m-1$. Then $\gamma(H_{2m+1,2n+1}) = 2t+2$. Since $r \ge 2$, $2r \ge 4$, $2n+1 \ge (2t+1)(2m+2)+4+1$ and $2n \ge (2t+1)(2m+2)+4$. It is easy to see that $\gamma(H_{2m+1,2n+1} - v) = 2t+2$ for any vertex v and given n.
Let $2 \le r \le m+1$, $t+r \ge m+1$. Now Lemma 4 and its proof show that $H_{2m+1,2n+1}$ is not γ-critical. □

Theorem 12 *The Harary graph $H_{2m+1,2n+1}$ where $2n+1 = 2l(2m+1)+2r+1$ and $0 \le 2r \le 4m$ is not γ_t-critical.*

Proof. Let $r = 0$ then $2n+1 = 2l(2m+1)+1$ and $\gamma_t(H_{2m+1,2n+1}) = 2l+1$. We show that $\gamma_t(H_{2m+1,2n+1} - (n+1)) = 2l+1$, where $n+1$ is the vertex with maximum degree. To the contrary, let $\gamma_t(H_{2m+1,2n+1} - (n+1)) = 2l$. So we

have to choose these $2l$ vertices so that any 2 vertices are such as $\{i, i+n\}$ for some $1 \le i \le n-m$, $|D_1| = l$ vertices must dominate $\{1, 2, \cdots, n-1, n\} \cup \{i+n | \, i \in D_1\}$ and $|D_2| = l$ other vertices must dominate $\{n+2, n+3, \cdots, 2n, 2n+1\} \cup \{i | \, i \in D_1\}$. If $i \in \{1, 2, \cdots, m, m+1\}$, then $i+n$ dominates $n+1$ and so $2l$ vertices cannot dominate $H_{2m+1,2n+1} - (n+1)$. If $i \ge m+2$, then it cannot find l vertices in $\{m+2, \cdots, n-m\}$ so that dominate $\{1, 2, \cdots, n-1, n\}$. Anyway, $\gamma_t(H_{2m+1,2n+1} - (n+1)) = 2l+1$ for $r = 0$.
Let $r = 1$ then $2n+1 = 2l(2m+1)+3$ and $\gamma_t(H_{2m+1,2n}) = 2l+2$. By Lemma 2, $2l$ vertices of the set $S = \{v+(2m+1)i \mid 0 \le i \le l-1\} \cup \{v+(2m+1)i+n \mid 0 \le i \le l-1\}$ totally dominate $2l(2m+1)$ vertices and three vertices $\{v+(2m+1)(l-1)+m+1, v+(2m+1)(l-1)+m+1+n, v+(2m+1)(l-1)+m+2+n\}$ is totally dominated by two adjacent vertices $v+(2m+1)(l-1)+m+1$ and $v+(2m+1)(l-1)+m+1+n$. Now if the vertex $2n+1$ is deleted and $2n+1$ is not adjacent to any vertex of S, then two other non-dominated vertices are $\{1, n+1\}$ or $\{n, 2n\}$. Each of two sets cannot be dominated by a vertex not belong to S but adjacent to a vertex of S. So, it must be dominated by itself. Hence, $\gamma_t(H_{2m+1,2n+1} - (2n+1)) = 2l+2$. If the vertex $2n+1$ is adjacent to a vertex of S it is easily seen that $H_{2m+1,2n+1}$ is not γ_t-critical.
Let $2r \ge m+1$. Then $\gamma_t(H_{2m+1,2n+1}) = 2l+2$ and $2n+1 = 2l(2m+1)+2r+1 \ge 2l(2m+1)+m+2$. Since $2n \ge 2l(2m+1)+m+1 = 2(l-1)(2m+1)+5m+3$. So $2l+1$ vertices cannot dominate $2n$ vertices of $H_{2m+1,2n+1}$, and hence $\gamma_t(H_{2m+1,2n+1} - v) = 2l+2$.
Let $4 \le 2r \le m$. It is sufficient to use the method of the proof of Theorem 7 for the case $4 \le 2r \le m+1$, then it will be seen that $H_{2m+1,2n+1}$ is not γ_t-critical for the given n. □

By using Theorems 11 and 12 we have:

Corollary 13 *Let k be any non negative integer. The graph $H_{2m+1,2n+1}$ for*
(i) $2n = 2l(2m+1) = (2t+1)(2m+2) + 2r$ where $t + r = m$ is γ-critical but is not γ_t-critical.

(ii) $2n = 2l(2m+1) + 2 = (2t+1)(2m+2) + 2$ where $l = (m+1)(2k+1)$ and $t = m + k(2m+1)$ is not γ_t-critical and γ-critical.

4 Answer the questions

Now we fully answer the question 2 of [1] with the family of Harary graphs
.

Question 2 of [1]. Which graphs are domination vertex critical and total domination vertex critical or one but not the other?

Solution. The Corollaries 7 and 10 (i) show that, there are many graphs $H_{2m,n}$ and $H_{2m+1,2n}$ so that they are γ_t-critical and γ-critical.
The Corollaries 7 and 10 (ii) show that, there are many graphs $H_{2m,n}$ and $H_{2m+1,2n}$ so that they are γ_t-critical but are not γ-critical.
The Corollaries 7, 10 (iii) and 13 (i) show that, there are many graphs $H_{2m,n}$, $H_{2m+1,2n}$ and $H_{2m+1,2n+1}$ so that they are not γ_t-critical but they are γ-critical.
The Corollaries 7, 10 (iv) and 13 (ii) show that, there are many graphs $H_{2m,n}$, $H_{2m+1,2n}$ and $H_{2m+1,2n+1}$ so that they are not both γ_t-critical and γ-critical.

Now we study the part 2 of Question 1 of [1].

Part 2 of Question 1 of [1]. Does there exist a 4-γ_t-critical graph with diameter 2?

The graph $H_{2m,n}$ has the diameter $k+1$ if $2km+2 \leq n \leq 2(k+1)m+1$. The path $1, m+1, 2m+1, \cdots, km+1, km+1+\lceil \frac{r}{2} \rceil$ where $2 \leq r \leq 2m$ is a diameter of $H_{2m,n}$ for the given n. For n belongs to this interval, for some k and m, $H_{2m,n}$ may be a domination vertex critical or total domination vertex critical or one but not the other. In general the diameter does not affect on (total) domination vertex criticality. For example, see below.
Let $m = 4$.
1. For $74 \leq n \leq 81$ the graph $H_{2m,n}$ is not γ-critical but it is γ_t-critical for $n = 79$.
2. For $58 \leq n \leq 65$, $H_{2m,n}$ is not γ_t-critical but it is γ-critical for $n = 64$.
3. For $42 \leq n \leq 49$, $H_{2m,n}$ is γ_t-critical for $n = 44$ and it is γ-critical for $n = 46$.
4. For $(2 \times 72 \times 4) + 2 = 578 \leq n \leq 585 = (2 \times 73 \times 4) + 1$ the graph $H_{2m,n}$ is not γ_t-critical and it is not γ-critical.
The Harary graph $H_{2m+1,2n}$ for $2n = 2l(2m+1) + 2r$ has a diameter $l+1 \leq k \leq l+2$ for $1 \leq r \leq 2m$ and the path $1, m+1, 2m+1, \cdots, lm+1, (l+1)m+1$ or $1, m+1, \cdots, lm+1, (l-1)m+1, (l-1)m+1+s$ where $1 \leq s \leq m$ is a

diameter in $H_{2m+1,2n}$.

Solution of part 2 of Question 1 of [1].

In Harary graph $H_{2m+1,2n}$ for $2n = 2(2m+1)+2$, the diameter is 2 and by Theorem 9, it is 4-γ_t-critical. This is an answer to the part 2 of Question 1 of [1].

In Harary graph $H_{2m,n}$, there is no answer for Question 1 of [1].

In Harary graph $H_{2m+1,2n+1}$ for $2n = 2(2m+1)+2$, the diameter is 2, $\gamma_t = 4$ and by Theorem 12 it is not γ_t-critical.

In general, the Question 2 is open for other than Harary graphs and the part 1 of Question 1 is open right now.

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