A note on the vertex-distinguishing proper total coloring of graphs*

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Abstract

Let G be a simple graph of order $p \geq 2$.A proper k-total coloring of a simple graph G is called a *k-vertex distinguishing proper total coloring* (k-VDTC) if for any two distinct vertices u and v of G, the set of colors assigned to u and its incident edges differs from the set of colors assigned to v and its incident edges. The notation $\chi_{vt}(G)$ indicates the smallest number of colors required for which G admits a k-VDTC with $k \geq \chi_{vt}(G)$. For every integer $m \geq 3$, we will present a graph G of maximum degree m such that $\chi_{vt}(G) < \chi_{vt}(H)$ for some proper subgraph $H \subset G$.

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1. Introduction

Let $G = (V, E)$ be a simple graph. A proper total k-coloring of G is a mapping $f : (V \cup E) \rightarrow [k]$ such that no two adjacent vertices receive the same color, no two incident edges receive the same color, and no vertex and incident edge receive the same color. Given such a coloring f, for any vertex $v \in V$,let $C(v) = \{f(v)\} \cup \{f(uv) : uv \in E(G)\}$. For every pair of adjacent vertices $uv \in E$, if $C(u) \neq C(v)$ then we say that f is an adjacent vertex distinguishing total coloring (AVDTC). We call the smallest k for which such a coloring of G exists the adjacent vertex distinguishing total chromatic number, denoted by $\chi_{at}(G)$ [1][2][3].

Conjecture 1[1]. Let G be a connected graph of order $n(\geq 2)$, then $\chi_{at}(G) \leq \Delta(G) + 3$.

Given a k-proper total coloring f of G,for any distinct vertices $u, v \in V$, if $C(u) \neq C(v)$ then we say that f is a vertex distinguishing total coloring (VDTC). We call the smallest k for which such a coloring of G exists the vertex distinguishing total chromatic number, denoted by $\chi_{vt}(G)$. The concept of VDTC was introduced independently by Zhang et.al [4].The other terminologies and marks refer to [5].

All the graph mentioned in this paper are simple and finite. Let $n_d(G)$ denote the number of vertices of degree d. It is clear that $\binom{\chi_{vt}(G)}{d+1} \geq n_d$ for all d with respect to $\delta(G) \leq d \leq \Delta(G)$, where $\delta(G), \Delta(G)$ denote the minimum and maximum degrees of G, respectively.In this paper, we will show

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Theorem *For every integer $m \geq 3$, there exists a simple graph G of maximum degree d such that $\chi_{vt}(H) > \chi_{vt}(G)$ for some proper subgraph H of G.*

2. The proof of theorem

The notation $[\alpha, \beta]$ stands for an integer set $\{\alpha, \alpha+1, \ldots, \beta\}$, where integers $\beta > \alpha \geq 0$. We need the following lemmas.

Lemma 1[1]. *For a path P_n of order $n(\geq 4)$, if $\binom{m-1}{3} < n-2 \leq \binom{m}{3}$, then $\chi_{vt}(P_n) = m$.*

Lemma 2[1]. *Let F_m be a fan of order $m+1$, then $\chi_{vt}(F_m) = 5$ when $m = 3$ and $\chi_{vt}(F_m) = m+1$ when $m \geq 4$.*

The proof of Theorem. The symbol F_m will be used to denote a fan on $m+1$ vertices, where $V(F_m) = \{u_i : i \in [0, m]\}$ and $E(F_m) = \{u_0u_i : i \in [1, m]\} \cup \{u_iu_{i+1} : i \in [1, m-1]\}$.

Case 1. $\Delta(G) = 3$.

We have a connected graph G obtained by adding a one-degree vertex v_4 and an edge v_1v_4 to the 3-cycle $C_3 = v_1v_2v_3v_1$. So $E(G) = \{v_1v_4\} \cup E(C_3)$. It is easy to show $\chi_{vt}(G) = 4$ by the following total coloring ϕ: $\phi(v_1) = 1, \phi(v_2) = 4, \phi(v_3) = 2, \phi(v_4) = 2, \phi(v_1v_2) = 2$, $\phi(v_2v_3) = 1$, $\phi(v_3v_1) = 3$ and $\phi(v_1v_4) = 4$. Thus, $C_3 = G - v_1v_4$ and $\chi_{vt}(C_3) = 5$ in [1], so $\chi_{vt}(C_3) > \chi_{vt}(G)$.

Case 2. $\Delta(G) = 4$.

We get a connected graph G obtained by adding a 4-degree vertex w and 4 edges wv_1, wv_2, wv_3, wv_4 to the 12-cycle $C_{12} = v_1v_2 \cdots v_{12}v_1$. So $E(G) = \{wv_1, wv_2, wv_3, wv_4\} \cup E(C_{12}), V(G) = \{w\} \cup V(G)$. It is easy to show $\chi_{vt}(G) = 5$ by the following total coloring ϕ:

$\phi(w) = \phi(v_6) = \phi(v_{12}) = \phi(v_1v_2) = \phi(v_4v_5) = \phi(v_7v_8) = 1$;

$\phi(v_2) = \phi(v_4) = \phi(v_9) = \phi(wv_3) = \phi(v_5v_6) = \phi(v_{10}v_{11}) = \phi(v_{12}v_1) = 2$;

$\phi(v_5) = \phi(v_7) = \phi(v_{11}) = \phi(wv_4) = \phi(v_2v_3) = \phi(v_8v_9) = 3$;

$\phi(wv_1) = \phi(v_3v_4) = \phi(v_6v_7) = \phi(v_9v_{10}) = 4$;

$\phi(v_1) = \phi(v_3) = \phi(v_8) = \phi(v_{10}) = \phi(wv_2) = \phi(v_{11}v_{12}) = 5$.

Thus, $C_{12} = G - \{w, wv_1, wv_2, wv_3, wv_4\}$ and $\chi_{vt}(C_{12}) = 6$ in [1], so $\chi_{vt}(C_{12}) > \chi_{vt}(G)$.

Case 3. $\Delta(G) \geq 5$.

Case 3.1. G is disconnected.

To show the result, we take the union $G = P_n \cup F_m$ of P_n and F_m, where $V(P_n) \cap V(F_m) = \emptyset$ and $n = \binom{m+1}{3} + 2$. Clearly, $\chi_{vt}(P_n) = m+1$ by Lemma 1 and $\chi_{vt}(F_m) = m+1$ by Lemma 2, so $\chi_{vt}(G) = m+1$.

On the other hand, we have $\chi'_{vt}(G - S) \geq m+2$ since $G - S$ is the union of disjoint path and cycle P_n, C_m, where $S = \{u_0\} \cup \{u_0u_i : i \in [1, m]\}$ and $n+m = \binom{m+1}{3} + m + 2$.

Case 3.2. G is connected.

Take integer $m \geq 5$. G is constructed from $P_n = v_1v_2 \cdots v_n$ and F_m in the following way, where $n = \binom{m+1}{3} + 1$. We join the vertex v_1 of P_n to the vertex u_1 of F_m. Clearly, $\Delta(G) = m$.

Let π be a $(m+1)$-VDTC of $P_{n+1} = v_1v_2 \cdots v_nu_1$ with $m+1 = \chi_{vt}(P_{n+1})$. Without loss of the generality, we assume that $\pi(v_1u_1) = 4, \pi(u_1) = 2$. It is straightforward to define a total coloring θ of G as: $\theta(e) = \pi(e)$ if $e \in E(P_n)$; $\theta(v) = \pi(v)$ if $v \in V(P_n)$; $\theta(v_1u_1) = 4$; $\theta(u_i) = i+1, i \in [0, m]$; $\theta(u_0u_i) = i+2, i \in [1, m-1], \theta(u_0u_m) = 2$; $\theta(u_iu_{i+1}) = i+4, i \in [1, m-3]$; and $\theta(u_{m-2}u_{m-1}) = 1, \theta(u_{m-1}u_m) = 3, \theta(u_mu_1) = 1$.

Hence, the total coloring θ shows that $\chi_{vt}(G) = m + 1$. Notice that $\chi_{vt}(G - u_0) = \chi_{vt}(P_{m+n}) = m + 2$ according to Lemma 1 for $n + m = \binom{m+1}{3} + m + 2$.

The proof of Theorem is finished.

By our experience, we propose the following conjecture:

Conjecture 2. For any k-regular graph G with $k \geq 2$, there do not exist subgraphs H of G such that $\Delta(H) = k$ and $\chi_{vt}(H) > \chi_{vt}(G)$.

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