A brute-force method for studying the chromatic properties of homeomorphic graphs

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Abstract

Let M be a graph, and let $\mathcal{H}(M)$ denote the homeomorphism class of M, that is, the set of all graphs obtained from M by replacing every edge by a 'chain' of edges in series.. Given M it is possible, either using the 'chain polynomial' introduced by E. G. Whitehead and myself (Discrete Math. 204 (1999) 337-356) or by *ad hoc* methods, to obtain an expression which subsumes the chromatic polynomials of all the graphs in $\mathcal{H}(M)$. It is a function of the number of colors and the lengths of the chains replacing the edges of M. This function contains complete information about the chromatic properties of these graphs. In particular it holds the answer to the question "Which pairs of graphs in $\mathcal{H}(M)$ are chromatically equivalent". However, extracting this information is not an easy task.

In this paper I present a method for answering this question. Although at first sight it appears to be wildly impractical, it can be persuaded to yield results for some small graphs. Specific results are given, as well as some general theorems. Among the latter is the theorem that, for any given integer γ, almost all cyclically 3-connected graphs with cyclomatic number γ are chromatically unique.

The analogous problem for the Tutte polynomial is also discussed, and some results are given.

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Section 1. Introduction.

In this paper graphs will be allowed to have multiple edges, but unless otherwise stated will be without cut-vertices and hence without loops.

Let M be a graph whose edges are labelled with labels $a, b, c, \ldots$. If we replace every edge of M by a "chain" of edges in series, replacing edge a by a chain of length n_a (i.e., having n_a edges), b by a chain of length n_b, and so on, we obtain a "homeomorph" of M. Figure 1 shows such a labelled graph and one particular homeomorph — that obtained by setting $n_a = 3$, $n_b = 3$, $n_c = 2$, $n_d = 5$, $n_e = 1$, $n_f = 2$ and $n_g = 4$.

Conversely, if we start with a graph G and successively "suppress" all vertices of degree 2 we arrive at a homeomorphically reduced graph M. For convenience we shall call this the "reduced graph" of G.

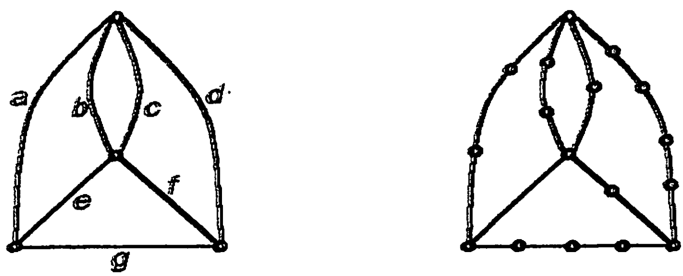

Figure 1

In [23] Whitehead and I showed how to get a general expression for the chromatic polynomial of such a "chain graph" for general values of the chain lengths, and to that end introduced a "chain polynomial" for the graph M, defined as follows

$$Ch(M; \omega; a, b, c, \ldots) = \sum F(Y, \omega)\, \pi_U$$

where U is a subset of the labels, π_U is the formal product of the labels in U, and $F(Y, \omega)$ is the flow polynomial of the graph induced by the edges in Y, the set of edges whose labels are not in U. The sum is over all subsets U of the set of labels, and ω, here and elsewhere, stands for $1 - \lambda$, where λ is the usual variable for the flow and chromatic polynomials. For more information on the flow polynomial see [23,27]; for the chromatic polynomial see [19,22].

A property of the flow polynomial that we shall need is the following

Theorem 1. When expressed in terms of ω the flow polynomial of a graph has the form $(-1)^{q-p+1} f(\omega)$ where $f(\omega)$ is a polynomial in ω in which all coefficients are positive and in which there is no constant term. Moreover, if the graph has c cut-vertices then the lowest power of ω is $c+1$. In particular, if the graph is 2-connected the lowest power of ω is 1.

In practice, the labels are interpreted as powers of a variable x. In the notation used in [23] a stands for x^{n_a}, b stands for x^{n_b}, c for x^{n_c}, and so on. If we do not need to specify all the details we can write the chain polynomial as $Ch(M;\ \omega, x)$, or just $Ch(M)$. To within a factor, the general chromatic polynomial for a homeomorphism class of graphs is obtained by setting $x = \omega$ in $Ch(M)$. More precisely, it is

$$(1.1) \qquad P(G) = (-1)^Q\, Ch(M, \omega, \omega) / (\lambda - 1)^{q-p}$$

where Q denotes the sum of the lengths of the chains in M. Here and elsewhere p and q stand for the numbers of vertices and edges respectively in whatever graph is under discussion.

As an example take the case of K_4 -- the complete graph on 4 vertices, labelled as in the left-hand graph of figure 2 below. Its chain polynomial [23] is

$$abcdef - \omega(abc + aef + bdf + cde + ad + be + cf) + (\omega^2 + \omega)(a + b + c + d + e + f) + \omega(\omega + 1)(\omega + 2)$$

Hence the chromatic polynomials of its homeomorphs are given by the general formula

$$\begin{aligned}
&(-1)^Q [\omega^{n_a + n_b + n_c + n_d + n_e + n_f} \\
&\quad - \omega(\omega^{n_a + n_b + n_c} + \omega^{n_a + n_e + n_f} + \omega^{n_b + n_d + n_f} + \omega^{n_c + n_d + n_e} \\
(1.2) &\qquad + \omega(\omega^{n_a + n_d} + \omega^{n_b + n_e} + \omega^{n_c + n_f}) \\
&\quad + (\omega^2 + \omega)(\omega^{n_a} + \omega^{n_b} + \omega^{n_c} + \omega^{n_d} + \omega^{n_e} + \omega^{n_f}) \\
&\qquad\qquad + \omega(\omega + 1)(\omega + 2)]/(\lambda - 1)^2
\end{aligned}$$

A general chromatic polynomial like (1.2) contains complete information about the chromatic properties of the graphs in $\mathcal{H}(M)$, the set of all homeomorphs of M. In particular it holds the answer to the question "Which pairs of graphs in $\mathcal{H}(M)$ are chromatically equivalent, i.e. have the same chromatic polynomial?" Although this is true in theory there seems to be no easy way to extract this information. In this paper I give a method whereby this can be done for small graphs. The investigation of K_4 by this method will be described in detail.

This same chain polynomial can be used to obtain the general Tutte polynomial for the graphs in $\mathcal{H}(M)$. This raises the analogous question of which pairs of these graphs are 'Tutte equivalent', i.e. have the same Tutte polynomial. This question too can be answered for small graphs, and is treated later in the paper.

Section 2. An example.

Let us take the case of K_4, the complete graph on 4 vertices. Homeomorphs of K_4 have been extensively studied (see, for example, [2,7,8,10,11,12,13,16, 17,26,28,29,33]). We note the following definitions.

Definition. A graph is "chromatically unique" if it does not have the same chromatic polynomial as any other (nonisomorphic) graph

A useful term introduced by E. G. Whitehead is that of *chromatic distinctness*.

Definition. A graph is "chromatically distinct" if it is not chromatically equivalent to any graph homeomorphic to it.

Thus a chromatically unique graph is automatically chromatically distinct, but a chromatically distinct graph may fail to be chromatically unique by virtue of being chromatically equivalent to some graph to which it is not homeomorphic.

We note also the following theorem.

Theorem 2 (see [4]) If a graph is chromatically equivalent to a homeomorph of K_4 then it is itself a homeomorph of K_4.

Consider two homeomorphs of K_4, say G_1 and G_2, labelled as in figure 2, and whose chain lengths are therefore $n_a, n_b, \ldots\ldots$ Assume that these two graphs have the same chromatic polynomial.

As we shall be working almost exclusively with chain lengths it will be convenient to simplify the notation by using for them the corresponding upper

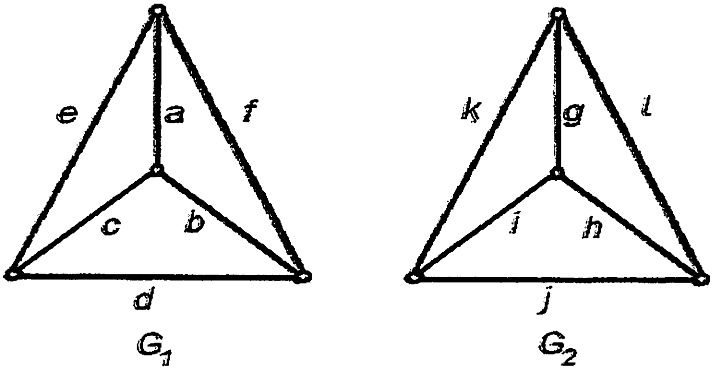

Figure 2.

case letters; that is, we use A for n_a, B for n_b, and so on.

. We can then rewrite (1.2) as

$$
\begin{aligned}
&(-1)^Q [\omega^{A+B+C+D+E+F} \\
&\quad - \omega(\omega^{A+B+C} + \omega^{A+E+F} + \omega^{A+D+F} + \omega^{C+D+E} \\
(2.1)\qquad &\qquad + \omega(\omega^{A+D} + \omega^{B+E} + \omega^{C+F}) \\
&\quad + (\omega^2 + \omega)(\omega^A + \omega^B + \omega^C + \omega^D + \omega^E + \omega^F) \\
&\qquad\qquad + \omega(\omega+1)(\omega+2)]/(\lambda-1)^2
\end{aligned}
$$

The chromatic polynomial of G_2 will, of course, be a similar expression in the variables *G, H, I, J, K* and *L*.

Note: This use of the letters *A* to *L*, in order to avoid a more cumbersome notation, introduces a conflict with the very natural use of *G* (and, later, *H*) for a general graph. Although I deem it unlikely that this slight clash of symbols will cause any confusion I have distinguished the two usages typographically.

Consider two homeomorphs of K_4, say G_1 and G_2, labelled as in figure 2, and whose chain lengths are therefore n_a, n_b, ... We want the two chromatic polynomials to be identically equal. Now the division by $\pm(\lambda - 1)^2$ is common to both polynomials and can therefore be ignored. In what remains, the leading terms will be the same, since $A+B+C+D+E+F$ and $G+H+I+J+K+L$ give the numbers of edges in G_1 and G_2, and these must be equal. The term independent of the chain lengths is also the same for both polynomials. Removing these terms and dividing by ω we obtain

$$(2.2) \quad \begin{aligned} &-(\omega^{A+B+C} + \omega^{A+E+F} + \omega^{B+D+F} + \omega^{C+D+E} + \omega^{A+D} ++ \\ &\omega^{B+E} + \omega^{C+F}) + (\omega+1)(\omega^A + \omega^B + \omega^C + \omega^D + \omega^E + \omega^F) \end{aligned}$$

for G_1, and a similar expression for G_2. If we now equate these two polynomials and manipulate the terms so that all coefficients are positive, we finally arrive at the following equation involving the 12 chain lengths.

$$(2.3) \quad \begin{aligned} &(\omega+1)(\omega^A + \omega^B + \omega^C + \omega^D + \omega^E + \omega^F) \\ &+ \omega^{G+H+I} + \omega^{G+K+L} + \omega^{H+J+L} + \omega^{I+J+K} + \omega^{G+J} + \omega^{H+K} + \omega^{I+L} \\ &= (\omega+1)(\omega^G + \omega^H + \omega^I + \omega^J + \omega^K + \omega^L) \\ &+ \omega^{A+B+C} + \omega^{A+E+F} + \omega^{B+D+F} + \omega^{C+D+E} + \omega^{A+D} + \omega^{B+E} + \omega^{C+F} \end{aligned}$$

It is clear that for (2.3) to be an identity each term on the left-hand side must cancel with a term on the right-hand side. This requirement can be succinctly expressed by making a "tableau" listing, in two columns, the 19 powers on the left and right, as in Tableau I below

. It will be convenient to separate by a line the first 12 expressions on each side from the rest, and to refer to expressions as being "above the line" or "below the line".

For G_1 and G_2 to be chromatically equivalent we must find positive integer values for the variables A to L such that the numbers in the left-hand column of the tableau are the same as those in the right-hand column, though in a different order. In other words, there must be a bijection from the left-hand expressions to the right-hand expressions which maps equals onto equals

The way is now clear to the solution of the problem of finding all possible pairs of chromatically equivalent homeomorphs of K_4. All we have to do is to run through the 19! possible bijections. Each such bijection will give us a set of 19 linear equations in the 12 variables. If these equations have a solution then we will have found at least one pair. In general we will get a vector space

A	G		
$A+1$	$G+1$		
B	H	B	H
$B+1$	$H+1$	$B+1$	$H+1$
C	I	C	I
$C+1$	$I+1$	$C+1$	$I+1$
D	J	D	J
$D+1$	$J+1$	$D+1$	$J+1$
E	K	E	K
$E+1$	$K+1$	$E+1$	$K+1$
F	L	F	L
$F+1$	$L+1$	$F+1$	$L+1$
$G+H+I$	$A+B+C$	$A+H+I$	$A+B+C$
$G+K+L$	$A+E+F$	$A+K+L$	$A+E+F$
$H+J+L$	$B+D+F$	$H+J+L$	$B+D+F$
$I+J+K$	$C+D+E$	$I+J+K$	$C+D+E$
$G+J$	$A+D$	$A+J$	$A+D$
$H+K$	$B+E$	$H+K$	$B+E$
$I+L$	$C+F$	$I+L$	$C+F$
Tableau I		Tableau 2	

of solutions. (Strictly speaking we get the subset of a vector space for which all the vector elements are positive integers. This usage will be assumed in what follows.) Each such vector space gives a family of pairs depending on one or more parameters. Furthermore, any given pair of chromatically equivalent homeomorphs will give values in the tableau for which a bijection is possible and hence will be found by this exhaustive procedure.

Section 3. Practicality

There is one slight snag with the method just outlined. The number of bijections, 19!, is rather large[1.] It would be quite impossible to generate each one and deal with the resulting equations. Fortunately it is possible to devise a backtrack procedure of a fairly standard kind which enables the majority of the bijections to be eliminated without their being generated at all. I shall now briefly describe this procedure, starting with Tableau I (though, as we shall see later, we can profitably simplify the problem first.)

We start by asking what the first expression on the left can map onto, and choose the first possibility on the right. We therefore equate A with G.. This

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[1] In case anyone is interested, the exact number is 121,645,100,408,832,000.

enables us to put G equal to A in all the other expressions. The expressions A and G are deleted from the two sides of the tableau, as are any other pairs of expressions that may turn out to be equal as a result of the substitution. In the present case the second expressions on each side, $A+1$ and $G+1$, are now equal and can therefore be deleted.. We thus obtain a second tableau, Tableau II, in which the variable G does not occur.. We now proceed similarly with this new tableau .

In making a mapping of a left-hand expression onto a right-hand expression one of two things can happen. Either we get a valid equation, in which case we use it to eliminate another variable in the other expressions, and continue with the revised tableau; or we get an impossible equation, as would happen, for example, if we attempted to equate $B + 1$ with $B + D + F$ (since all variables are strictly positive integers). In the latter case we try the next expression on the right-hand side. If this is not possible, because we have reached the end of the right-hand side of the current tableau, then we have to backtrack to the previous tableau and try the next possibility there.

If we reach a stage where there are no expressions left in the tableau, then we have a feasible set of equations. Note that we have, at this stage, performed the "forward substitution" part of the usual routine for solving a set of linear equations and it remains only to perform the back substitutions to get the required solution.

If the program has backtracked to the first tableau and finds that there is no continuation (the first expression on the left was previously mapped onto the last expression on the right) then the program ends; all possibilities have been considered.

It is to be expected that a program such as this would take a long time to execute and would produce a lot of output. Thus, when we obtain a solution it is in our best interests to output it promptly. Nevertheless, two things are worth doing. One is to test whether the two graphs that we now have are isomorphic; for if they are then the solution is of no interest. The other is to manipulate the solution into a canonical form such that if the same solution is obtained again it will be output as an identical record. This will facilitate the elimination of duplicates, of which we expect a vast number.

A method for doing this is the following. First replace each chain length expression by an integer. For example, a two-parameter chain length $2u + 3v + 1$ could be replaced by 20301. Then, in each graph separately, consider the 24 permutations of the vector, (A, B, C, D, E, F) by the 24 automorphisms of the edges of K_4 , and of the resulting vectors choose the one which comes first in lexical order. This gives a canonical form for each graph separately. Now compare these two vectors and interchange them, if necessary, so that the first vector precedes the second in lexical order. This gives the canonical form for the solution.

Note that this procedure will also detect if the two graphs are isomorphic, so the other matter is taken care of. This is not precisely the method that I used in the program, but is along the same lines. The details do not matter.

A program like this, starting with Tableau I would probably have taken something like 48 hours to run on the rather slow (1999 vintage) personal computer that I used then. This is not necessarily important. This program needs to be run only once, so there is no incentive to hone it into optimum efficiency. It would, for example, be counterproductive to spend two weeks improving it so that it would take only six hours to run. Only simple modifications would be worth introducing. As it happens, there is one very good one available.

Section 4. Preliminary simplification

Let us look at the shortest chain lengths in G_1 and G_2. Without loss of generality we can take them to be A and G. From Tableau I it is clear that A cannot be mapped onto anything below the line on the right-hand side. It is also easy to see that if A is mapped onto anything above the line other than G then nothing on the left can map onto G. Hence A must be mapped onto G. We get the following theorem, first stated by Whitehead and Zhao [28]

Theorem 3. If two homeomorphs of K_4 are chromatically equivalent then their shortest chains are of the same length.

We can therefore put $A = G$, start with Tableau II and work with 17 expressions on each side, and 11 variables in all. (Note that this argument is not materially affected if there is a tie for shortest chain.) This is an improvement; but we can go further by considering the next shortest chains on each side. The same argument applies, and we obtain

Theorem 4. If two homeomorphs of K_4 are chromatically equivalent then their two shortest chains are of the same lengths.

. However, there is now a complication in that, in each graph, the two shortest chains may be adjacent (having a common vertex) or non-adjacent. We therefore have three cases to consider, according as the two shortest chains are adjacent in both graphs, non-adjacent in both graphs, or adjacent in one and non-adjacent in the other. Thus the previous tableau can be simplified in three different ways, and we shall have to run the program three times. This is worth it, since we now have only 15 expressions on each side and only 10 variables.

We could go further and consider the third shortest chains. But we would have many more cases to consider, including those arising from the possibility, for example, that if A and D are the shortest chains in G_1 then the length of the third shortest chain in G_2 might be $A + D$. For this reason I chose to stop at the consideration of the two shortest chains.

Note that because the chain lengths are, in general, expressed in terms of parameters, it is not usually possible to say what are the shortest chains (which is smaller, $2u + 3v + 1$ or $3u + 2v + 2$?) Hence we have to be content with a weaker form of Theorem 3, which simply states that G_1 and G_2 share a common chain length. A similar remark applies to Theorem 4.

Section 5. The results

A C++ program along the above lines was written and run three times with slightly modified input as just described. The runs took a total of roughly seven hours of computer time, and the solutions obtained were put into separate files according to the number of parameters. These files contained many duplicates, and had to be processed to eliminate all but one occurrence of each solution. This was done by first sorting each file and then using a simple program which read each record and discarded any record which was the same as the previous one. It turned out that there are no solutions with more than 3 parameters and just a single one with exactly 3 parameters

There remained the task of eliminating those solutions which were special cases of a family with a larger number of parameters. The programs that did this were straightforward and of no particular interest, so I shall not describe them.

The final result was that in addition to the unique 3-parameter family there are seven 2-parameter families, six 1-parameter families and two isolated pairs. These are summarized in Table I, in which each pair is exhibited as two rows of six chain lengths, viz

$$\begin{matrix} A & B & C & D & E & F \\ G & H & I & J & K & L \end{matrix}$$

The identifier on the left gives the number of parameters and an arbitrary identifying letter[2]

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[2] In presenting these results I am, of course, making the assumption that my programs were free of errors and ran correctly. I believe this to be the case, but would enthusiastically welcome any independent corroboration. Evidence of errors would also be welcomed, though with less enthusiasm!

3a	1	x	y	y+2	y+z	z
	1	x	y+z+1	y	y+1	z
2a	1	2	v	u+v	v+2	u
	1	2	v+1	v	u+v+1	u
2b	1	v	v+1	u	u+v+1	u+1
	1	v	v+1	u+2	u	u+v
2c	1	v	u	v+2	u+v+1	u+1
	1	v+1	u+1	v	u+v+2	u
2d	v	v+1	2v+2	u	v+2	u+v
	v	v+2	2v+1	v+1	u	u+v+1
2e	v	v+1	u	u+v+1	u+2v+1	v+2
	v	v+1	u+2v+2	u	u+v	v+2
2f	v	v+1	u	u+2v+1	u+v+1	v+2
	v	v+1	u+v	u	u+2v+2	v+2
2g	v	v+1	u+v	u	v+2	2v+2
	v	v+2	u+v+1	v+1	u	2v+1
1a	1	2	4	t+2	t	6
	1	3	t	2	t+4	5
1b	1	t	t+1	t+4	2t+2	t+2
	1	t+1	t+3	t	2t+4	t+1
1c		t	t+2	t+4	t+2	2t+2
	1	t+1	t+2	t	2t+4	t+3
1d	1	t	t+2	t+4	3t+5	2t+3
	1	t+1	t+3	t	3t+6	2t+4
1e	1	t	2t+2	t+4	t+2	3t+3
	1	t+1	2t+3	t	t+3	3t+4
1f	2	t	t+1	t+2	2t+2	3t+2
	2	t	t+3	t+2	t+1	2t+1
0a	2	3	4	10	8	6
	2	4	7	3	12	5
0b	2	3	6	8	6	10
	2	5	6	3	7	12

Table I

Section 6. Observations on graphs in general.

As a result of this investigation we now have a complete picture of the relationships between homeomorphs of K_4 *vis-à-vis* the relation of chromatic equivalence — a picture which I like to call the "chromatic landscape" of K_4. From this it becomes a matter of routine to answer most, if not all, the questions that have from time to time been asked about the chromatic properties of K_4 homeomorphs. In particular, any homeomorph of K_4 which does not correspond to one of the 32 vectors appearing in Table I is chromatically distinct (and therefore, by Theorem 2, chromatically unique.)

Mention should be made here of some work of E. G. Whitehead, Jr.[29] He computed the chromatic polynomials of all homeomorphs of K_4 with up to 36 edges, and by observing regularities among the data he was able to deduce and confirm a number of infinite families of equivalent pairs, including the 3-parameter family mentioned above and many, but not all, of the others.

Since this investigation has turned out to be feasible, despite the daunting prospect of handling 19! permutations, we are led to wonder whether other graphs might be similarly investigated. The outlook is not encouraging. The next interesting graph to investigate would be W_5, the wheel on 5 vertices. Unfortunately there are 91 rows in the tableau for this graph and it does not seem that much can be done to reduce this number.

We can, however, make some general observations. As already remarked, any homeomorph of K_4 that is not chromatically distinct belongs to one or more of a finite number of vector spaces of solutions, none of which is of dimension greater than 3. It follows that the number of such graphs is asymptotically small compared with the set of all homeomorphs of K_4, which depends on 6 parameters.

Note. It used to be that a sentence like "Almost all graphs have property P" meant that all but a finite number had that property. More recently (see, for example, [1] or [12]) "almost all" has come to be used in an asymptotic sense, meaning that the ratio of the number of graphs having property P to the number of all graphs tends to 1 as the graphs become large. This asymptotic sense will be assumed in what follows.

We have therefore arrived at the following result.

Theorem 5. Almost all homeomorphs of K_4 are chromatically distinct.

From this and Theorem 2 we derive

Theorem 6. Almost all homeomorphs of K_4 are chromatically unique.

This is a well-known theorem, first proved by Li [12] by other means. We now observe that the above argument can be used to obtain a more general result.

Take any reduced graph, M, and consider the set $\mathcal{H}$ of all its homeomorphs. From the chain polynomial of M we obtain a general expression for the chromatic polynomials of these homeomorphs. In theory, if not in practice, we can set up the appropriate tableau and follow the method given for K_4 to find all possible families of chromatically equivalent pairs. The outcome will be a finite number of vector spaces, and any graph in $\mathcal{H}$ that is not chromatically distinct will belong to at least one of these spaces.

Now it is intuitively obvious that the requirement that the two sides of the tableau must match up one-to-one in a consistent manner must impose ***some*** restriction on the chain lengths and that therefore none of these vector spaces will have dimension as high as q, the number of edges in M. This, however, needs to be proved. If we can succeed in proving it then we can readily deduce:

Conjecture For any graph M, almost all homeomorphs of M are chromatically distinct.

In the next section we examine the dimensions of the vector spaces of solutions.

Section 7. The general problem.

Take M to be a graph, which, for now, we assume to be 3-connected.

Take two graphs, G_1 and G_2 each homeomorphic to M and with general chain lengths, given by two sets of symbols, S_1 and S_2. (In the case of K_4 we had $S_1 = \{A, B, C, D, E, F\}$ and $S_2 = \{G, H, I, J, K, L\}$; we shall use these letters for the purpose of illustration). For convenience let "S_i - expression" stand for "expression in the variables in S_i ").

Consider a bijection between the two sides. If any S_1-expression on the left maps onto an S_1-expression on the right, the result is an equation in variables in S_1 alone. This is sufficient to show that the graphs G_1 for this particular bijection correspond to elements belonging to a vector space of dimension less than q. Clearly the same thing applies to the graph G_2. This establishes the desired result in this case. In the contrary case, every S_1-expression on the left maps onto an S_2-expression on the right, and vice versa . This means that every equation obtained is of the form

(7.1) (An S_1-expression) = (An S_2-expression)

We first observe that for each variable, A say, there is at least one expression which is just A itself. For the coefficient of x^{n_a} in the chain polynomial is the flow polynomial of the graph obtained by deleting a from G_1. By the connectedness requirement this graph does not have a bridge (so the flow polynomial does not vanish) and does not have a cutvertex (so, by Theorem 1, the flow polynomial has a term in ω, becoming a constant when we divide by ω).

Suppose that one such single-variable S_1-expression (say A again) maps onto an expression in more than one variable in S_2. Suppose, for example, that we have something like

$$A + 2 = G + I + L + 1$$

Now G, I and L will each equal an S_1-expression. Since there is no cancellation (all signs are plus) the equivalent of the sum $G + I + L + 1$ cannot reduce identically to $A + 2$. Hence we have an equation in the variables in S_1 and again we have our desired result. We therefore consider the contrary case where each single-variable expression maps onto a single-variable expression.

The expressions in A alone will be of the form A, $A + 1$, $A + 2$, etc., possibly with repetitions. Let us call this a "batch" of expressions. Suppose that one expression in a batch maps onto an expression in, say, J, while another maps onto an expression in some other variable, say L. Then we shall have two equations, for example

$$A + 2 = J + 1 \quad \text{and} \quad A + 5 = L + 7$$

These imply an equation connecting J and L. Again we have the desired result.

The contrary possibility is now that every such batch on the left maps exactly onto a batch on the right. This gives us a bijection from the set S_1 to the set S_2. It follows that, G_1 and G_2 have the same set of chain lengths.

It may seem unlikely that two homeomorphs related in this close way could be chromatically equivalent no matter what the chain lengths were, unless they were, in fact, isomorphic; but we note that this can certainly happen if we ignore the connectedness requirement. The two homeomorphs in figure 3 will have the same chromatic polynomial no matter what the chain lengths A . B . C . etc. may be (this follows from Theorem 4 in [23]). Yet these graphs are not, in general, isomorphic. They are, however, "2-isomorphic". This is a concept that we shall need later, and for our purposes it can be loosely defined as follows.

Definition. Two graphs, G_1 and G_2, each of connectivity 2, are said to be "2-isomorphic" if G_2 can be obtained from G_1 by a sequence of "twisting operations". A twisting operation on, say, G_1 relative to a pair of cut-vertices u and v is that of taking G_1 apart at u and v, reversing one of the "pieces" thus obtained, and then reuniting the two pieces. The example in figure 3 should clarify this informal description.

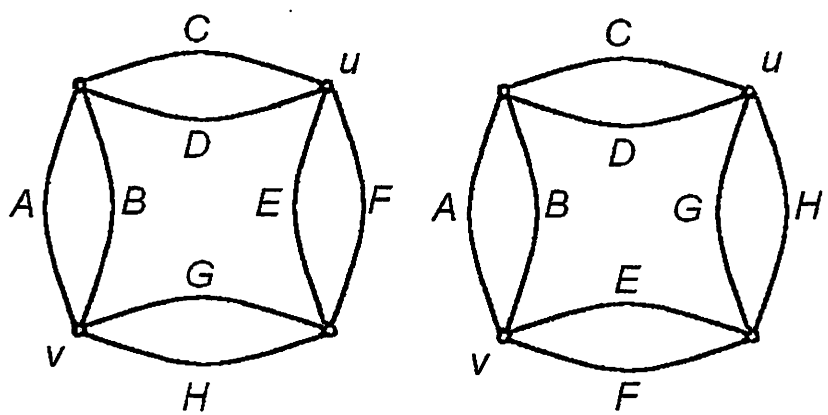

Figure 3.

A general definition of "2-isomorphic", including its application to graphs of connectivity less than 2 (not relevant here) can be found in [30]. An important property of 2-isomorphic graphs is that they are chromatically equivalent.

The situation is now as follows. G_1 is represented by M with edge labels a, b, c, etc. -- call this representation $M^{(1)}$. G_2 is represented by M with the same labels, standing for the same edge lengths, but possibly attached to different edges. Call this $M^{(2)}$.

Consider any cycle $\mathcal{C}$ of length m, say, in $M^{(1)}$ and let $\mathcal{D}$ be the set of edges in $M^{(2)}$ with the same labels as the edges in $\mathcal{C}$. There are three possibilities:

(i) Some proper subset of the edges of $\mathcal{D}$ forms a cycle in $M^{(2)}$
(ii) The edges of $\mathcal{D}$ form a cycle of length m in $M^{(2)}$
(iii) No cycles are formed by any edges of $\mathcal{D}$.

We now give specific values to the chain lengths, obtaining two particular versions of G_1 and G_2, and use the theorem that chromatically equivalent graphs have the same girth (see [14]).

Make the edges of $\mathcal{C}$ into chains of length 2, and make all other edges into "long" chains whose length is some sufficiently large integer. This defines two graphs $G_1^{\#}$ and $G_2^{\#}$ which are chromatically equivalent. Furthermore, the girth of $G_1^{\#}$ is $2m$, since, except for $\mathcal{C}$, every cycle in $G_1^{\#}$ contains at least one long chain.

If possibility (i) holds then $G_2^{\#}$ contains a cycle of length less than $2m$, so its girth will be less than that of $G_1^{\#}$. This is not possible since the two graphs are chromatically equivalent.

If possibility (iii) holds then every cycle of $G_2^{\#}$ will contain at least one long chain. Hence the girth of $G_2^{\#}$ will be greater than $2m$. This too is not possible.

Note that the reason for making the edges in $\mathcal{C}$ into chains of length 2 (rather than 1) was to avoid any possible difficulty arising from the fact that, as far as chromatic polynomials are concerned, a double edge is the same as a single edge.

We deduce that (ii) must hold. In other words, if a subset of labels forms a cycle in $M^{(1)}$ then the same subset forms a cycle in $M^{(2)}$ and conversely. This means that the labelled graphs $M^{(1)}$ and $M^{(2)}$ have the same cycle matroid. By Whitney's 2-isomorphism theorem (see, for example, [30]) $M^{(1)}$ and $M^{(2)}$ are 2-isomorphic. But since we have assumed M to be 3-connected it follows that $M^{(1)}$ and $M^{(2)}$ are in fact isomorphic.

We have thus established that we can have no families of chromatically equivalent pairs with dimension as high as q. We can therefore upgrade our conjecture to a theorem, though only when M is 3-connected.

Theorem 7. For any 3-connected graph M almost all homeomorphs of M are chromatically distinct.

Section 8. The broader horizon

Hitherto we have considered graphs that are homeomorphs of the same reduced graph. Let us broaden our horizon and consider two graphs G_1 and G_2 that are homeomorphs of two different reduced graphs M_1 and M_2. For what values of their chain lengths are G_1 and G_2 chromatically equivalent? An example will make it clear that there is a brute force method, much as before, for answering this question, and to that end we briefly consider the two graphs shown in Figure 4. S is the graph obtained from K_4 by shrinking one edge; T is the 4-theta graph (see [25] for this nomenclature).

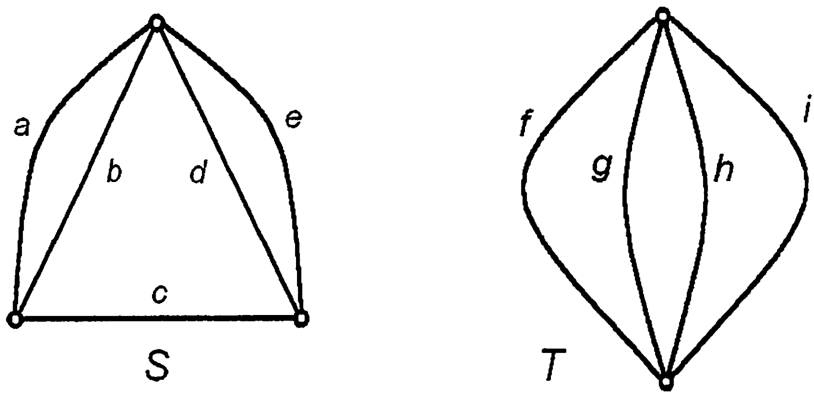

Figure 4

The general chromatic polynomial of S, after the outside divisor and the term *abcde* have been removed and the result divided by ω, reduces to

(8.1)
$$-(abc + cde + ad + ae + bd + be) + (\omega + 1)(a + b + c + d) + \omega c - (\omega + 1)^2$$

The corresponding result for T is

(8.2)
$$-(fg + fh + fi + gh + gi + hi) + (\omega + 1)(f + g + h + i) - (\omega^2 + \omega + 1)$$

and rearranging the terms as before, we obtain.

$$(\omega + 1)(\omega^A + \omega^B + \omega^C + \omega^D) + \omega^{E+1} + \omega^{F+G} + \omega^{F+H} + \omega^{F+I} + \omega^{G+H} + \omega^{G+I} + \omega^{H+I}$$

$$(8.3) \qquad = (\omega + 1)(\omega^F + \omega^G + \omega^H + \omega^I) + \omega^{A+B+C} + \omega^{C+D+E} + \omega^{A+D} + \omega^{A+E} + \omega^{B+D} + \omega^{B+E} + 1$$

The tableau corresponding to (8.3) is shown in Tableau III. Note that the terms independent of the chain lengths do not cancel completely. In consequence the tableau contains a constant term "1" (which corresponds to the term ω since the entries in the tableau are powers of ω).

A	F
$A+1$	$F+1$
B	G
$B+1$	$G+1$
$C+1$	H
D	$H+1$
$D+1$	I
E	$I+1$
$E+1$	$A+B+C$
$F+G$	$C+D+E$
$F+H$	$A+D$
$F+I$	$A+E$
$G+H$	$B+D$
$G+I$	$B+E$
$H+I$	1

Tableau III

2	F
B	$F+1$
$B+1$	G
$C+1$	$G+1$
D	H
E	$H+1$
$F+G$	I
$F+H$	$I+1$
$F+I$	$B+C+1$
$G+H$	$C+D+E$
$G+I$	$B+D$
$H+I$	$B+E$

Tableau IV

Since $C > 0$, $C+1$ cannot map onto this "1" so, without loss of generality, we can take $A = 1$. This condenses Tableau III to Tableau IV, to which the programs described earlier can be applied.

The output from the computer program contains just two solutions, each in two forms. In terms of the vectors (A, B, C, D, E) and (F, G, H, I) the first solution is that

$(1, 3, 1, 3, 3)$ and $(1, 3, 2, 2, 3)$ are each equivalent to $(2, 2, 3, 4)$

However, the two instances of S are two different edge-gluings of the cycle on 4 edges and the 3-theta graph with chain lengths 2, 3 and 3, and hence, while distinct, are only trivially so. (An edge-gluing of two graphs is the result of identifying an edge in one with an edge in the other).

This example is already known; it was discovered by Peng (see [18], or the reference in [11])

The other solution is that

(1, 4, 3, 3, 3) and (1, 4, 2, 3, 4) are each equivalent to (2, 3, 4, 5)

Again the two instances of S result from different edge-gluings, this time of a cycle on 5 vertices with the 3-theta graph with chain lengths 3, 3 and 4. As far as I know this chromatically equivalent pair has not previously appeared in the literature.

For completeness we note one further solution, a highly degenerate 3-parameter family, resulting from edge-gluings in both graphs, namely

(1, u, v, 1, w) and (1, u, v+1, w).

The "cyclomatic number" γ of a connected graph is $q - p + 1$, and hence is the same for two chromatically equivalent graphs. Moreover it is invariant under suppression or insertion of vertices of degree 2. Now K_4, S and T are, with one exception, the only reduced graphs with cyclomatic number 3. and by Theorem 6 and what has just been done we have examined all relevant pairs of these except for the cases of (a) two homeomorphs of the graph S and (b) two homeomorphs of the graph T. Case (b) was settled in [6, Lemma 5.1] where it was shown that all k-theta graphs are chromatically distinct. Case (a) is similar to that of two homeomorphs of K_4 with which we started; but as it is simpler, I shall merely quote the result of the corresponding computer investigation. There are two 2-parameter families, namely

$u+v+2$	$v+1$	u	v	$u+2$	and	$u+v+2$	$u+2$	u	v	$v+1$
$u+v+1$	v	$u+2$	$v+1$	$u+1$		$u+v+1$	$u+1$	$u+2$	$v+1$	$u+1$;

two single-parameter families, namely

3	3	1	$t+1$	$t+3$	and	3	$t+3$	1	3	$t+1$
3	$t+2$	t	2	4		3	4	t	2	$t+2$;

and one isolated pair, namely

3	5	1	5	8
2	6	5	4	5

Purposely omitted are the many graphs in which one, at least, of A, B, D or E has value 1. These graphs are edge-gluings of a cycle and a 3-theta graph. Chromatic equivalents of these graphs are obtained by performing the edge-gluing in a different way. If the 3-theta graph has chains of length P, Q and R and the cycle is of length Z, then there are, in general, three ways of performing the edge-gluing, giving rise to the triplet

1	$Z-1$	$P-1$	Q	R
1	$Z-1$	$Q-1$	P	R
1	$Z-1$	$R-1$	P	Q

of chromatically equivalent graphs.

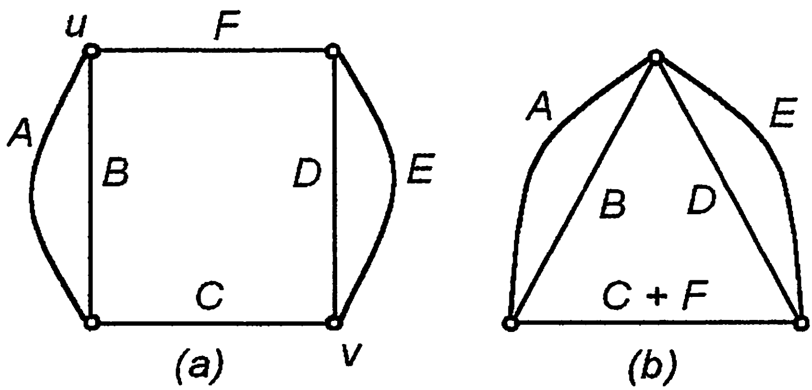

Figure 5

Also omitted are graphs like that in figure 5(a). (This is the exception mentioned above) Although not isomorphic to the graph of figure 5(b), it is a trivial variation, being, 2-isomorphic to it (perform the twist operation with vertices u and v).. These two graphs will therefore be chromatically equivalent no matter what the indicated chain lengths are.

From the results in this section we have complete information concerning chromatic equivalence between graphs with cyclomatic number 3. Every such graph must belong to at least one family of graphs, depending on at most three parameters. This is true even for the graphs in figure 5(a); for $C + F$ and the other chain lengths depend on at most two parameters (those for 5(b)) and one other parameter determines the values of C and F individually. Hence we have the following theorem.

Theorem 8 Almost all graphs with cyclomatic number 3 are chromatically unique.

Note. We can say "chromatically unique" here rather than just "chromatically distinct" because we have, in effect, compared every graph with every other graph to which it could possibly be equivalent, namely those having the same cyclomatic number.

Section 9. Graphs with a given cyclomatic number.

If a graph G has any vertices of degree 2 it is, *ipso facto*, only 2-connected; for the two vertices adjacent to the vertex of degree 2 form a two-vertex cut-set, as do any two vertices of one and the same chain. If this is the only reason for G failing to be 3-connected we shall say that G is "cyclically 3-connected". (This is by analogy with the term "cyclically 4-connected'). It is easy to verify the following:

Theorem 9 A graph is cyclically 3-connected if, and only if, its reduced graph is 3-connected.

We note that for cyclically 3-connected graphs "2-isomorphism" is the same as "isomorphism". Any twisting operation leaves the graph unchanged since one of the pieces is just a path.

Those graphs that are 2-connected but are not cyclically 3-connected will be said to be "properly 2-connected". They are the graphs whose reduced graphs are only 2-connected.

Consider the set, $\mathscr{K}$, of graphs with cyclomatic number γ. Since the suppression or insertion of vertices of degree 2 does not change the cyclomatic number every graph in $\mathscr{K}$ is a homeomorph of some reduced graph with cyclomatic number γ. Let $\mathscr{R}$ be the set of these reduced graphs.

Theorem 10 The largest graphs in $\mathscr{R}$ are the cubic graphs on $2n$ vertices, where $n = \gamma - 1$.

Proof. If a graph G in $\mathscr{R}$ has a vertex v of degree > 3 we can construct a larger graph as follows. Delete v and divide the vertices formerly adjacent to it into two sets, each set having at least two elements. Join the vertices in one of these sets to one end of a new edge and those in the other set to the other end. The resulting graph has one more vertices than G and has the same cyclomatic number.

Repeat this operation until all vertices have degree 3. The resulting cubic graph has an even number, say $2n$, of vertices and $3n$ edges. Hence $\gamma = q - p + 1 = n + 1$. It follows immediately that $\mathscr{R}$ is a finite set.

If a reduced graph M has less than $3n$ edges its family of homeomorphic graphs will be given by a vector space of chain lengths of dimension less than $3n$. The number of these homeomorphs will therefore be small in comparison with the number of cubic reduced graphs. For this reason, as we shall see, we can restrict our attention to graphs whose reduced graphs are cubic.

Consider such a graph G_1, whose reduced graph M_1 is cubic and 3-connected. Suppose that G_1 is chromatically equivalent to some other graph G_2 whose reduced graph M_2 is cubic. At this stage we cannot rule out the possibility that M_2 might not be 3-connected. However, if M_2 is only 2-connected then it has two edges which form a cutset. For in any splitting of the graph into two pieces, at vertices u and v say, two edges at u will be in the same piece, while the third belongs to the other piece. This latter edge and the corresponding one for vertex v form the edge cutset.

By the "bead-on-a-string" principle [23] or otherwise, we know that the chain lengths, say J and L, of these edges will always occur together, that is, in any expression in the tableau which we shall form for G_1 and G_2 these variables will always be together as $J + L$.

We now proceed much as in the proof of Theorem 7. As before, we want to show that we cannot have a family of solutions which is a vector space of dimension as high as $3n$. We can then deduce that the number of homeomorphs of M_1 that are chromatically equivalent to some homeomorph of M_2 is small compared to the number of all homeomorphs of M_1. Put another way, we want to show that being chromatically equivalent to some G_2 imposes at least one linear equation between the chain lengths in G_1.

As before we find that all equations resulting from the tableau are of the form of equation (7.1), and that each 'batch' of single variable expressions for G_1 must map onto a batch of expressions for G_2 in a one-to-one manner. Now if G_2 is properly 2-connected then, as noted above, there will be two variables that always occur together, and the number of batches for G_2 will be less than $3n$. Hence at this stage we can rule out this possibility and assume that G_2 also is cyclically 3-connected.

From here on the proof proceeds exactly as for Theorem 7. We show that M_1 and M_2 have the same circuit matroid and hence are isomorphic.

We are now in a position to prove our main result.

Theorem 11 For any integer γ almost all cyclically 3-connected graphs with cyclomatic number γ are chromatically unique.

Proof. There is only a finite number of reduced graphs with cyclomatic number γ and hence only a finite number of pairs. Consider the tableau for each pair and derive from it the corresponding families of pairs of chromatically equivalent graphs. If the reduced graphs in a pair have less than $3n$ edges each family of solutions will be a vector space of dimension $< 3n$. We have just shown that this is also true if one reduced graph is cyclically 3-connected while the other is properly 2-connected. (Note: we do not need to know whether it is possible for a cyclically 3-connected graph to be chromatically equivalent to a properly 2-connected graph).

Hence every cyclically 3-connected graph with cyclomatic number γ which is chromatically equivalent to some other graph belongs to one or more families of graphs. All these families are given by vector spaces of dimension $< 3n$ and hence, since there is only a finite number of them, the sum total of all graphs contained in them, even with repetitions, is small compared to the number of all graphs, since these correspond to the elements of a vector space of dimension $3n$. This completes the proof of Theorem 11

The situation with regard to properly 2-connected graphs is quite different. The twisting operation will give a chromatically equivalent graph that is 2-isomorphic to, but, in general, not isomorphic to the original graph. A 2-connected graph can be chromatically unique but clearly this is the exception rather than the rule.

This behaviour on the part of properly 2-connected graphs is unfortunate. We can bring them in line with the other graphs if we redefine 'chromatic uniqueness'.

Definition. A graph G is "chromatically 2-unique" if every graph with the same chromatic polynomial as G is 2-isomorphic to G.

We then have

Theorem 12. For every positive integer γ, almost all graphs with cyclomatic number γ are chromatically 2-unique.

The proof is left as an exercise for the reader.

Section 10. Tutte equivalence of homeomorphs.

We now look at the corresponding problem for the Tutte polynomial. For a given graph M we seek a complete description of those pairs of homeomorphs of M that are Tutte equivalent, i.e. that have the same Tutte polynomial. We can call the sum total of this information the "Tutte landscape" of M.

In [24] Whitehead and I showed that the Tutte polynomial of a homeomorph, G, of M can be obtained by essentially substituting $x + y - xy$ in the chain polynomial of M. To be precise, the Tutte polynomial of G is

$$\chi(G; x, y) = (x-1)^{-(q-p+1)}\, Ch(M; x+y-xy, x^{n_a}, x^{n_b}, x^{n_c}, \ldots)$$

Hence, if two homeomorphs, G_1 and G_2, of M are Tutte equivalent then $Ch(M; x+y-xy, x^{n_a}, x^{n_b}, x^{n_c}, \ldots)$ must be the same when the chain lengths of G_1 are used as when those of G_2 are used. Thus we have a situation similar to that discussed in section 2, but there are two important differences.

(a) The powers of x in the general Tutte polynomial depend on the chain lengths, but the powers of y do not. Hence we can equate the coefficients of the various powers of y on the left and right sides of the equation between the general Tutte polynomials. Thus instead of a single equation, like equation (2.2), we shall in general have several equations, each giving rise to an independent tableau.

(b) Instead of two sets of variables, one for each graph, we shall have only a single set. This is a consequence of the following theorem

Theorem 13. [25] If two homeomorphic graphs are Tutte equivalent then they have the same multiset of chain lengths.

Put another way this means that the chain lengths in one graph will be the same as those in the other graph, but in different positions. This change in positions will correspond in an obvious way to a permutation π of the symbols for the chain lengths.

These points are best illustrated by an example. It is already known (see [25]) that all homeomorphs of K_4 are Tutte unique, so we shall look at the "double triangle", DT, shown in figure 6. Its chain polynomial was found (using MAPLE) and turned out to be a formidable-looking polynomial which nevertheless could be greatly simplified by defining the following three expressions:

$$S = a + b + c + d + e + f$$

$$T = ab + ac + ad + ae + af + bc + bd + be + bf + cd + ce + cf + de + df + ef$$

$$U = ab+cd+ef + ace+acf+abe+abf+bce+bcf+bde+bdf + abcd+abef+cdef$$

where, as before, a stands for x^A, that is, x^{n_a}, and so on.

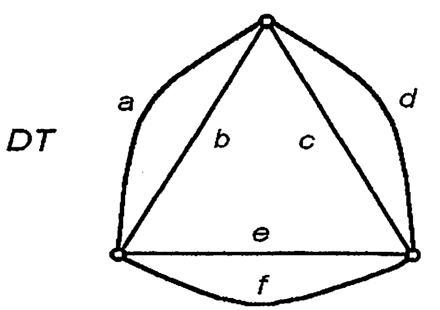

Figure 6

Take the chain polynomial of DT and, as in section 2, omit the term $abcdef$ and the term independent of the chain lengths. Divide the resulting expression by ω and then substitute $\omega = x + y - xy$. The resulting polynomial, which must be the same for two homeomorphs that are Tutte equivalent, reduces to the following simple expression

$$(10.1) \qquad (1-x)(1-y)T - (1 + 2x + 2y + x^2 + y^2 - 2xy^2 - 2x^2y + x^2y^2)S - U$$

Now S and T will be the same for the two graphs since they are invariant under permutations of the symbols a, b, c, d, e, f. It follows from (10.1) that a necessary and sufficient condition for the two homeomorphs to have the same Tutte polynomial it that U be the same for both.

Hence for each permutation π that we want to try we set up a tableau with the expressions occurring in U on the left-hand side and the permuted expressions on the right-hand side. Clearly this example has turned out to be particularly simple: all expressions are above the line (U has all positive terms) and there is only one tableau. Thus for the permutation

$$\begin{matrix} A & B & C & D & E & F \\ A & D & B & E & C & F \end{matrix}$$

we get Tableau V

$A+B$	$A+D$
$C+D$	$B+E$
$E+F$	$C+F$
$A+C+F$	$A+B+C$
$A+D+E$	$A+B+F$
$A+D+F$	$A+E+F$
$B+C+E$	$B+C+D$
$B+C+F$	$C+D+E$
$B+D+E$	$D+E+F$
$A+B+C+D$	$A+D+B+E$
$A+B+E+F$	$A+D+C+F$
$C+D+E+F$	$B+E+C+F$

Tableau V

Note that the expressions $A + C + E$ and $B + D + F$ occur on both sides and have therefore been omitted from the tableau, since they cancel each other.

It is not necessary to run through all 720 permutations of the six symbols; it suffices to take one element of each coset of the automorphism group of DT. There are thus just 15 permutations to try, and for each of these we form the appropriate version of Tableau V and process it by the same techniques that were used in section 3. This is a task of manageable proportions.

Note here that all the equations that we shall get by equating expressions on the left to those on the rig00ht will be homogeneous - there are no constants in the tableau. Consequently if G_1 and G_2 are Tutte-equivalent graphs and k is any positive integer, then the graphs obtained from G_1 and G_2 by multiplying all chain lengths by k will also be Tutte equivalent. This is not peculiar to the graph DT, but is true for all graphs. This, in turn, is a special case of a more general theorem which, while not directly germane to the present discussion, is of sufficient interest to warrant the digression in the next section.

Section 11. Uniform inflation

In [20] I showed that if we have two chain graphs with a distinguished edge in each, as illustrated in figure 7 with graphs G and H, then we can easily compute the chain polynomial of the graph N in the same figure, formed by identifying the ends of the edge z in G and H, each with z deleted.

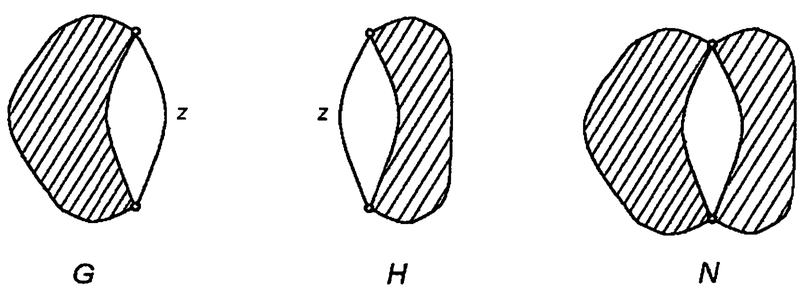

Figure 7

To do this we separate out the terms in z, writing, say,

$$Ch(G) = zP + Q \quad \text{and} \quad Ch(H) = zA + B.$$

Theorem 14. [20, p. 238] The chain polynomial of N is
$Ch(N) = PA - QB/\omega$

We can think of N as having been formed from G by "inflating" the edge z of G into the shaded part of the graph H, and if we write $Ch(N)$ in the form

(11.1) $$(-B/\omega)\,[P.(-\omega A/B) + Q]$$

then the process of finding the chain polynomial of the inflated graph can be boiled down to the following prescription: In the chain polynomial of G replace every occurrence of z (the label for the edge being inflated) by $-\omega A/B$, and multiply the resulting expression by $-B/\omega$.

We have seen that the Tutte polynomial of a graph G can be derived from the chain polynomial by means of the substitution $\omega = x + y - xy$. What happens if we go the other way, starting with the Tutte polynomial and making the inverse substitution $y = (\omega - x)/(1 - x)$? We shall get a function of ω and x (call it Φ) which will be a chain polynomial of sorts. But since information on individual chain lengths is lost in going from the chain polynomial to the Tutte polynomial, Φ will be a chain polynomial in which every 'x' stands for a chain; that is, it is what the chain polynomial of G becomes if we regard every edge as a chain of unit length.

Suppose we now want to inflate the edges of G. Using Φ alone we cannot inflate two edges by different graphs since we do not know which 'x' belongs to which edge; but it is quite possible to inflate all the edges by the same graph – an operation that can be called "uniform inflation" -- and the prescription given above shows how to do it. Using (11.1) we replace every occurrence of x by the appropriate $-\omega A/B$, and multiply the resulting expression by $(-B/\omega)^q$, where q is the number of edges in G. To get the Tutte polynomial of the inflated graph we now make the substitution $\omega = x + y - xy$. Here I have, for simplicity, ignored the need to multiply or divide by factors like $(x - 1)^{q-p+1}$. This makes no difference to the main argument.

Hence the Tutte polynomial of any uniform inflation of a graph G can be computed given only the Tutte polynomial of G. We thus have the following consequence:

Theorem 15 If G_1 and G_2 are Tutte equivalent then so are the uniform inflations of G_1 and G_2 by any given graph (the same for both).

In the special case when the inflating graph is the path on k edges we have the result given in section 10, namely, that multiplying the chain lengths by k preserves Tutte equivalence.

Results similar to the above have been given by Huggett [9] and Woodall [31]

Section 12. Result for the double triangle

It remains to give the result of the investigation of the double triangle, a result which turns out to be unexpectedly simple, doubtless due to the many symmetries of the graph. There is just one family, and it depends on three parameters, k, u and v. It is shown diagrammatically in figure 8.

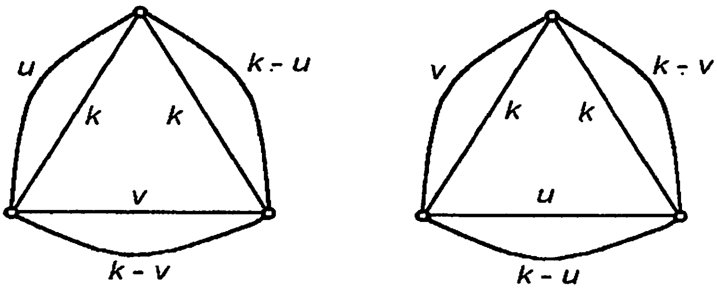

Figure 8

Section 13. Questions and conjectures.

An obvious general question is whether there are ways of improving the brute-force methods described here so that they can be applied to larger graphs. At the moment I have no suggestions.

In the proof of Theorem 7 we had to entertain the possibility that a cyclically 3-connected graph might be chromatically equivalent to a properly 2-connected graph. As it turned out we did not need to know whether this was possible. So we have an outstanding question.

Question 1. Can a cyclically 3-connected graph be chromatically equivalent to a properly 2-connected graph?

This is equivalent to

Question 2. Can one always tell from a chromatic polynomial whether the graph is cyclically 3-connected or properly 2-connected?

An obvious companion question is

Question 3. If so, is there a practical method of doing this?

Note that if we replace 'cyclically 3-connected' and 'properly 2-connected' by '3-connected' and '2-connected' then the corresponding question is answered in the negative. In [21] I gave two graphs with the same chromatic polynomial; one of them was 3-connected, the other had a vertex of degree 2 and was therefore only 2-connected. However, the latter graph *was* cyclically 3-connected.

Theorem 11 suggests that the following conjecture might be true

Conjecture 1. All cyclically 3-connected graphs are chromatically unique.

We have seen that almost all homeomorphs of 3-connected graphs with cyclomatic number γ derive from cubic reduced graphs, and the same is true of the properly 2-connected graphs. Wormald [32] has shown that the number of 2-connected cubic graphs is small compared with the number of 3-connected cubic graphs. This, in conjunction with conjecture 1 would imply

Conjecture 2. Almost all graphs are chromatically unique.

Maybe so; but there appears to be a wide gap between Theorem 11 and these conjectures.

Section 14. Acknowledgement

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References

1. Bollobas, B. Graph Theory - An introductory course
Springer Verlag, New York 1979

2. Brito, M. R. On supercycle C(a, b, c)
Ars Combinatoria 25A (1988) 165 – 171

3. Chao, C. Y. Whitehead, E. G. Jr. Chromatically unique graphs
Discrete Math. 27 (1979) no. 2 171 – 177

4. Chao, C. Y., Zhao, L. C. Chromatic polynomials of a family of graphs
Ars Combinatoria 15 (1983) 111 – 129

5. Chen, X. E., Ouyang, K. Z. Chromatic classes of certain 2-connected (n,n+2) graphs homeomorphic to K_4.
Discrete Math. 172 (1977) 17 – 29

6. Dong, F.M.,Teo, K.L., Little, C.H.C., Hendy, M., Koh, K.M.
Chromatically Unique Multibridge Graphs
The Electronic Journal of Combinatorics 11 (2004), #R12

7. Giudici, R. E., Margaglio. C. Chromatically equivalent graphs
Report No. 88-03, Dpto. de Mat. y Ciencia de la Comp. Univ. Simon Bolivar (1988)

8. Giudici, R. E., Melián M. Y. Chromatic uniqueness of 3-face graphs.
Report No. 88-05, Dpto de Mat. y Ciencia de la Comp. Univ. Simon Bolivar (1988)

9. Huggett, S. On tangles and matroids
Journal of Knot Theory and Its Ramifications, 14 (2005) no. 7. 919 - 929

10. Koh, K.M., Teo, K.L.. The search for chromatically unique graphs.
Graphs and Combin. 6 (1990) 259 – 285

11. Koh, K.M., Teo, K.L.. The search for chromatically unique graphs. II
Discrete Math. 172 (1997) 59 – 78

12. Li, W. M. Almost every K_4-homeomorph is chromatically unique
Ars Combinatoria 23 (1987) 13 – 36

13. Margaglio, C. Chromatic polynomials and classes of homeomorphic graphs. Report No. 90-03 Dpto. de Mat. y Ciencia de la Comp. Univ. Simon Bolivar (1990)

14. Meredith, G. H. J. Coefficients of chromatic polynomials
J. Comb. Theory Ser. B 13 (1972) 14 – 17

15. Oxley, J. Matroids
Graph Connections (eds. L. W. Beineke, R, J. Wilson)
Oxford Lecture Series in Mathematics.and its Applications, 5
Oxford Science Publications. Clarendon Press, Oxford (1997)

16. Peng, Yanling, Liu, Ruying Chromaticity of a family of K_4 homeomorphs
Discrete Math. 258 (2002) 161 -177

17. Peng, Yanling, Some new results on chromatic uniqueness of K_4 homeomorphs
Discrete Math. 288 (2004) 177 – 183

18. Peng, Yanling, Chromatic uniqueness of certain K(2,4) homeomorphs
(Bahasa Malaysia) Mathematika 7 (1991) 191 – 111

19. Read, R. C. An introduction to chromatic polynomials
J. Combinat. Theory 4 (1968) 52 – 71

20. Read, R. C. Chain polynomials of graphs
Discrete Math. 265 (1-3) (2003) 213 – 235

21. Read, R. C. Reviewer's remarks. Math. Reviews, MR50: 6906 (1975)

22. Read, R. C. Tutte, W. T. Chromatic Polynomials
Selected Topics in Graph Theory 3. Academic Press (1988) 15 – 42

23. Read, R. C., Whitehead, E. G. Jr. Chromatic polynomials of homeomorphism classes of graphs.
Discrete Math. 204 (1999) 337-356

24. Read, R. C. Whitehead, E. G. Jr. The Tutte polynomial for homeomorphism classes of graphs.
Discrete Math.243 (2002) 267-272

25. Read, R. C. Whitehead, E. G. Jr. A note on chain lengths and the Tutte polynomial.
Discrete Math. 308 (2008) 1826-1829

26. Ren, Haizhen, On the chromaticity of K_4 homeomorphs
Discrete Math. 252 (2002) 247 – 257

27. Tutte, W. T. Graph Theory
Encyclopedia of Mathematics and its Applications, vol. 21,
Addison-Wesley Publishing Co., Reading, Mass., 1984

28. Whitehead, E. G. Jr., Zhao, L. C. Chromatic uniqueness and equivalence of K_4 homeomorphs
J. Graph theory 8 (1984) 355 – 364

29. Whitehead, E. G. Jr. Chromatically equivalent K_4 homeomorphs.
Combinatorics, graph theory and algorithms. Vol. 1, 2
(Kalamazoo, MI, 1992), 867 - 872
New Issues Press, Kalamazoo, MI (1999)

30. Whitney, H. 2-isomorphism of graphs
Amer. J. Math. 55 (1933) 245 – 254

31. Woodall, D. R. Tutte polynomial expansions for 2-separable graphs
Discrete Math. 247 (2002) 201 – 213

32. Wormald, N. Some problems in the enumeration of labelled graphs
Ph. D. Thesis. University of Newcastle, New South Wales. 1978

33. Xu, S. Chromaticity of a family of K_4 homeomorphs
Discrete Math. 117 (1993) 293 - 297