Lattices Generated by Subspaces under Symplectic Group over a Finite Field

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Abstract. For $1 \leq d \leq \nu - 1$.Let V denote the 2ν-dimensional symplectic space over a finite field F_q, and fix a $(\nu - d)$-dimensional totally isotropic subspace W of V. Let $L(d, 2\nu) = P \cup \{V\}$, where $P = \{A | A$ is a subspace of V, $A \cap W = \{0\}$ and $A \subset W^{\perp}\}$. Partially ordered by ordinary or reverse inclusion, two families of finite atomic lattices are obtained. This article discusses their geometricity, and computes their characteristic polynomials.

Key words: Finite field; Symplectic space; Lattices; Characteristic polynomials.

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1. Introduction

Let F_q be a finite field with q elements, where q is a prime power. Let V be the 2ν-dimensional symplectic space over a finite field F_q, and fix a $(\nu - d)$-dimensional totally isotropic subspace W of V. Let $L(d, 2\nu) = P \cup \{V\}$, where $P = \{A | A$ is a subspace of V, $A \cap W = \{0\}$ and $A \subset W^{\perp}\}$. Partially ordered by ordinary or reverse inclusion, two families of finite atomic lattices are obtained. denoted by $L_O(d, 2\nu)$ or $L_R(d, 2\nu)$, respectively. For any two elements $A, B \in L_O(d, 2\nu)$,

$$A \vee B = \begin{cases} V & \text{if } W \cap (A + B) \neq \{0\}, \\ A + B & \text{otherwise.} \end{cases}$$

$$A \wedge B = A \cap B.$$

Similarly, for any two elements $A, B \in L_R(d, 2\nu)$,

$$A \wedge B = \begin{cases} V & \text{if } W \cap (A + B) \neq \{0\}, \\ A + B & \text{otherwise.} \end{cases}$$

$$A \vee B = A \cap B$$

Therefore, both $L_O(d, 2\nu)$ and $L_R(d, 2\nu)$ are finite lattices. This article discusses their geometricity, and computes their characteristic polynomials.

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2. Prelimianries

Now we recall some terminologies and definitions about finite posets and lattices, which can be found in [1, 2] for details.

Let P denote a finite set. A *partial order* on P is a binary relation $\leq$ on P such that

(1) $a \leq a$ for any $a \in P$.
(2) $a \leq b$ and $b \leq a$ implies $a = b$.
(3) $a \leq b$ and $b \leq c$ implies $a \leq c$.

By a *partial ordered set* (or *poset* for short), we mean a pair $(P, \leq)$, where P is a finite set and $\leq$ is a partial order on P. As usual, we write $a < b$ whenever $a \leq b$ and $a \neq b$. By abusing notation, we will suppress reference to $\leq$, and just write P instead of $(P, \leq)$.

Let P be a poset and let R be a commutative ring with the identical element. A binary function $\mu(a, b)$ on P with values in R is said to be the *Möbius function* of P if

$$\sum_{a \leq c \leq b} \mu(a, c) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

For any two elements $a, b \in P$, we say *a covers b*, denoted by $b < \cdot a$, if $b < a$ and there exists no $c \in P$ such that $b < c < a$. An element m of P is said to be *minimal* (resp. *maximal*) whenever there is no element $a \in P$ such that $a < m$ (resp. $a > m$). If P has a unique minimal (resp. maximal) element, then we denote it by 0 (resp. 1) and say that P is a poset with 0 (resp. 1). Let P be a finite poset with 0. By a *rank function* on P, we mean a function r from P to the set of all the non-negative integers such that

(1) $r(0) = 0$.
(2) $r(a) = r(b) + 1$ whenever $b < \cdot a$.

Let P be a finite poset with 0 and 1. The polynomial

$$\chi(P, x) = \sum_{a \in P} \mu(0, a) x^{r(1) - r(a)},$$

is called the *characteristic polynomial* of P, where r is the rank function of P.

A poset P is said to be a *lattice* if both $a \vee b := \sup\{a, b\}$ and $a \wedge b := \inf\{a, b\}$ exist for any two elements $a, b \in P$. Let P be a finite lattice with 0. By an *atom* in P, we mean an element in P covering 0. We say P is *atomic lattice* if any element in $P \setminus \{0\}$ is a union of atoms. A finite atomic lattice P is said to be a *geometric lattice* if P admits a rank function r satisfying

$$r(a \wedge b) + r(a \vee b) \leq r(a) + r(b), \forall a, b \in P. \quad (1)$$

for any two distinct elements $a, b \in L$. In a finite atomic lattice, (1) is equivalent to

$$a \wedge b < \cdot a \Rightarrow b < \cdot a \vee b.$$

The set of all the flats of a combinatorial geometry ordered by inclusion is a geometric lattice. Conversely, given a geometric lattice L, the incidence structure $(L, \{L_y \mid y \in L\})$ is a combinatorial geometry, where $L_y\{x \in L \mid x \leq y\}$.

Let L and L' be two lattices. If there exists a bijection σ from L to L' such that

$$\sigma(a \vee b) = \sigma(a) \vee \sigma(b), \sigma(a \wedge b) = \sigma(a) \wedge \sigma(b).$$

then σ is said to be an isomorphism from L to L'. In this case we call L is isomorphic to L', denoted by $L \simeq L'$. It is well known that two isomorphic lattices have the same characteristic polynomial and the geometricity.

The results on the lattices generated by the orbits of subspaces under finite classical groups have been obtained in a series of papers by Guo and Nan [3], Wang and Guo [4], Wang and Li [5],Gao and Xu [6, 7],Huo and Wan [8, 9].Very recently, lattices generated by strongly closed subgraphs in d-bounded distance-regular graphs have been obtained in Gao, Guo and Liu [10], Guo, Gao and Wang [11].

We recall some terminologies and definitions about symplectic space, which can be found in [12] for details.

We assume that

$$K = \begin{pmatrix} 0 & I^{(\nu)} \\ -I^{(\nu)} & 0 \end{pmatrix}.$$

The symplectic group of degree 2ν over F_q,denoted by $Sp_{2\nu}(F_q)$, consists of all $2\nu \times 2\nu$ matrices T over F_q satisfying $TKT^T = K$. The row vector space $F_q^{(2\nu)}$ together with the right multiplication action of $Sp_{2\nu}(F_q)$ is called the 2ν-dimensional symplectic space over F_q. An m-dimensional subspace P is said to be of type (m, s), if PKP^T is of rank $2s$. A subspace of type $(m, 0)$ is called the m-dimensional totally isotropic subspace. Denote by $P^\perp$ the dual subspace of P, i.e.,

$$P^\perp = \{y \in F_q^{2\nu} \mid yKx^T = 0, \ \forall x \in P\}.$$

Let $L_{(R,i)}$ denote the set of all subspaces in a i-dimensional symplectic space. If we partially order $L_{(R,i)}$by reverse inclusion, then $L_{(R,i)}$ is a well known lattice.
For $0 \leq m \leq 2\nu$, suppose $\mathcal{M}(m, s, 2\nu)$ denotes the set of all subspaces of type (m, s) in other 2ν -dimensional symplectic spaces. As for the size of $\mathcal{M}(m, s, 2\nu)$, see [12, Theorem 3.18].

Lemma 2.1. For $1 \leq d \leq \nu - 1$.Let V denote the 2ν-dimensional symplectic space over the finite field F_q, and fix an $(\nu - d)$-dimensional totally isotropic subspace W of V . Then the number of type (i, s) subspace intersecting trivially with W in $W^{\perp}$ is $|\mathcal{M}(i, s, 2d) \mid q^{i(\nu-d)}$,where $2s \leq i \leq d+s$.

Proof. Since the symplectic group acts transitively on the set of subspaces of the same type, we may assume that W has the matrix representation of the form

$$W = {}_i\begin{pmatrix} I^{(\nu-d)} & 0 & 0 & 0 \end{pmatrix}_{\substack{\nu-d \quad d \quad \nu-d \quad d}},$$

If U is a subspaces of type (i, s) of V, satisfying $U \cap W = \{0\}$ and $U \subset W^{\perp}$, then U has a matrix representation of the form

$$U = {}_i\begin{pmatrix} Y_{11} & Y_{12} & 0 & Y_{14} \end{pmatrix}_{\substack{\nu-d \quad d \quad \nu-d \quad d}},$$

where Y_{11} is an $i \times (\nu - d)$ matrix and

$${}_i\begin{pmatrix} Y_{12} & Y_{14} \end{pmatrix}_{\substack{d \quad d}},$$

is an type (i, s) subspace. Hence the desired result follows. □

3. Main results

Theorem 3.1 *Let* $1 \leq d \leq \nu - 1$.Then
(1)$L_O(d, 2\nu)$ is atomic.
(2) $L_O(d, 2\nu)$ is not a geometric lattice .
Proof. (1) Since the set of all the atoms of $L_O(d, 2\nu)$ consists of all the 1-dimensional totally isotropic subspaces, which is intersecting trivially with W and orthogonal to W, then $L_O(d, 2\nu)$ is atomic.
(2) For any $A \in L_O(d, 2\nu)$, define

$$r(A) = \begin{cases} \dim A & \text{if } A \neq V, \\ 2d+1 & \text{otherwise.} \end{cases}$$

Then r is the rank function of $L_O(d, 2\nu)$.

Now suppose that $1 \le d \le \nu - 1$, let

$$A = (\underbrace{1\,0\,0\cdots 0}_{\nu-d}\ \underbrace{1\,0\,0\cdots 0}_{d}\ \underbrace{0\,0\,0\cdots 0}_{\nu-d}\ \underbrace{0\,0\,0\cdots 0}_{d})$$

$$B = (\underbrace{0\,0\,0\cdots 0}_{\nu-d}\ \underbrace{1\,0\,0\cdots 0}_{d}\ \underbrace{0\,0\,0\cdots 0}_{\nu-d}\ \underbrace{0\,0\,0\cdots 0}_{d})$$

then $r(A \vee B) = 2d + 1$ and $r(A \wedge B) = 0$. It follows that

$$r(A \wedge B) + r(A \vee B) = r(A \vee B) = 2d + 1 > r(A) + r(B) = 2.$$

Hence $L_O(d, 2\nu)$ is not a geometric lattice. □

Theorem 3.2 Let $1 \le d \le \nu - 1$.Then

(1)$L_R(d, 2\nu)$ is atomic.

(2) $L_R(d, 2\nu)$is a geometric lattice if and only if $d = 1$ or $d = \nu - 1$.

Proof. (1) Since the symplectic group acts transitively on the set of subspaces of the same type, we may assume that W has the matrix representation of the form

$$W = \begin{pmatrix} I^{(\nu-d)} & 0 & 0 & 0 \end{pmatrix} . \atop \quad \nu-d \quad\ d \quad \nu-d \quad d$$

For each element $A \in L_R(d, 2\nu)$ with $\dim A = l$, since the subgroup fixing W of classical group acts transitively on the set $\{U \mid U \in P \text{ and } \dim U = l\}$, we may assume that

$$A = \begin{pmatrix} 0 & I^{(s)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I^{(s)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I^{(l-2s)} & 0 \end{pmatrix} . \atop {\scriptstyle \nu-d \quad s \quad l-2s \quad d+s-l \quad \nu-d \quad s \quad l-2s \quad d+s-l}$$

Let B and $C_1, C_2, \ldots, C_{l-2s}, D_1, D_2, \ldots, D_{d+s-l}, E_1, E_2, \ldots, E_{d+s-l}$ be the subspaces of type $(2d, d)$ of V with the following matrix representations

$$B = \begin{pmatrix} 0 & I^{(s)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I^{(l-2s)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I^{(d+s-l)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I^{(s)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I^{(l-2s)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(d+s-l)} \end{pmatrix} . \atop {\scriptstyle \nu-d \quad s \quad l-2s \quad d+s-l \quad \nu-d \quad s \quad l-2s \quad d+s-l}$$

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$$C_i = \begin{pmatrix} 0 & I^{(s)} & 0 & 0 & 0 & 0 & 0 & 0 \\ L_i & 0 & I^{(l-2s)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I^{(d+s-l)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I^{(s)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I^{(l-2s)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(d+s-l)} \end{pmatrix}.$$
$$\nu-d \quad s \quad l-2s \quad d+s-l \quad \nu-d \quad s \quad l-2s \quad d+s-l$$

where $1 \leq i \leq l-2s$ and satisfying

$$(L_i)_{p,t} = \begin{cases} 1 & \text{if } p = i, t = 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$D_j = \begin{pmatrix} 0 & I^{(s)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I^{(l-2s)} & 0 & 0 & 0 & 0 & 0 \\ M_j & 0 & 0 & I^{(d+s-l)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I^{(s)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I^{(l-2s)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(d+s-l)} \end{pmatrix}.$$
$$\nu-d \quad s \quad l-2s \quad d+s-l \quad \nu-d \quad s \quad l-2s \quad d+s-l$$

where $1 \leq j \leq d+s-l$ and satisfying

$$(M_j)_{p,t} = \begin{cases} 1 & \text{if } p = j, t = 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$E_k = \begin{pmatrix} 0 & I^{(s)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I^{(l-2s)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I^{(d+s-l)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I^{(s)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I^{(l-2s)} & 0 \\ N_k & 0 & 0 & 0 & 0 & 0 & 0 & I^{(d+s-l)} \end{pmatrix}.$$
$$\nu-d \quad s \quad l-2s \quad d+s-l \quad \nu-d \quad s \quad l-2s \quad d+s-l$$

where $1 \leq k \leq d+s-l$ and satisfying

$$(N_k)_{p,t} = \begin{cases} 1 & \text{if } p = k, t = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then B and $C_1, C_2, \ldots, C_{l-2s}, D_1, D_2, \ldots, D_{d+s-l}, E_1, E_2, \ldots, E_{d+s-l}$ are atoms of $L_R(m, s, d, 2\nu)$ satisfying

$$B \vee C_1 \vee C_2 \vee \ldots \vee C_{l-2s} \vee D_1 \vee D_2 \vee \ldots \vee D_{d+s-l} \vee E_1 \vee E_2 \vee \ldots \vee E_{d+s-l} = A.$$

Therefore, $L_R(d,2\nu)$ is an atomic lattice.

(2) For any $A \in L_R(d,2\nu)$, define

$$r'(A) = \begin{cases} 2d+1-\dim A & \text{if } A \neq V, \\ 0 & A = V \end{cases}$$

Then r' is the rank function of $L_R(d,2\nu)$.

If $d=1$,it is clear that $L_R(d,2\nu)$ is a geometric.

If $d=\nu-1$,for any $A,B \in L_R(d,2\nu)$.

If $\dim(A+B) \leq 2d$ and $W \cap (A+B) \neq \{0\}$, it is clear that $L_R(d,2\nu)$ is a geometric lattice.

If $\dim(A+B) \leq 2d$ and $W \cap (A+B) = \{0\}$,$r(A \wedge B) = 2d+1-\dim(A+B)$,$r(A \vee B) = 2d+1-\dim(A \cap B)$. It follows that

$$r(A \wedge B) + r(A \vee B) = r(A) + r(B).$$

Hence the desired result follows.

If $\dim(A+B) = 2d+1$,then $r(A \wedge B) = r(V) = 0$,$r(A \vee B) = 2d+1-\dim(A \cap B)$. It follows that

$$r(A \wedge B)+r(A \vee B) = 2d+1-\dim(A \cap B) = 4d+2-\dim A-\dim B = r(A)+r(B).$$

Hence the desired result follows.

Now suppose that $2 \leq d \leq \nu-2$.

Let B and C be two subspaces of type $(2d,d)$ of V with the following matrix representations of the form

$$B = \begin{pmatrix} 0 & I^{(d)} & 0 & 0 \\ 0 & 0 & 0 & I^{(d)} \end{pmatrix}.$$
$$\begin{matrix} \nu-d & d & \nu-d & d \end{matrix}$$

and

$$C = \begin{pmatrix} I^{(2)} & 0 & I^{(2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & I^{(d-2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I^{(d)} \end{pmatrix}.$$
$$\begin{matrix} 2 & \nu-d-2 & 2 & d-2 & \nu-d & d \end{matrix}$$

Then B and C are the elements of $L_R(d,2\nu)$ satisfying $r(B \wedge C) = r(V) = 0, r(B \vee C) = 2d+1-(2d-2) = 3$ and

$$r(B \vee C) + r(B \wedge C) = 3 > 2 = r(B) + r(C).$$

It follows that $L_R(d,2\nu)$ is not a geometric lattice whenever $2 \leq d \leq \nu-2$.□

Theorem 3.3. Let $1 \leq d \leq \nu-1$.The Characteristic polynomial of $L_R(d,2\nu)$ is

$$\chi(L_R(d,2\nu),t) = t^{2d+1} - \sum_{i=0}^{2d}\sum_{s=0}^{d} \mid \mathcal{M}(i,s,2d) \mid q^{i(\nu-d)}\chi(L_{(R,i)},t).$$

Proof. For convenience, we write $L = L_R(d, 2\nu)$. For subspaces of type (i, s) $U \in L_R(d, 2\nu)$ let

$$L^U = \{Q \in L \mid Q \geq U\}.$$

Note that $L^V = L$. Since $L^U \simeq L_{(R,i)}, \chi(L^U, t) = \chi(L_{(R,i)}, t)$. Since $\{0\}$-subspace is the maximum element and V is the minimum element in L, the characteristic polynomial of L is

$$\chi(L, t) = \sum_{U \in L} \mu(V, U) t^{2d+1-r(U)}.$$

By the *Möbius* inversion formula

$$t^{2d+1} = \sum_{U \in L} \chi(L^U, t).$$

Hence, By Lemma 2.1 we obtain as

$$\begin{aligned}
\chi(L, t) &= t^{2d+1} - \sum_{U \in L \setminus V} \chi(L^U, t) \\
&= t^{2d+1} - \sum_{U \in L, \dim(U) \leq 2d} \chi(L^U, t) \\
&= t^{2d+1} - \sum_{i=0}^{2d} \sum_{s=0}^{d} \mid \mathcal{M}(i, s, 2d) \mid q^{i(\nu-d)} \chi(L_{(R,i)}, t).
\end{aligned}$$

as desired. □

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References

[1] M. Aigner, Combinatorial Theory, Springer-Verlag, Berlin, 1979.

[2] Z.Wan, Y. Huo, Lattices generated by orbits of subspaces under finite classical groups (in Chinese), 2nd Edition, Science Press, Beijing, 2002.

[3] J. Guo, J. Nan, Lattices generated by orbits of flats under finite affine-symplectic groups,Linear Algebra Appl (2009) doi:10.1016/j.laa.2009.03.002.

[4] K. Wang and J. Guo, Lattices generated by orbits of totally isotropic flats under finite affine-classical groups, Finite Fields Appl 14 (3) (2008) 571-578.

[5] K. Wang and Z. Li, Lattices associated with vector spaces over a finite field, Linear Algebra Appl 429 (2008), 439-446.
[6] Y. Gao, J. Xu, Lattices generated by orbits of subspaces under finite singular pseudo- symplectic groups II, Finite Fields Appl 15(3) (2009) 360-374.
[7] Y. Gao, Lattices generated by orbits of subspaces under finite singular unitary group and its characteristic polynomials, Linear Algebra Appl 368 (2003), 243-268.
[8] Y. Huo and Z. Wan, Lattices generated by transitive sets of subspaces under finite pseudo-symplectic groups, Comm. Algebra 23 (1995), 3753-3777.
[9] Y. Huo and Z. Wan, On the geometricity of lattices generated by orbits of subspaces under finite classical groups, J. Algebra 243 (2001), 339-359.
[10] S. Gao, J. Guo and W. Liu, Lattices generated by strongly closed subgraphs in d- bounded distance-regular graphs, Europ. J. of Combin 28 (2007) 1800-1813.
[11] J. Guo, S. Gao and K. Wang, Lattices generated by subspaces in d-bounded distance-regular graphs, Discrete Math 308 (22)(2008), 5260-5264.
[12] Z. Wan, Geometry of Classical Groups over Finite Fields, 2nd edition, Science Press,Beijing/New York, 2002.