On geodetic and k-geodetic sets in graphs

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Abstract

We investigate the relationship between geodetic sets, k-geodetic sets, dominating sets and independent sets in arbitrary graphs. As a consequence of the study we provide several tight bounds on the geodetic number of a graph.

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1 Introduction

The concepts of a geodetic set and the geodetic number of a graph were introduced by Harary *et al.*, [12] and studied further by several authors [3–7, 9, 10, 16–19]. Some of those works are devoted to study some variations of the concept of a geodetic set. We cite, for instance, studies on k-geodetic sets [10, 17], edge geodetic sets [1], Steiner geodetic sets [19] and geodetic sets in digraphs [8, 9, 14]. Other published articles on this issue are devoted to study geodetic sets in some particular kind of graphs, for instance, the geodetic number of the Cartesian product of graphs was recently studied in [3, 10, 11]. Among the applications of geodetic sets (or geodominating sets) we emphasize the article [18] on communication overlap in networks.

In this paper we contribute to increase the knowledge about geodetic sets. Particularly, we investigate the relationship between geodetic sets, k-geodetic sets, dominating sets and independent sets in arbitrary graphs. As a consequence of the study, we provide several tight bounds on the geodetic number of a graph.

We begin by stating some notation and terminology. In this paper $\Gamma = (V, E)$ denotes a connected simple graph of order $n = |V|$ and size $m = |E|$, and $\overline{\Gamma}$ denotes the complement graph of Γ. We denote two adjacent vertices u and v by $u \sim v$, the degree of a vertex $v \in V$ by $\delta(v)$ and the minimum and maximum degree of Γ by δ and Δ, respectively. For a vertex $v \in V$ and a set $X \subset V$, the number of neighbors v has in X is denoted by $\delta_X(v)$, that is, $\delta_X(v) = |\{u \in X : u \sim v\}|$. The distance $d(u,v)$ between two vertices u and v is the length of a shortest $u-v$ path in Γ. A $u-v$ path of length $d(u,v)$ is called $u-v$ *geodesic.* We define $I[u,v]$ to be the set of all vertices lying on some $u-v$ geodesic of Γ, and for a nonempty set $S \subseteq V$,

$$I[S] = \bigcup_{u,v \in S} I[u,v].$$

A set $S \subseteq V$ is a *geodetic set* in Γ if $I[S] = V$ and a geodetic set of minimum cardinality is called a *minimum geodetic set.* The cardinality of a minimum geodetic set in Γ is called the *geodetic number* of Γ and it is denoted by $g(\Gamma)$. A vertex $v \in V$ is geodominated by a pair $x, y \in V$ if v lies on an $x-y$ geodesic of Γ. Analogously, for an integer $k \geq 1$, a vertex v is k-geodominated by a pair $x, y \in V$ if v lies on an $x-y$ geodesic of Γ and $d(x,y) = k$. A subset $S \subseteq V$ is a (*total*) *k-geodetic set* if each vertex ($v \in V$) $v \in \overline{S} = V \setminus S$ is k-geodominated by some pair of vertices of ($S \setminus \{v\}$) S. The minimum cardinality of a (total) k-geodetic set of Γ is the (*total*) *k-geodetic number* of Γ and it is denoted by ($g_k^t(\Gamma)$) $g_k(\Gamma)$. Note that $g(\Gamma) \leq g_k(\Gamma)$ for every $k \in \{1, ..., D\}$, where D denotes the diameter of Γ. Moreover, if Γ contains total k-geodetic sets, then $g_k(\Gamma) \leq g_k^t(\Gamma)$. A subset $S \subseteq V$ is called a *dominating set* if every vertex $v \in \overline{S}$ is adjacent to an element of S. The minimum cardinality of a dominating set is the *domination number* of Γ and it is denoted by $\gamma(\Gamma)$. The maximum cardinality of an independent set in Γ, which is called the *independence number* of Γ, is denoted by $\alpha(\Gamma)$ and the minimum cardinality of an independent dominating set of Γ is denoted by $i(\Gamma)$. We refer to the book [2] for concepts on graph theory.

2 Results

Theorem 1. *Let Γ be a graph of minimum degree $\delta \geq 2$. If Γ does not contain cycles of length* 3 *or* 5, *then every independent dominating set in Γ is a geodetic set.*

Proof. Let S be an independent dominating set in Γ and let $v \in \overline{S}$. If $\delta_S(v) \geq 2$, then v is 2-geodominated by S. Otherwise, $\delta_S(v) = 1$ and, as a consequence, there exist $u \in S$ and $w \in \overline{S}$ such that $v \sim u$ and $v \sim w$. Moreover, as Γ is a triangle free graph and S is an independent dominating set, there exists $z \in S$ different from u such that $z \sim w$. If $d(u,z) = 2$, then there exists $y \in \overline{S}$ different from v and w which is adjacent to u and z. Therefore, $uyzwvu$ is a cycle of length 5, a contradiction. In consequence, $d(u,z) = 3$ and v is 3-geodominated by the vertices u and z. □

Corollary 2. *Let Γ be a graph of minimum degree $\delta \geq 2$. If Γ does not contain cycles of length* 3 *or* 5, *then $g(\Gamma) \leq i(\Gamma)$.*

One example of a graph where $g(\Gamma) = i(\Gamma) = 2$ is the 3-cube graph $\Gamma = K_2 \times K_2 \times K_2$.

Let D denote the diameter of Γ. It was shown in [4] that $g(\Gamma) \leq n - D + 1$. Let us see the following result which, imposing an additional condition, improves this bound.

Remark 3. *Let Γ be a graph of order n and diameter D, and let A be the set of all vertices on a path which determines the diameter. If $V \setminus A$ contains subsets S_i, $i \in \{1, ..., k\}$, such that $S_i \cap S_j = \emptyset$, $i \neq j$, and the induced subgraphs $\langle S_i \rangle$ are isomorphic to P_3, then*

$$g(\Gamma) \leq n - D - k + 1.$$

Proof. Let $A = \{u_0, ..., u_D\}$ and let $S_i = \{v_i, w_i, z_i\}$, $i \in \{1, ..., k\}$, such that $v_i \sim w_i \sim z_i$. Then $S = V \setminus \{w_1, ..., w_k, u_1, u_2, ..., u_{D-1}\}$ is a geodetic set of cardinality $n - D - k + 1$. □

Recall that a vertex v is an *extreme vertex* in a graph if the subgraph induced by its neighborhood is complete. It is known that every geodetic set contains all extreme vertices of a graph.

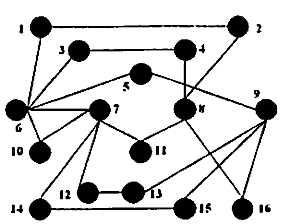

Figure 1: $\{6, 7, 8, 9, 10\}$ is a minimum geodetic set, but its complement is not a dominating set. Note that the vertex 10 is an extreme vertex.

Remark 4. *Let Γ be a graph without extreme vertices. If S is a minimum geodetic set (or a minimum 2-geodetic set) in Γ, then $\overline{S}$ is a dominating set and, as a consequence, $g_2(\Gamma) \leq n - \gamma(\Gamma)$.*

Proof. We suppose that $\overline{S}$ is not a dominating set, that is, there exists $v \in S$ such that $N(v) = \{u \in V : u \sim v\} \subset S$. Since v is not an extreme vertex, there exist two nonadjacent vertices $x, y \in N(v)$, in consequence, v is geodominated (2-geodominated) by x and y. Hence, $S' = S \setminus \{v\}$ is a geodetic set (2-geodetic set), a contradiction. □

For the cycle graph C_n of order $n = 5, 7$ we have $g_2(C_5) = 3$ and $g_2(C_7) = 4$, in consequence, $g_2(C_n) = n - \gamma(C_n)$.

Remark 5. *For any nonempty graph Γ of order n without extreme vertices,*

$$g_2(\Gamma) \leq \left\lfloor \frac{n\Delta}{\Delta + 1} \right\rfloor.$$

Proof. It is well-known (see, for instance, [13]) that for any graph of maximum degree Δ and order n, the domination number is bounded by $\gamma(\Gamma) \geq \frac{n}{\Delta+1}$. Therefore, by Remark 4 we have, $g_2(\Gamma) \leq n - \gamma(\Gamma) \leq n - \frac{n}{\Delta+1} = \frac{n\Delta}{\Delta+1}$. □

The above bound is tight. It is achieved for the cycle graph $\Gamma = C_5$ where $\Delta = 2$ and $g_2(\Gamma) = 3$. As the next result shows, the above bound can be improved for the case of triangle free graphs.

Theorem 6. *For any triangle free graph Γ of order n, minimum degree $\delta \geq 2$ and independence number $\alpha(\Gamma)$,*

(*i*) $g_2(\Gamma) \leq \min\left\{n - \alpha(\Gamma), \left\lfloor \frac{n+\alpha(\Gamma)}{2} \right\rfloor\right\}$,

(*ii*) $g_2(\Gamma) \leq \lfloor \frac{2n}{3} \rfloor$.

Proof. Let S be an independent set in Γ. Since Γ is a triangle free graph and $\delta \geq 2$, every vertex in S has two nonadjacent neighbors in $\overline{S}$. Therefore, $\overline{S}$ is a 2-geodetic set. As a consequence,

$$g_2(\Gamma) \leq n - \alpha(\Gamma). \tag{1}$$

On the other hand, we can consider a partition $\{X, Y\}$ of $\overline{S}$ such that the edge-cut set between X and Y has maximum cardinality. Suppose $|X| \leq |Y|$. If we take $W = S \cup X$, for every $y \in Y$ it is satisfied $\delta_Y(y) + \delta_W(y) = \delta(y) \geq 2$ and, as S is a dominating set, $\delta_W(y) = \delta_X(y) + \delta_S(y) \geq \delta_Y(y) + 1$, therefore, $\delta_W(y) \geq 2$. Since Γ is a triangle free graph, W is a 2-geodetic set, in consequence, $g_2(\Gamma) \leq |W| = |X| + \alpha(\Gamma)$. Using that $n = |X| + |Y| + \alpha(\Gamma) \geq 2|X| + \alpha(\Gamma)$, we obtain $2g_2(\Gamma) - \alpha(\Gamma) \leq n$. Hence,

$$g_2(\Gamma) \leq \frac{n + \alpha(\Gamma)}{2}. \tag{2}$$

Therefore, (i) follows. We note that (ii) is a direct consequence of combining (1) and (2). □

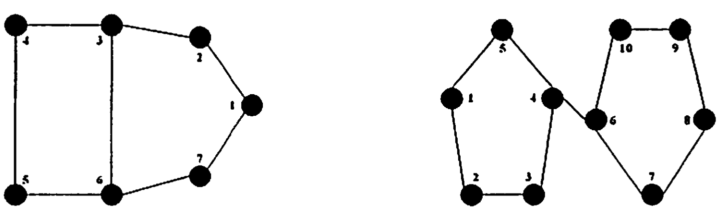

Figure 2: $\{2, 4, 5, 7\}$ is a 2-geodetic set in the left hand side graph and $\{1, 2, 4, 6, 8, 9\}$ is a 2-geodetic set in the right hand side graph.

The above bounds are tight. For instance, (i) and (ii) are achieved for the graphs in Figure 2. For the complete bipartite graphs $K_{r,s}$ ($r, s \geq 2$) we have $\alpha(\Gamma) = \max\{r, s\}$ and $g_2(\Gamma) = \min\{r, s, 4\}$. Hence, if $\min\{r, s\} \leq 4$, then $g_2(K_{r,s}) = n - \alpha(K_{r,s})$.

Note that in the above proof we have shown the following useful remark.

Remark 7. *Let Γ be a triangle free graph Γ of order n and minimum degree $\delta \geq 2$. If S is an independent set in Γ, then $\overline{S}$ is a 2-geodetic set.*

Theorem 8. *Let Γ be a graph of minimum degree $\delta \geq 2$. If Γ does not contain cycles of length 3 or 5, then the vertex set of Γ is partitionable into a geodetic set and a 2-geodetic set.*

Proof. The result is a direct consequence of combining Remark 7 and Theorem 1. □

Note that, from the above results, we obtain the following inequality chain for graphs of minimum degree $\delta \geq 2$ without cycles of length 3 or 5:

$$g(\Gamma) \leq i(\Gamma) \leq \alpha(\Gamma) \leq n - g_2(\Gamma) \leq n - g(\Gamma).$$

As a consequence,

$$g(\Gamma) \leq \frac{n}{2}.$$

We consider next the case of total 2-geodetic sets in a graph.

Theorem 9. *Let Γ be a graph of order n, minimum degree δ and maximum degree Δ.*

(i) If S is a total 2-geodetic set in Γ and $\overline{S}$ is an independent set, then $|S| \geq \left\lceil \frac{n\delta}{\Delta+\delta-2} \right\rceil$.

(ii) If S is an independent 2-geodetic set in Γ, then $|S| \geq \left\lceil \frac{2n}{\Delta+2} \right\rceil$.

(iii) $g_2^t(\Gamma) \geq \left\lceil \frac{2n}{\Delta} \right\rceil$.

Proof. If S is a total 2-geodetic set, then $\delta_S(v) \geq 2$, for every $v \in S$. Moreover, if $\overline{S}$ is an independent set, then $\delta_S(v) = \delta(v)$, for every $v \in \overline{S}$. Thus,

$$\begin{aligned}
\Delta|S| &\geq \sum_{v\in S} \delta(v) \\
&= \sum_{v\in S} \delta_S(v) + \sum_{v\in S} \delta_{\overline{S}}(v) \\
&= \sum_{v\in S} \delta_S(v) + \sum_{v\in \overline{S}} \delta_S(v) \\
&= \sum_{v\in S} \delta_S(v) + \sum_{v\in \overline{S}} \delta(v) \\
&\geq 2|S| + \delta(n - |S|).
\end{aligned}$$

Therefore, (i) follows. On the other hand, if S is an independent set, then $\delta_{\overline{S}}(v) = \delta(v)$, for every $v \in S$ and, if S is a 2-geodetic set, then $\delta_S(v) \geq 2$, for every $v \in \overline{S}$. Hence,

$$\Delta|S| \geq \sum_{v \in S} \delta(v) = \sum_{v \in S} \delta_{\overline{S}}(v) = \sum_{v \in \overline{S}} \delta_S(v) \geq 2(n - |S|).$$

Thus, (ii) follows. Finally, if S is a total 2-geodetic set, then $\delta_S(v) \geq 2$, for every $v \in V$. Thus,

$$\Delta|S| \geq \sum_{v \in S} \delta(v) = \sum_{v \in S} \delta_S(v) + \sum_{v \in \overline{S}} \delta_S(v) \geq 2|S| + 2(n - |S|).$$

Therefore, (iii) follows. □

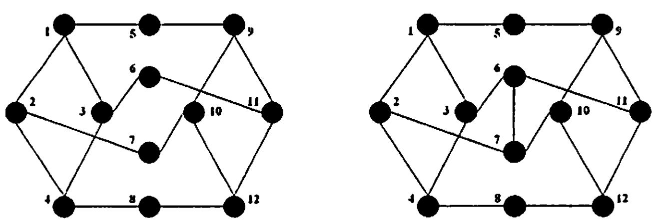

Figure 3: In both graphs $S = \{1, 2, 3, 4, 9, 10, 11, 12\}$ is a total 2-geodetic set. In the left hand side graph, $\overline{S}$ is an independent set.

In the left hand side graph of Figure 3, the set $S = \{1, 2, 3, 4, 9, 10, 11, 12\}$ is a total 2-geodetic set and $\overline{S}$ is an independent set. In this case, Theorem 9 (i) leads to the exact value of $|S|$. Bound (ii) is achieved, for instance, for any cycle graph C_{2t}, $t \geq 2$, and bound (iii) is achieved for both graphs in Figure 3.

Theorem 10. *For any graph* Γ *of order* n *and maximum degree* Δ,

$$g_k(\Gamma) \geq \left\lceil \frac{2n}{\Delta(\Delta - 1)^{k-1}(k - 1) + 2} \right\rceil.$$

Proof. Let S be a k-geodetic set in Γ. We know that every vertex in $\overline{S}$ lies on a path of length k which begins and ends in S. For every vertex u, the number of paths of length k beginning in u is bounded above by $\Delta(\Delta - 1)^{k-1}$. If we consider all vertices in S, we have that the number of path of length k beginning and ending in S is at most $\frac{|S|\Delta(\Delta-1)^{k-1}}{2}$, and the number of vertices of $\overline{S}$ which can lie on those paths is bounded above by $\frac{|S|\Delta(\Delta-1)^{k-1}(k-1)}{2}$. Therefore, $n - |S| \leq \frac{|S|\Delta(\Delta-1)^{k-1}(k-1)}{2}$. □

The above bound is achieved for the cycle graph C_{kt} of order kt where $g_k(C_{kt}) = t$.

2.1 On geodetic sets in Γ and dominating sets in $\overline{\Gamma}$

Theorem 11. *If S is a geodetic set in Γ and it is not a 2-geodetic set, then S is a dominating set in $\overline{\Gamma}$.*

Proof. We suppose that S is not a dominating set in $\overline{\Gamma}$. Then, there exists $v \in \overline{S}$ adjacent to every vertex of S in Γ, so the distance between any two vertices in S is at most 2. As a consequence, S is a 2-geodetic set in Γ, a contradiction. □

Corollary 12. *If $g(\Gamma) \neq g_2(\Gamma)$, then $g(\Gamma) \geq \gamma(\overline{\Gamma})$.*

Corollary 13. *For any graph Γ of diameter $D \geq 3$ and every $k \in \{3, ..., D\}$, $g_k(\Gamma) \geq \gamma(\overline{\Gamma})$.*

The above two bounds are tight. For the cycle graph of order 6 we have $g_3(C_6) = g(C_6) = 2 = \gamma(\overline{C_6})$ and, for the 3-cube graph, we have $g_3(Q_3) = g(Q_3) = 2 = \gamma(\overline{Q_3})$.

One example of a graph where, $g_2(\Gamma) < \gamma(\overline{\Gamma})$ is the graph obtained by removing one edge from the complete graph K_n, $n \geq 5$. The resultant graph satisfies $g_2(\Gamma) = 2$ and $\gamma(\overline{\Gamma}) = n - 1$.

It is easy to see that if a graph Γ has diameter two or three, then every geodetic set in Γ is a dominating set. As a consequence, $g(\Gamma) \geq \gamma(\Gamma)$. For instance, we take the 3-cube graph Q_3 and the cycle graph C_4, where $g(Q_3) = \gamma(Q_3) = 2 = g(C_4) = \gamma(C_4)$.

Theorem 14. *If Γ is a graph of diameter $D > 4$, then every geodetic set in Γ is a dominating set in $\overline{\Gamma}$.*

Proof. We suppose that $S \subset V$ is not a dominating set in $\overline{\Gamma}$. Then there exists a vertex of $\overline{S}$ adjacent, in Γ, to every vertex of S. As a consequence, for every $x, y \in S$, $d(x, y) \leq 2$, and, if S is a geodetic set in Γ, then every vertex in $\overline{S}$ is adjacent to some vertex of S. Thus, for every $a, b \in \overline{S}$, $d(a, b) \leq 4$. As a consequence, $D \leq 4$, a contradiction. □

Corollary 15. *For any graph Γ of diameter $D > 4$, $g(\Gamma) \geq \gamma(\overline{\Gamma})$.*

Examples of equality in Corollary 15 are the cycle graphs C_{2t} with $t \geq 5$ and the grid graphs $P_s \times P_q$ with $s + q \geq 7$. For both families of graphs we have $g(\Gamma) = 2 = \gamma(\overline{\Gamma})$.

2.2 Geodetic sets in Cartesian products of graphs

We recall that the Cartesian product of two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ is the graph $\Gamma_1 \times \Gamma_2 = (V, E)$, such that $V = \{(a, b) : a \in V_1,\ b \in V_2\}$ and two vertices $(a, b) \in V$ and $(c, d) \in V$ are adjacent in $\Gamma_1 \times \Gamma_2$ if and only if, either ($a = c$ and $bd \in E_2$) or ($b = d$ and $ac \in E_1$).

The geodetic number of the Cartesian product of graphs has been studied in [3, 11]. In the refereed articles the authors obtain relationships between the geodetic number of the Cartesian product of graphs and the geodetic number of its factors. In this section we obtain an upper bound on the geodetic number of the Cartesian product of graphs in terms of the order of its factors. We begin with the following well known lemma [15].

Lemma 16. *Let $\Gamma_i = (V_i, E_i)$ be a connected graph, $i \in \{1,2\}$. For any $(a,b),(c,d) \in V_1 \times V_2$,*

- $I_{\Gamma_1 \times \Gamma_2}[(a,b),(c,d)] = I_{\Gamma_1}[a,c] \times I_{\Gamma_2}[b,d]$.
- $I_{\Gamma_1 \times \Gamma_2}[(a,b),(c,d)] = I_{\Gamma_1 \times \Gamma_2}[(a,d),(c,b)]$.

Theorem 17. *Let $\Gamma_i = (V_i, E_i)$ be a connected (nontrivial) graph, $i \in \{1,2\}$. For any $u \in V_1$ and $v \in V_2$,*

$$S = (V_1 \times \{v\}) \cup (\{u\} \times V_2) \setminus \{(u,v)\} \subset V_1 \times V_2$$

is a geodetic set in $\Gamma_1 \times \Gamma_2$.

Proof. Let $u \in V_1$, $v \in V_2$ and $S' = (V_1 \times \{v\}) \cup (\{u\} \times V_2)$. By Lemma 16 we have,

$$\bigcup_{(u,b),(a,v)\in S'} I_{\Gamma_1 \times \Gamma_2}[(u,b),(a,v)] = \bigcup_{a\in V_1} I_{\Gamma_1}[u,a] \times \bigcup_{b\in V_2} I_{\Gamma_2}[b,v] = V_1 \times V_2.$$

Now, let $S = S' \setminus \{(u,v)\}$ and suppose $(a,b) \in \overline{S}$. If $(a,b) \neq (u,v)$, from $d_{\Gamma_1\times\Gamma_2}((a,v),(u,b)) = d_{\Gamma_1}(a,u) + d_{\Gamma_2}(v,b)$ we have $(a,b) \in I_{\Gamma_1\times\Gamma_2}[(a,v),(u,b)]$. Now suppose $(a,b) = (u,v)$. Since Γ_1 and Γ_2 are connected graphs, there exist $x \in V_1$ and $y \in V_2$, such that $u \sim x$ and $v \sim y$, so $(u,v) \in I_{\Gamma_1\times\Gamma_2}[(x,v),(u,y)]$. Therefore, S is a geodetic set in $\Gamma_1 \times \Gamma_2$. □

Corollary 18. *For any connected graph Γ_i of order $n_i \geq 2$, $i \in \{1,2\}$,*

$$g(\Gamma_1 \times \Gamma_2) \leq n_1 + n_2 - 2.$$

One example where $g(\Gamma_1 \times \Gamma_2) = n_1 + n_2 - 2$ is $\Gamma = K_n \times K_2$, $n \geq 2$, since $g(K_n) = n$ and $g(K_n \times K_2) = n$.

2.3 Geodetic sets in line graphs

The line graph $\mathcal{L}(\Gamma)$ of a simple graph Γ is obtained by associating a vertex with each edge of the graph Γ and connecting two vertices with an edge if and only if the corresponding edges of Γ meet at one endvertex.

Theorem 19. *Let $\Gamma = (V,E)$ be a connected graph, let $S \subset E$ and $X = \cup_{e\in S}\{u \in V : u \in e\}$. If S is a geodetic set in $\mathcal{L}(\Gamma)$, then X is a geodetic set in Γ. Moreover, if S is a k-geodetic set in $\mathcal{L}(\Gamma)$, then X is a $(k-1)$-geodetic set in Γ.*

Proof. For two vertices e_1, e_2 of $\mathcal{L}(\Gamma)$, such that $d(e_1,e_2) = k$, we define the interval $I^k_{\mathcal{L}(\Gamma)}[e_1,e_2]$ to be the set of all vertices lying on some $e_1 - e_2$ geodesic of $\mathcal{L}(\Gamma)$. As Γ is connected, for every $u \in V \setminus X$ there exists $e = uv \in E \setminus S$. Since S is a geodetic set in $\mathcal{L}(\Gamma)$, there exists an integer k and edges $e_1, e_2 \in S$, such that $e \in I^k_{\mathcal{L}(\Gamma)}[e_1,e_2]$. Thus, there exist an endvertex in e_1 and other endvertex in e_2, say u_1, u_2, such that $u \in I^{k-1}_{\Gamma}[u_1,u_2]$. Therefore, the results follow. □

Corollary 20. *For any graph* Γ, $g(\Gamma) \leq 2g(\mathcal{L}(\Gamma))$ *and* $g_{k-1}(\Gamma) \leq 2g_k(\mathcal{L}(\Gamma))$.

The above bounds are attained, for instance, in the case of the graph Γ of Figure 4, where $g(\Gamma) = g_3(\Gamma) = 4 = 2g_4(\mathcal{L}(\Gamma)) = 2g(\mathcal{L}(\Gamma))$. Other examples are the cycle graphs of order $n = 2t$, where $g_{t-1}(C_{2t}) = 4 = 2g_t(\mathcal{L}(C_{2t}))$.

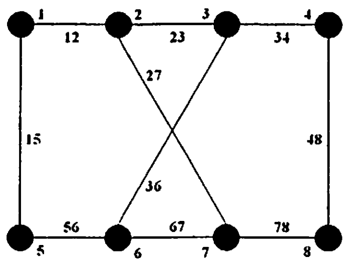

Figure 4: $\{1,4,5,8\}$ is a 3-geodetic set in Γ and $\{15,48\}$ is a 4-geodetic set in $\mathcal{L}(\Gamma)$.

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