The sum numbers and the integral sum numbers of $\overline{C_n}$ and $\overline{W_n}$

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Abstract Let $G = (V, E)$ be a simple graph. N and Z denote the set of all positive integers and the set of all integers respectively. The *sum graph* $G^+(S)$ of a finite subset $S \subset N$ is the graph (S, E) with $uv \in E$ if and only if $u + v \in S$. G is a sum graph if it is isomorphic to the sum graph of some $S \subset N$. The *sum number* $\sigma(G)$ of G is the smallest number of isolated vertices, which result in a sum graph when added to G. By extending N to Z, the notions of the *integral sum graph* and the *integral sum number* of G are got respectively. In this paper, we prove that $\zeta(\overline{C_n}) = \sigma(\overline{C_n}) = 2n - 7$ and that $\zeta(\overline{W_n}) = \sigma(\overline{W_n}) = 2n - 8$ for $n \geq 7$.

Keywords The sum graph; The integral sum graph; The sum number; The integral sum number; Cycle; Path; Wheel.

1. Introduction

Let $G = (V, E)$ be a simple graph with the vertex set V and the edge set E. The concepts of the *sum graph* and *integral sum graph* were introduced by F.Harary in [2] and [3]. N and Z denote the set of all positive integers and the set of all integers respectively. The *sum graph* $G^+(S)$ of a finite subset $S \subset N$ is the graph (S, E) with $uv \in E$ if and only if $u + v \in S$. G is said to be a *sum graph* if it is isomorphic to a sum graph of some $S \subset N$ and it is said that S

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gives a *sum labelling* for G in [12]. The *sum number* $\sigma(G)$ of G is the smallest number of isolated vertices, which result in a sum graph when added to G.

F.Harary also introduced the corresponding notions of the *integral sum graph* and the *integral sum number* $\zeta(G)$ of G, by extending N to Z in the above definitions. Obviously $\zeta(G) \leq \sigma(G)$.

To simplify the notations, throughout this paper we may assume that the vertices of G are identified with their labels.

The vertex $w \in V$ is *working* if the label of w corresponds to an edge $uv \in E$. G is *exclusive* if none of the vertices in V is working.

Besides, the complement $\overline{G}$ of G with order n is the graph $K_n - E(G)$ with the vertex set V and the edge set $E(K_n) - E(G)$. A cycle $C_n = a_1a_2\cdots a_na_1$ is a graph with the vertex side $\{a_1, a_2, a_3, \cdots, a_{n-1}, a_n\}$ and the edge side $\{a_1a_2, a_2a_3, \cdots, a_{n-1}a_n, a_na_1\}$. A wheel W_n is a graph with the vertex set $\{c, a_1, a_2, a_3, \cdots, a_{n-1}, a_n\}$ and the edge set $\{ca_1, ca_2, \cdots, ca_n\} \cup \{a_1a_2, a_2a_3, \cdots, a_{n-1}a_n, a_na_1\}$. It is obvious that $\overline{W_n} = \overline{C_n} \cup K_1$.

Finally, some useful results are obtained as follow.

Lemma 1([8]) $\zeta(K_n) = \sigma(K_n) = 2n - 3$ for $n \geq 4$.

Lemma 2([8]) $\zeta(C_n) = \begin{cases} 3, & n = 4, \\ 0, & n \neq 4. \end{cases}$; $\zeta(W_n) = \begin{cases} 5, & n = 3, \\ 0, & n \neq 3. \end{cases}$

Lemma 3([2]) For $n \geq 3$, $\quad \sigma(C_n) = \begin{cases} 2, & n \neq 4, \\ 3, & n = 4. \end{cases}$

Lemma 4([11][7]) For $n \geq 3$, $\sigma(W_n) = \begin{cases} \frac{n}{2} + 2, & n \text{ even}, \\ n, & n \text{ odd}. \end{cases}$

Lemma 5([1]) $\sigma(P_n) = 1$ and $\zeta(P_n) = 0$ for $n \geq 2$.

In this paper, we determine the sum numbers and the integral sum numbers of $\overline{C_n}$ and $\overline{W_n}$ for $n \geq 7$.

2. Main results

Let $\overline{C_n} = (V, E)$ and $S = V \cup C$, where $C_n = a_1a_2\cdots a_na_1$ and C is the isolated vertex set. Let $|a_n| = \max\{|a| : a \in V\}$ with $a_n \in V$. It is clear that $\overline{C_i} = \overline{K_i}$ $(i = 1, 2, 3)$, $\overline{C_4} = 2K_2$ and $\overline{C_5} = C_5$. So $\zeta(\overline{C_i}) = \sigma(\overline{C_i}) = 0$ $(i = 1, 2, 3)$; $\sigma(\overline{C_4}) = 1$, $\zeta(\overline{C_4}) = 0$ and $\sigma(\overline{C_5}) = 2$, $\zeta(\overline{C_5}) = 0$. Moreover, we obtained the result that $\zeta(\overline{C_6}) = \sigma(\overline{C_6}) = 4$ in [13]. In this section, we only consider $n \geq 7$.

Lemma 2.1 $\overline{C_n}$ is not an integral sum graph for $n \geq 7$.

Proof: By contradiction. If $\overline{C_n}$ is an integral sum graph for $n \geq 6$ then $a_i + a_j \in V$ for any $a_ia_j \in E$. Since $a_1a_2 \notin E$, $0 \notin V$.

Assume that $a_n > 0$ (A similar argument work for $a_n < 0$). According to the choice of a_n, $a_n + a_k \in V$, $a_n + a_k > 0$ and $a_k < 0$ for $k = 2, 3, \cdots, n-2$. So there exist at least $n-2$ distinct positive vertices $a_n, a_n + a_2, \cdots, a_n + a_{n-2}$ in V. On the other hand, there also exist at least $n-3$ distinct negative vertices $a_2, a_3, \cdots, a_{n-2}$ in V. So there are at most three positive vertices in V. Thus, $n - 2 \leq 3$, contradicting $n \geq 7$. Therefore, $\overline{C_n}$ is not an integral sum graph for $n \geq 7$. □

Lemma 2.2 There exists at least one edge $a_n a_{j_0} \in E$ such that $a_n + a_{j_0} \in C$ for $n \geq 7$.

Proof: By contradiction. Suppose to the contrary that $a_n + a_j \in V$ for any $a_j \in V - \{a_1, a_{n-1}, a_n\}$. Assume that $a_n > 0$ (A similar argument work for $a_n < 0$). According to the choice of a_n, $a_n + a_j > 0$ and $a_j < 0$. Then there are at least $n-2$ distinct positive vertices $\{a_n, a_n + a_2, \cdots, a_n + a_{n-2}\}$. Meanwhile, there are at least $n-3$ distinct negative vertices $\{a_2, a_3, \cdots, a_{n-2}\}$. So $(n-2) + (n-3) \leq n$, i.e., $n \leq 5$, contradicting $n \geq 7$. □

Lemma 2.3 $\{a_n + a_2, \cdots, a_n + a_{n-2}\} \subseteq C$ for $n \geq 7$.

Proof: By contradiction. If not, then there exist $a_n a_{j_1} \in E$ and $a_{k_1} \in V$ such that $a_n + a_{j_1} = a_{k_1}$ for $n \geq 7$. Assume that $a_n > 0$ (A similar argument work for $a_n < 0$). According to the choice of a_n, $a_{j_1} < 0$ and $a_{k_1} > 0$.

By Lemma 2.2, there exists one edge $a_n a_{j_0} \in E$ such that $a_n + a_{j_0} \in C$ for $n \geq 7$. For any $a_j \in V - \{a_1, a_n, a_{n-1}\}$, $(a_n + a_j) + a_{j_0} = (a_n + a_{j_0}) + a_j \notin S$. Thus, $a_n + a_j \in \{a_{j_0-1}, a_{j_0}, a_{j_0+1}\} \cup C$. Then there are at most three distinct vertices $a_{j'}, a_{j''}, a_{j'''} \in V - \{a_1, a_n, a_{n-1}, a_{j_0}\}$ such that $\{a_n + a_{j'}, a_n + a_{j''}, a_n + a_{j'''}\} \subseteq \{a_{j_0-1}, a_{j_0}, a_{j_0+1}\}$.

Let $V_0 = \{a_1, a_n, a_{n-1}\} \cup \{a_{k_1-1}, a_{k_1}, a_{k_1+1}\}$. For any $a_l \in V - V_0$, $a_{k_1} + a_l \in S$ and $a_n + a_l \in S$. Since $a_{k_1} + a_l = (a_n + a_{j_1}) + a_l = (a_n + a_l) + a_{j_1} \in S$, $a_n + a_l \in V - \{a_{j_1-1}, a_{j_1}, a_{j_1+1}\}$. That is, there are at least $n-5$ distinct positive vertices a_n and $a_n + a_l$ with $a_l \in V - V_0$. Meanwhile, there are at least $n-6$ distinct negative vertices $a_l \in V - V_0$. So $n - 6 \leq 3$, i.e., $n \leq 9$.

By the above, $a_n + a_l \in \{a_{j_0-1}, a_{j_0}, a_{j_0+1}\}$ for any $a_l \in V - V_0$. So we only need to prove $|\{a_{j'}, a_{j''}, a_{j'''}\}| = 0$ for $n \leq 9$.

If $n = 9$ then we assume that $a_n + a_x = a_y$ with $a_x \in V - \{a_1, a_n, a_{n-1}, a_{j_0}\}$ and $a_y \in \{a_{j_0-1}, a_{j_0}, a_{j_0+1}\}$. Since $n = 9$, there exists one vertex $a_z \in V - (\{a_1, a_n, a_{n-1}, a_{j_0}\} \cup \{a_{j'}, a_{j''}, a_{j'''}\})$ such that $a_n + a_z \in C$ and $a_z a_y \in E$. So $a_z + a_y = a_z + (a_n + a_x) = (a_n + a_z) + a_x \in S$, contradicting $a_n + a_z \in C$.

If $n = 8$ then $|V_0| \neq 4$.

If $|V_0| = 6$ then $|V - V_0| \geq 3$, that is, $n \geq 9$, a contradiction.

If $|V_0| \in \{3, 5\}$, then there exist two vertices $a_{x_1}, a_{x_2} \in V - (V_0 \cup \{a_{j_0}, a_{j_1}\})$. If $a_n + a_{x_1} \in V$ and $a_n + a_{x_2} \in V$ then we may assume that $a_n + a_{x_1} = a_{n-1}$ and $a_{n-1} a_{k_1-1} \in E$. So $(a_{n-1} + a_{k_1-1}) + a_{x_1} = (a_n + a_{x_1}) + a_{k_1-1} = a_{n-1} + a_{k_1-1} \in S$, contradicting $a_{n-1} + a_{k_1 - 1} \in C$. If there exists one vertex $a_x \in \{a_{x_1}, a_{x_2}\}$ such that $a_n + a_x \in C$ then $a_x a_{k_1} \in E$. So $a_x + a_{k_1} = a_x + (a_n + a_{j_1}) = (a_n + a_x) + a_{j_1} \in S$, contradicting $a_n + a_x \in C$.

If $n = 7$ then $|V_0| \neq 3$.

If $|V_0| = 6$ then $|V - V_0| \geq 3$, that is, $n \geq 9$, a contradiction.

If $|V_0| = 4$ then there exists $a_x \in V - V_0$ such that $a_n + a_x \in C$ and $a_x a_{k_1} \in E$. So $a_x + a_{k_1} = a_x + (a_n + a_{j_1}) = (a_n + a_x) + a_{j_1} \in S$, contradicting $a_n + a_x \in C$.

If $|V_0| = 5$ then $a_{k_1} = a_{j_0}$ and $a_{k_1+1} = a_{n-1}$. So there exists only one vertex $a_x \in V - V_0$ such that $a_n + a_x \in \{a_{j_0-1}, a_{j_0+1}\}$. By the symmetry of $\overline{C_7}$ and the above results, we can assume that $a_{k_1} = a_{j_0} = a_5$, $a_{j_1} = a_2$ and $a_x = a_3$ in Figure 1. So $a_7 + a_3 \in \{a_4, a_6\}$.

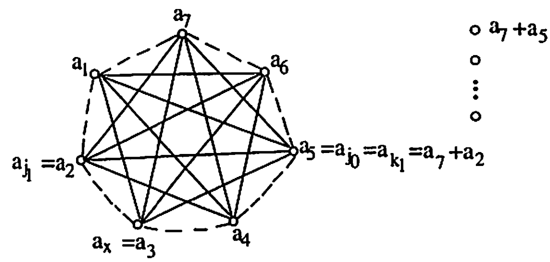

Figure 1

(1) Since $a_7 + a_5 \in C$, $(a_7 + a_5) + a_1 = (a_1 + a_5) + a_7 \notin S$. So $a_1 + a_5 \in \{a_6, a_7\} \cup C$.

(2) Since $a_7 + a_5 \in C$, $(a_7 + a_5) + a_3 = (a_3 + a_5) + a_7 \notin S$. So $a_3 + a_5 \in \{a_1, a_6, a_7\} \cup C$.

(3) Since $a_7 + a_5 \in C$, $(a_7 + a_5) + a_4 = (a_7 + a_4) + a_5 \notin S$. So $a_7 + a_4 \in \{a_5, a_6\} \cup C$.

If $a_7 + a_3 = a_6$ then $a_6 + a_4 = (a_7 + a_3) + a_4 = (a_7 + a_4) + a_3 \in S$. By (3), $a_7 + a_4 = a_5$, contradicting $a_7 + a_2 = a_5$.

If $a_7 + a_3 = a_4 > 0$ then $a_7 + a_4 \in C$.

(4) Since $a_6 + a_4 = a_6 + (a_7 + a_3) = a_7 + (a_6 + a_3) \in S$, $a_6 + a_3 = a_2$.

(5) Since $a_1 + a_4 = a_1 + (a_7 + a_3) = a_7 + (a_1 + a_3) \in S$, $a_1 + a_3 \in \{a_2, a_5\}$.

If $a_1 + a_3 = a_2$ then $a_1 + a_4 = a_2 + a_7 = a_5$ (since $a_7 + a_3 = a_4$). So $(a_7 + a_4) + a_1 = a_7 + (a_1 + a_4) = a_7 + a_5 \in S$, contradicting $a_7 + a_4 \in C$. Thus, $a_1 + a_3 = a_5$.

Since $a_7 + a_3 = a_4$ and $a_1 + a_3 = a_5$, $a_7 + a_5 = a_1 + a_4 \in C$. So $(a_1 + a_4) + a_5 = (a_1 + a_5) + a_4 \notin S$. By (1), $a_1 + a_5 \in C$. So $(a_1 + a_5) + a_3 = (a_3 + a_5) + a_1 \notin S$. By (2), $a_3 + a_5 \in \{a_7\} \cup C$. If $a_3 + a_5 = a_7$ then $a_4 + a_5 = 2a_7$, contradicting the choice of a_7. Thus, $a_3 + a_5 \in C$.

Since $a_6 + a_3 = a_2$, $a_2 + a_5 = (a_3 + a_6) + a_5 = (a_3 + a_5) + a_6 \in S$, contradicting $a_3 + a_5 \in C$.

Thus, Lemma 2.3 holds. □

Lemma 2.4 For each $a_k a_l \in E - \{a_n a_2, \cdots, a_n a_{n-2}\}$, $a_k + a_l \in C$ with $n \geq 7$.

Proof: Assume that $a_n > 0$ (A similar argument work for $a_n < 0$). Then $a_n > 0$. Let $E_n = \{a_n a_2, \cdots, a_n a_{n-2}\}$.

(i) Try to prove $a_k + a_l \in C$ for any $a_k a_l \in E - (E_n \cup \{a_1 a_{n-1}\})$ with $n \geq 7$. In fact, if $a_k a_l \neq a_1 a_{n-1}$ then there must exist at least one vertex which is adjacent to a_n in $\overline{C_n}$ for any $a_k a_l \in E - E_n$. We may assume $a_k a_n \in E$ by the symmetry of $\overline{C_n}$. So $a_k + a_l \in S$. By lemma 2.3, $a_n + a_k \in C$. Since $(a_n + a_k) + a_l = a_n + (a_k + a_l) \notin S$, $a_k + a_l \in \{a_1, a_{n-1}, a_n\} \cup C$. So there exist at most three edges $a_k a_{l_i} \in E - E_n$ $(i = 1, 2, 3)$ such that $\{a_k + a_{l_1}, a_k + a_{l_2}, a_k + a_{l_3}\} \subseteq \{a_1, a_{n-1}, a_n\}$.

If $n \geq 8$ then there exists at most one $a_{l'} \in V$ such that $a_k a_{l'} \in E - \{a_k a_{l_1}, a_k a_{l_2}, a_k a_{l_3}\}$. And there exists $a_x \in \{a_1, a_{n-1}, a_n\}$ with $i \in \{1, 2, 3\}$ such that $a_x = a_k + a_{l_i}$ and $a_x a_{l'} \in E$. Since $a_x + a_{l'} = (a_k + a_{l_i}) + a_{l'} = (a_k + a_{l'}) + a_{l_i} \in S$, contradicting $a_k + a_{l'} \in C$.

If $n = 7$ then we will prove the claims first. Let $C_7 = a_1 a_2 \cdots a_7 a_1$.

Claim 1 $a_k + a_l \in C$ for any $a_k a_l \in E - E_n$ with $a_k \in \{a_2, a_5\}$.

In fact, by the symmetry of $\overline{C_n}$, we only consider $a_k = a_2$. By the above, we only need to prove that $\{a_{l_1}, a_{l_2}, a_{l_3}\} = \emptyset$. If not, then $|\{a_{l_1}, a_{l_2}, a_{l_3}\}| \in \{1, 2, 3\}$.

(a) If $|\{a_{l_1}, a_{l_2}, a_{l_3}\}| = 3$ then $\{a_{l_1}, a_{l_2}, a_{l_3}\} = \{a_4, a_5, a_6\}$. Suppose that $a_2 + a_{l_1} = a_1$, $a_2 + a_{l_2} = a_7$ and $a_2 + a_{l_3} = a_6$, denoted (1), (2) and (3) respectively. By (1) − (2), $a_1 + a_{l_2} = a_7 + a_{l_1}$. By (1) − (3), $a_6 + a_{l_1} = a_1 + a_{l_3}$. By (2) − (3), $a_6 + a_{l_2} = a_7 + a_{l_3}$. Since $a_1 + a_{l_i} \in S$ for any $i \in \{1, 2, 3\}$ and $a_6 + a_7 \notin S$ and $a_1 + a_7 \notin S$, $a_{l_1} = a_4$, $a_{l_2} = a_6$ and $a_{l_3} = a_5$, which imply that $a_1 + a_6 = a_7 + a_4 \in C$. Since $a_7 + a_3 = (a_2 + a_6) + a_3 = a_2 + (a_6 + a_3) \in S$ and $(a_1 + a_6) + a_3 = a_1 + (a_6 + a_3) \notin S$, $a_6 + a_3 = a_2$. So $a_2 + a_7 = (a_6 + a_3) + a_7 = (a_3 + a_7) + a_6 \in S$, contradicting $a_3 + a_7 \in C$. Thus, $|\{a_{l_1}, a_{l_2}, a_{l_3}\}| \neq 3$.

(b) If $|\{a_{l_1}, a_{l_2}, a_{l_3}\}| = 2$ then we assume that $\{a_{l_1}, a_{l_2}\} \subset \{a_4, a_5, a_6\}$.

(b.1) Suppose that $a_2 + a_{l_1} = a_1$ and $a_2 + a_{l_2} = a_7$, denoted (1) and (2) respectively. By (1) − (2), $a_1 + a_{l_2} = a_7 + a_{l_1}$. Since $a_{l_2} \in \{a_4, a_5, a_6\}$, $a_7 + a_{l_1} = a_1 + a_{l_2} \in S$. So $a_{l_1} \in \{a_4, a_5\} - \{a_{l_2}\}$. Let $\alpha \in \{a_4, a_5, a_6\} - \{a_{l_1}, a_{l_2}\}$. Then $a_1 + \alpha \in S$ and $a_2 + \alpha \in C$. $a_1 + \alpha = (a_2 + a_{l_1}) + \alpha = (a_2 + \alpha) + a_{l_1} \in S$, contradicting $a_2 + \alpha \in C$.

(b.2) Suppose that $a_2 + a_{l_1} = a_7$ and $a_2 + a_{l_2} = a_6$, denoted (1) and (2) respectively. By (1) − (2), $a_6 + a_{l_1} = a_7 + a_{l_2}$. Note that $a_{l_1} \in \{a_4, a_5, a_6\}$.

If $a_{l_1} = a_4$ then $a_{l_2} = a_5$ and $a_6 + a_4 = a_7 + a_5 \in C$. So $a_2 + a_6 \in C$. Since $a_1 + a_6 = a_1 + (a_2 + a_5) = (a_1 + a_5) + a_2 \in S$, $a_1 + a_5 \in \{a_2, a_4, a_7\}$. Since $a_7 + a_5 \in C$, $a_1 + (a_7 + a_5) = (a_1 + a_5) + a_7 \notin S$. So $a_1 + a_5 = a_7$. Since $a_7 + a_3 = (a_1 + a_5) + a_3 = (a_3 + a_5) + a_1 \in S$, $a_3 + a_5 \in \{a_1, a_4\}$. Since $a_6 + a_3 = (a_2 + a_5) + a_3 = (a_3 + a_5) + a_2 \in S$, $a_3 + a_5 = a_4$. Thus, $a_3 + (a_7 + a_5) = (a_3 + a_5) + a_7 = a_4 + a_7 \in S$, contradicting $a_7 + a_5 \in C$.

If $a_{l_1} = a_5$ then $a_6 + a_5 = a_7 + a_{l_2} \notin S$, contradicting $a_7 + a_{l_2} \in S$ with $a_{l_2} \in \{a_4, a_5, a_6\}$.

If $a_{l_1} = a_6$ then $a_{l_2} \in \{a_4, a_5\}$. Let $\alpha \in \{a_4, a_5\} - \{a_{l_2}\}$. Then $a_7 + \alpha \in S$ and $a_2 + \alpha \in C$. So $a_7 + \alpha = (a_2 + a_6) + \alpha = (a_2 + \alpha) + a_6 \in S$, contradicting $a_2 + \alpha \in C$.

(b.3) Suppose that $a_2 + a_{l_1} = a_1$ and $a_2 + a_{l_2} = a_6$, denoted (1) and (2) respectively. By (1) − (2), $a_1 + a_{l_2} = a_6 + a_{l_1}$. Since $a_{l_2} \in \{a_4, a_5, a_6\}$, $a_6 + a_{l_1} = a_1 + a_{l_2} \in S$. So $a_{l_1} \neq a_5$, i.e., $a_{l_1} \in \{a_4, a_6\}$.

If $a_{l_1} = a_4$, then $a_{l_2} = a_5$ and $a_1 + a_5 = a_6 + a_4$. And $a_2 + a_6 \in C$ (since $|\{a_{l_1}, a_{l_2}, a_{l_3}\}| = 2$). Thus, $a_1 + a_6 = (a_2 + a_4) + a_6 = (a_2 + a_6) + a_4 \in S$, contradicting $a_2 + a_6 \in C$.

If $a_{l_1} = a_6$ then $a_{l_2} \in \{a_4, a_5\}$. Let $\alpha \in \{a_4, a_5\} - \{a_{l_2}\}$. Then $a_1 + \alpha \in S$ and $a_2 + \alpha \in C$. Thus, $a_1 + \alpha = (a_2 + a_6) + \alpha = (a_2 + \alpha) + a_6 \in S$, contradicting $a_2 + \alpha \in C$.

(c) If $|\{a_{l_1}, a_{l_2}, a_{l_3}\}| = 1$ then assume that $a_{l_1} \in \{a_4, a_5, a_6\}$, that is, $a_2 + a_{l_1} = a_x$ with $a_x \in \{a_1, a_6, a_7\}$.

(c.1) Suppose that $a_2 + a_{l_1} = a_x = a_7$, where $a_{l_1} \in \{a_4, a_5, a_6\}$. Then there exists one vertex $a_y \in \{a_4, a_5, a_6\} - (\{a_{l_1}\} \cup \{a_6\})$ such that $a_7 a_y \in E$ and $a_2 + a_y \in C$. So $a_7 + a_y = (a_2 + a_{l_1}) + a_y = (a_2 + a_y) + a_{l_1} \in S$, contradicting $a_2 + a_y \in C$.

(c.2.1) Suppose that $a_2 + a_{l_1} = a_x$ with $a_{l_1} \in \{a_4, a_6\}$ and $a_x \in \{a_1, a_6\}$. Then there exists one vertex $a_y \in \{a_4, a_5, a_6\} - \{a_{l_1}\}$ such that $a_x a_y \in E$ and $a_2 + a_y \in C$. So $a_x + a_y = (a_2 + a_{l_1}) + a_y = (a_2 + a_y) + a_{l_1} \in S$, contradicting $a_2 + a_y \in C$.

(c.2.2) Suppose that $a_2 + a_5 = a_x$ with $a_x \in \{a_1, a_6, a_7\}$. Then $a_x a_4 \in E$ and $a_2 + a_4 \in C$. So $a_x + a_4 = (a_2 + a_5) + a_4 = (a_2 + a_4) + a_5 \in S$, contradicting $a_2 + a_4 \in C$.

Thus, Claim 1 holds.

Claim 2 $a_k + a_l \in C$ for any $a_k a_l \in E - E_n$ with $a_k \in \{a_3, a_4\}$.

In fact, by the symmetry of $\overline{C_n}$, we only consider $a_k = a_3$. By the above, we only prove that for any $a_3 a_l \in E - E_n$, $a_3 + a_l \notin \{a_1, a_6, a_7\}$. By Claim 1, $a_2 + a_6 \in C$ and $a_i + a_5 \in C$ for all $i \in \{1, 2, 3\}$. Since $(a_3 + a_5) + a_6 = (a_3 + a_6) + a_5 \notin S$, $a_3 + a_6 \notin \{a_1, a_6, a_7\}$, i.e. $a_3 + a_6 \in C$. Using the same methods as above, it is easy to prove Claim 2.

Claim 3 $a_k + a_l \in C$ for any $a_k a_l \in E - E_n - \{a_1 a_6\}$ with $a_k \in \{a_1, a_6\}$.

In fact, it is easy to prove Claim 3 by Claim 1 and Claim 2.

(ii) Try to prove $a_1 + a_{n-1} \in C$ for $n \geq 7$. By contradiction. Suppose that there exists $a_y \in V - \{a_1, a_{n-1}\}$ such that $a_1 + a_{n-1} = a_y$. Since $n \geq 7$ then there exists one vertex $a_{y'} \in V$ such that $a_y a_{y'} \in E$ and either $a_1 a_{y'} \in E$ or $a_{n-1} a_{y'} \in E$. Then $a_y + a_{y'} \in S$. By lemma 2.3, $a_n + a_{y'} \in C$. By (i), $a_1 + a_{y'} \in C$. If $a_1 a_{y'} \in E$ then $a_y + a_{y'} = (a_1 + a_{n-1}) + a_{y'} = (a_1 + a_{y'}) + a_{n-1} \in S$, contradicting $a_1 + a_{y'} \in C$. If $a_{n-1} a_{y'} \in E$ then $a_y + a_{y'} = (a_1 + a_{n-1}) + a_{y'} = (a_{n-1} + a_{y'}) + a_1 \in S$, contradicting $a_1 + a_{y'} \in C$.

Thus, Lemma 2.4 holds. □

Lemma 2.6 $\overline{C_n}$ is exclusive for $n \geq 7$. □

Lemma 2.7 $\zeta(\overline{C_n}) \geq 2n-7$ for $n \geq 7$.

Proof: Let $V = \{b_1, b_2, b_3, \cdots, b_n\}$, $S = V \cup C$ and C is the isolated vertex set. Without loss of generality, we can assume that $b_1 < b_2 < \cdots < b_n$. So $b_1+b_2 < b_1+b_3 < b_1+b_4 < \cdots < b_1+b_n < b_2+b_n < b_3+b_n < \cdots < b_{n-1}+b_n$. Let $C_0 = \{b_1+b_2, b_1+b_3, \cdots, b_1+b_n, b_2+b_n, \cdots, b_{n-1}+b_n\}$. Then there are at most four numbers which are not in S, but in C_0. On the other hand, the others in C_0 are the isolated vertices by Lemma 2.6. Thus, $\zeta(\overline{C_n}) \geq 2n-7$. □

Lemma 2.8 $\sigma(\overline{C_n}) \leq 2n-7$ for $n \geq 7$ and n odd.

Proof: We consider the sum labelling of $\overline{C_n} \cup (2n-7)K_1$ as follows:
$a_1 = \frac{n+1}{2} \times 10 + 1$,
$a_{2k} = (\frac{n-1}{2} - k + 1) \times 10 + 1,\ k = 1, 2, 3, \cdots, \frac{n-1}{2}$,
$a_{2k+1} = k \times 10 + 6,\ k = 1, 2, 3, \cdots, \frac{n-1}{2}$,
$c_k = (k+2) \times 10 + 2,\ k = 1, 2, 3, \cdots, n-3$,
$d_k = (k+1) \times 10 + 7,\ k = 1, 2, \cdots, \frac{n-5}{2}, \frac{n+1}{2}, \frac{n+3}{2}, \cdots, n-2$.
Let $S = V \cup C$, where C be the isolated vertex set and $C = \{c_k, d_l | k = 1, 2, \cdots, n-3; l = 1, 2, \cdots, \frac{n-5}{2}, \frac{n+1}{2}, \frac{n+3}{2}, \cdots, n-2\}$.
Let us verify that this labelling is the sum labelling in detail.
(1) The vertices in S are distinct .
(2) Consider the vertex a_1:
(i) For each $a_{2k} \in V$ $(k = 1, 2, 3, \cdots, \frac{n-1}{2})$, $a_1 + a_{2k} = [(n-k-1)+2] \times 10 + 2$.
When $k = 1$, $a_1 + a_2 = n \times 10 + 2 > (n-1) \times 10 + 2 = c_{n-3}$. So $a_1 + a_2 \notin S$.
When $2 \leq k \leq \frac{n-1}{2}$, $\frac{n-1}{2} \leq n-k-1 \leq n-3$ (since $2 \leq k \leq \frac{n-1}{2}$). So $a_1 + a_{2k} = c_{n-k-1} \in S$.
(ii) For each $a_{2k+1} \in V$ $(k = 1, 2, 3, \cdots, \frac{n-1}{2})$, $a_1 + a_{2k+1} = (\frac{n+1}{2} + k) \times 10 + 7$.
When $k = \frac{n-1}{2}$, $a_1 + a_{2k+1} = n \times 10 + 7 > (n-1) \times 10 + 7 = d_{n-2}$. So $a_1 + a_n \notin S$.
When $1 \leq k \leq \frac{n-3}{2}$, $a_1 + a_{2k+1} = [(\frac{n-1}{2} + k) + 1] \times 10 + 7$. Since $1 \leq k \leq \frac{n-3}{2}$, $\frac{n+1}{2} \leq \frac{n-1}{2} + k \leq n-2$. Hence, $a_1 + a_{2k+1} = [(\frac{n-1}{2} + k) + 1] \times 10 + 7 = d_{\frac{n-1}{2}+k} \in S$.
(3) Consider the other vertices $a_{2k}, a_{2l} \in V$ $(1 \leq k,\ l \leq \frac{n-1}{2};\ k \neq l)$. Clearly $a_{2k} + a_{2l} = [n+1-k-l] \times 10 + 2 = [(n-1-k-l)+2] \times 10 + 2$.
Since $1 \leq k, l \leq \frac{n-1}{2}$ and $k \neq l$, $1 \leq n-1-k-l \leq n-3$. Hence $a_{2k} + a_{2l} = c_m \in S$ with $m = n-1-k-l$ and $1 \leq k, l \leq \frac{n-1}{2}$.
(4) Consider the vertices a_{2k}, $a_{2l+1} \in V$ $(1 \leq k, l \leq \frac{n-1}{2})$. Clearly
$a_{2k} + a_{2l+1} = [(\frac{n-1}{2} - k + 1) \times 10 + 1] + [l \times 10 + 6] = [(\frac{n-1}{2} - k + 1) + 1] \times 10 + 7$.
Since $1 \leq k, l \leq \frac{n-1}{2}$, $1 \leq \frac{n-1}{2} - k + l \leq n-2$. Since $a_{2k} + a_{2l+1} \notin S \Longleftrightarrow \frac{n-1}{2} - k + l \in \{\frac{n-3}{2}, \frac{n-1}{2}\} \Longleftrightarrow l - k \in \{0, -1\} \Longleftrightarrow l - k \in \{0, -1\} \Longleftrightarrow l = k$ and $k = l+1$, $a_{2k} + a_{2l+1} \notin S \Longleftrightarrow a_{2k} + a_{2k+1} \notin S$; and $a_{2l+2} + a_{2l+1} \notin S$.
(5) Consider the vertices $a_{2k+1}, a_{2l+1} \in V$ $(1 \leq k,\ l \leq \frac{n-1}{2}$ and $k \neq l)$. Clearly,

$a_{2k+1}+a_{2l+1}=[k\times 10+6]+[l\times 10+6]=[k+l+1]\times 10+2=[(k+l-1)+2]\times 10+2$.

Since $1\le k,\, l\le \frac{n-1}{2}$, $1\le k+l-1\le n-2$. Hence

$a_{2k+1}+a_{2l+1}\notin S \Longleftrightarrow k+l-1=n-2 \Longleftrightarrow k+l=n-1 \Longleftrightarrow k=l=\frac{n-1}{2}$, which is a contradiction with $k\ne l$. So $a_{2k+1}+a_{2l+1}=c_{k+l-1}\in S$ ($1\le k,l\le \frac{n-1}{2}$ and $k\ne l$) for each $a_{2k+1},a_{2l+1}\in V$.

Thus, the above labelling is the sum labelling. So $\sigma(\overline{C_n})\le 2n-7$ for $n\ge 7$ and n odd. □

Lemma 2.9 $\sigma(\overline{C_n})\le 2n-7$ for $n\ge 8$ and n even.

Proof: We consider the following sum labelling of $\overline{C_n}\bigcup(2n-7)K_1$ for $n\ge 8$ and n even:

$a_{2k-1}=k\times 10+1,\ k=1,2,3,\cdots,\frac{n}{2}$,

$$a_{2k}=\begin{cases}(\frac{n}{2}-k-2)\times 10+6, & k=1,2,3,\cdots,\frac{n}{2}-3,\\ (n-k-2)\times 10+6, & k=\frac{n}{2}-2,\frac{n}{2}-1,\frac{n}{2},\end{cases}$$

$c_k=(k+2)\times 10+2,\ k=1,2,3,\cdots,n-2$,

$d_l=(l+1)\times 10+7,\ l\in\{1,2,\cdots n-1\}-\{\frac{n}{2}-3,\frac{n}{2}-2,n-3,n-2\}$.

Let $S=V\cup C$ and $C=\{c_k,d_l|k=1,2,\cdots,n-2;l\in\{1,2,\cdots n-1\}-\{\frac{n}{2}-3,\frac{n}{2}-2,n-3,n-2\}\}$, where C be the isolated vertex set.

Let us verify that this is the sum labelling in detail.

(1) The vertices in S are distinct.

(2). For $a_{2k-1},a_{2l-1}\in V$ ($k=1,2,\cdots,\frac{n-1}{2}$ and $k\ne l$), $a_{2k-1}+a_{2l-1}=[(k+l-2)+2]\times 10+2$. Since $k=1,2,\cdots,\frac{n-1}{2}$ and $k\ne l$, $0<k+l-2<n-2$. So $a_{2k-1}+a_{2l-1}=c_{k+l-2}\in S$.

(3) For $a_{2k-1},a_{2l}\in V$,

When $1\le l\le\frac{n}{2}-3$, $a_{2k-1}+a_{2l}=[(\frac{n}{2}-3+k-l)+1]\times 10+7$. Since $1\le k\le\frac{n}{2}$ and $1\le l\le\frac{n}{2}-3$, $1\le\frac{n}{2}+k-l-3\le n-4<n-3$. Clearly $a_{2k-1}+a_{2l}\notin S\Longleftrightarrow \frac{n}{2}-3+k-l\in\{\frac{n}{2}-3,\frac{n}{2}-2\}\Longleftrightarrow k-l\in\{0,1\}$. Hence $a_{2k-1}+a_{2l}\notin S\Longleftrightarrow a_{2l-1}+a_{2l}\notin S$ and $a_{2l+1}+a_{2l}\notin S$.

When $l\in\{\frac{n}{2}-2,\frac{n}{2}-1,\frac{n}{2}\}$, $a_{2k-1}+a_{2l}=[(n+k-l-3)+1]\times 10+7$. Since $1\le k\le\frac{n}{2}$ and $\frac{n}{2}-2\le l\le\frac{n}{2}$, $\frac{n}{2}-2\le n+k-l-3\le n-1$. Clearly $a_{2k-1}+a_{2l}\notin S\Longleftrightarrow n+k-l-3\in\{\frac{n}{2}-2,n-3,n-2\}\Longleftrightarrow l-k\in\{\frac{n}{2}-1,0,-1\}$. And $l-k=\frac{n}{2}-1\Longleftrightarrow(l,k)=(\frac{n}{2},1)$.

So $a_{2k-1}+a_{2l}\notin S\Longleftrightarrow(l,k)=(\frac{n}{2},1)$ and $k\in\{l,l+1\}\Longleftrightarrow a_1+a_n\notin S$ and $a_{2l-1}+a_{2l}\notin S$ and $a_{2l+1}+a_{2l}\notin S$. Thus, for $a_{2k-1},a_{2l}\in V$, if $a_{2k-1}a_{2l}\in E$ then $a_{2k-1}+a_{2l}\in S$; if $a_{2k-1}a_{2l}\notin E$ then $a_{2k-1}+a_{2l}\notin S$.

(4) For $a_{2k},a_{2l}\in V$ with $k\ne l$,

when $1\le k,l\le\frac{n}{2}-3$, $a_{2k}+a_{2l}=[n-5-(k+l)+2]\times 10+2$. Since $1\le k,l\le\frac{n}{2}-3$ and $k\ne l$, $1\le n-k-l-5\le n-7<n-2$. So $a_{2k}+a_{2l}=c_m\in S$ with $m=n-k-l-5$.

When $1\le k\le\frac{n}{2}-3$ and $l\in\{\frac{n}{2}-2,\frac{n}{2}-1,\frac{n}{2}\}$, $a_{2k}+a_{2l}=\{[\frac{3n}{2}-(k+l)-5]+2\}\times 10+2$. Since $1\le k\le\frac{n}{2}-3$ and $\frac{n}{2}-2\le l\le\frac{n}{2}$, $1<\frac{n}{2}-2\le\frac{3n}{2}-(k+l)-5\le n-4<n-2$. That is, $a_{2k}+a_{2l}=c_h\in S$ with $h=\frac{3n}{2}-(k+l)-5$.

When $k,l\in\{\frac{n}{2}-2,\frac{n}{2}-1,\frac{n}{2}\}$ and $k\ne l$, $a_{2k}+a_{2l}=\{[2n-(k+l)-5]+2\}\times 10+2$. Since $\frac{n}{2}-2\le k,l\le\frac{n}{2}$ and $k\ne l$, $1<n-5\le 2n-(k+l)-5\le n-2$.

Hence $a_{2k} + a_{2l} = c_g \in S$ with $g = 2n - (k + l) - 5$. So for $a_{2k}, a_{2l} \in V$ $(k \neq l)$, $a_{2k} + a_{2l} \in S$.

Thus, the labelling is the sum labelling. So $\sigma(\overline{C_n}) \leq 2n - 7$ for $n \geq 8$ and n even. □

Theorem 2.1 $\zeta(\overline{C_n}) = \sigma(\overline{C_n}) = 2n - 7$ for $n \geq 7$. □

Corollary 2.1 $\zeta(\overline{W_n}) = \sigma(\overline{W_n}) = 2n - 8$ for $n \geq 7$. □

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