Full cycle extendability of triangularly connected almost claw-free graphs

Mingquan Zhan*
Department of Mathematics
Millersville University, Millersville, PA 17551, USA

Abstract

This paper generalizes the concept of locally connected graphs. A graph G is triangularly connected if for every pair of edges $e_1, e_2 \in E(G)$, G has a sequence of 3-cycles $C_1, C_2, \cdots, C_l$ such that $e_1 \in C_1, e_2 \in C_l$ and $E(C_i) \cap E(C_{i+1}) \neq \emptyset$ for $1 \leq i \leq l-1$. In this paper, we show that every triangularly connected $K_{1,4}$-free almost claw-free graph on at least three vertices is fully cycle extendable.

Keywords: claw-free graphs, almost-free graphs, triangularly connected graphs, fully cycle extendability

1 Introduction

We use [1] for notations and terminology not defined here, and consider finite simple graphs only. The neighborhood of a vertex v in G and the subgraph induced by $A \subseteq V(G)$ are respectively denoted by $N_G(v)$ and $G[A]$. A graph G is locally connected if for each $v \in V(G)$, the subgraph $G[N_G(v)]$ induced by $N_G(v)$ is connected.

For an integer $k > 2$, a k-cycle is a 2-regular connected graph with k edges. If F is a graph, then we say that G is F-free if it does not contain an induced subgraph isomorphic to F. A $K_{1,3}$ is also called a claw, and a $K_{1,3}$-free graph is also called a claw-free graph. The vertex whose degree is r in $K_{1,r} (r \geq 3)$ is called the center of $K_{1,r}$.

*Research is partially supported by the Millersville University Faculty Released-Time grant

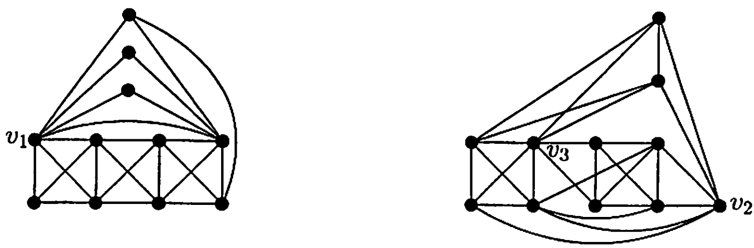
As a generalization of the class of claw-free graphs, the class of almost claw-free graphs was introduced by Ryjáček in [5]. A dominating set of G is a subset S of $V(G)$ such that every vertex of G belongs to S or is adjacent to a vertex of S. For $v \in S$, the vertices in $N_G(v) \setminus S$ are dominated by v. The domination number, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of G. If $\gamma(G) \leq k$, then we say that G is k-dominated. A graph G is almost claw-free if the set A of the vertices that are centers of claws in G is independent and $G[N_G(v)]$ is 2-dominated for each $v \in A$.

A graph G is pancyclic if for every integer k with $3 \leq k \leq |V(G)|$, G has a k-cycle. G is vertex pancyclic if for each vertex $v \in V(G)$, and for each integer k with $3 \leq k \leq |V(G)|$, G has a k-cycle C_k such that $v \in V(C_k)$. G is said to be fully cycle extendable if every vertex of G lies on a triangle and for every nonhamiltonian cycle C there is a cycle C' in G such that $V(C) \subseteq V(C')$ and $|V(C')| = |V(C)| + 1$. In [4], Oberly and Summer proved that every connected, locally connected claw-free graph on at least three vertices is hamiltonian. Clark [2] proved that, under these conditions, G is vertex pancyclic. Later, Hendry observed that Clark essentially proved the following stronger result.

Theorem 1.1 *(Hendry, [3]) If G is a connected, locally connected claw-free graph on at least three vertices, then G is fully cycle extendable.*

Theorem 1.2 *(Ryjáček, [5]) Every connected, locally connected $K_{1,4}$-free almost claw-free graph on at least three vertices is fully cycle extendable.*

As a generalization of the concept of locally connected graphs, triangularly connected graphs were introduced in [6]. A graph G is triangularly connected if for every pair of edges $e_1, e_2 \in E(G)$, G has a sequence of 3-cycles $C_1, C_2, \cdots, C_l$ such that $e_1 \in C_1, e_2 \in C_l$ and $E(C_i) \cap E(C_{i+1}) \neq \emptyset$ for $1 \leq i \leq l-1$. Clearly, every connected, locally connected graph is triangularly connected. But not every triangularly connected graph is locally connected. The graphs in Figure 1 are triangularly connected graphs which are not locally connected since the subgraphs induced by the neighborhoods of v_1, v_2 and v_3 are not connected. Graph A in Figure 1 is not almost claw-free, and Graph B is almost claw-free.



(A) (B)

Figure 1. Triangularly connected graphs

Let $\mathscr{C}_3(G)$ denote the graph whose vertex set $V(\mathscr{C}_3(G)) = \{C|C$ a 3-cycle of $G\}$ and edge set $E(\mathscr{C}_3(G)) = \{C_1C_2|C_1, C_2 \in V(\mathscr{C}_3(G))$, and $E(C_1) \cap E(C_2) \neq \emptyset\}$. By the definition of triangularly connected graphs, we have

Proposition 1.3 *A graph is triangularly connected if and only if both of the following hold:*
(i) For any $e \in E(G)$, there exists some $C_e \in V(\mathscr{C}_3(G))$ such that $e \in E(C_e)$, and
(ii) The graph $\mathscr{C}_3(G)$ is connected.

In [6], Lai et.al considered the hamiltonicity of triangularly connected claw-free graphs and proved the following.

Theorem 1.4 *(Shao, [6]) Every triangularly connected claw-free graph on at least three vertices is vertex pancyclic.*

Our goal here is to extend Theorems 1.1 and 1.2 to triangularly connected graphs.

Theorem 1.5 *Every triangularly connected, $K_{1,4}$-free almost claw-free graph on at least three vertices is fully cycle extendable.*

Since a claw-free graph is also a $K_{1,4}$-free almost claw-free graph, we have the following corollary.

Corollary 1.6 *Every triangularly connected claw-free graph on at least three vertices is fully cycle extendable.*

The graphs in Figure 2 show Theorem 1.5 is best possible. Graphs A, B in Figure 2 show that Theorem 1.5 fails if G is only locally 3-dominated, or the set of centers of claws is not independent. Graph C is a locally connected almost claw-free graph, Graph D is a triangularly connected graph which is not locally connected. Both Graph C and Graph D show that Theorem 1.5 fails if G is not $K_{1,4}$-free.

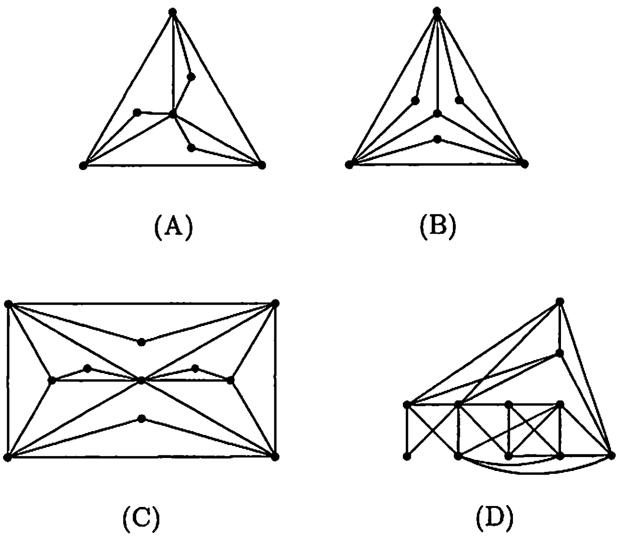


Figure 2

2 Theorem 1.5's proof

Since every vertex of G lies on a triangle, it is sufficient to prove that for every cycle C of length $r \leq |V(G)| - 1$ there is a cycle C' of length $r+1$ such that $V(C) \subset V(C')$. We argue it by contradiction, and throughout the proof, we suppose that for every cycle $C \subset G$, one of its orientations is chosen, and for any $u \in V(C)$, we denote by u^- and u^+ the predecessor and successor of u on C, respectively. Denote $u^{++} = (u^+)^+$ and $u^{--} = (u^-)^-$. For $u, v \in V(C)$, $C[u,v]$ or $\overleftarrow{C}[v,u]$ denotes the (u,v)-path of C with the same or opposite orientation with respect to the orientation of C; if $u = v$, then we define both $C[u,v]$ and $\overleftarrow{C}[v,u]$ as a single vertex. Whenever vertices of an induced $K_{1,3}$ or $K_{1,4}$ are listed, its center is always the first vertex of the list. Let A be the set of all centers of claws in G.

Let $C = v_1 v_2 \cdots v_r v_1$, and $\mathcal{B}(C) = \{B \in \mathscr{C}_3(G) | E(B) \cap E(C) \neq \emptyset\}$. Then

$E(C) \subseteq \bigcup_{B \in \mathcal{B}(C)} E(B)$. If there is some $B \subseteq \mathcal{B}(C)$ such that $|V(B) \cap V(C)| = 2$, it is clear that the subgraph of G induced by the edge set $E(C) \cup E(B) - (E(C) \cap E(B))$ extends C. So we assume that for each $B \in \mathcal{B}(C)$, $V(B) \subseteq V(C)$.

Let $e \in E(G)$ such that e is incident with exactly one vertex in $V(C)$ and C_e be a 3-cycle with $e \in C_e$. Clearly, $C_e \notin \mathcal{B}(C)$. As G is triangularly connected, there is a path P_e in $\mathscr{C}_3(G)$ from C_e to $\mathcal{B}(C)$. Let C, e, C_e and P_e be chosen in such a way that, among all cycles with vertex set $V(C)$, the path P_e is shortest. Let $P_e = Z_0 Z_1 \cdots Z_k$, where $Z_0 = C_e$ and $Z_k \in \mathcal{B}(C)$. Then $k \geq 1$. Moreover, we have the following.

Claim 1 *$|V(C_e) \cap V(C)| = 2$, $E(Z_i) \cap E(C) = \emptyset$ and $|V(Z_i) \cap V(C)| = 3$ for $i = 1, \cdots, k-1$.*

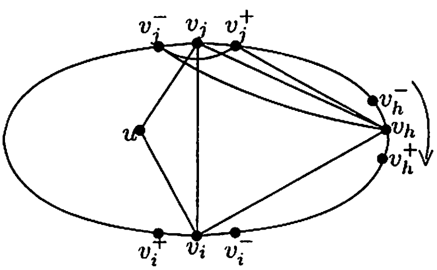


Figure 3

Let $Z_0 = uv_iv_ju$ and $Z_1 = v_hv_iv_jv_h$. Without loss of generality, we assume that $1 \leq h < i < j \leq r$ (see Figure 3). Obviously, $uv_j^+, uv_j^-, uv_i^+, uv_i^- \notin E(G)$. Since $v_iv_j \in E(G)$, we have either $v_i \notin A$ or $v_j \notin A$. Without loss of generality, we suppose that $v_j \notin A$. Then $v_j^+v_j^- \in E(G)$, which implies that $v_j^+ \neq v_i^-$ and $v_i^+ \neq v_j^-$. By the choices of e and P_e, $uv_h \notin E(G)$, otherwise, $P' = Z_0'Z_2 \cdots Z_k$ is shorter than P_e, where $Z_0' = uv_hv_j$ or $Z_0' = uv_hv_i$. As $G[\{v_j, u, v_h, v_j^-\}] \ncong K_{1,3}$, we have $v_j^-v_h \in E(G)$. Similarly, $v_j^+v_h \in E(G)$ if $v_h \neq v_j^+$.

Claim 2 $k = 1$.

By contradiction. Suppose that $k \geq 2$. Then $v_h \notin \{v_i^-, v_j^+\}$. We consider the following cases.

Case	Cycle C_1
$v_h^+v_h^- \in E(G)$	$v_jv_hC[v_j^+, v_h^-]C[v_h^+, v_j]$
$v_jv_h^- \in E(G)$	$v_j\overleftarrow{C}[v_h^-, v_j^+]\overleftarrow{C}[v_j^-, v_h]v_j$
$v_jv_h^+ \in E(G)$	$v_j\overleftarrow{C}[v_h, v_j^+]\overleftarrow{C}[v_j^-, v_h^+]v_j$

In each of these cases, v_j and v_h are consecutive on C_1 and the length of P_e is 1 corresponding to C_1 which is shorter than the length corresponding to C, a contradiction. So we have $v_h^+v_h^-, v_jv_h^-, v_jv_h^+ \notin E(G)$. Hence $G[\{v_h, v_h^+, v_h^-, v_j\}] \cong K_{1,3}$, which implies that $v_h \in A$. Since A is independent, we have $v_i \notin A$ and hence obviously $v_i^+v_i^- \in E(G)$. We now consider $G[\{v_h, v_h^+, v_h^-, v_i, v_j^-\}]$.

Case	Cycle C'
$v_iv_j^- \in E(G)$	$v_juv_i\overleftarrow{C}[v_j^-, v_i^+]\overleftarrow{C}[v_i^-, v_j]$
$v_iv_h^- \in E(G)$	$v_juv_i\overleftarrow{C}[v_h^-, v_j^+]\overleftarrow{C}[v_j^-, v_i^+]\overleftarrow{C}[v_i^-, v_h]v_j$
$v_iv_h^+ \in E(G)$	$v_juv_iC[v_h^+, v_i^-]C[v_i^+, v_j^-]\overleftarrow{C}[v_h, v_j]$
$v_j^-v_h^- \in E(G)$	$v_juv_iC[v_h, v_i^-]C[v_i^+, v_j^-]\overleftarrow{C}[v_h^-, v_j]$

In each case, the cycle C' extends C. So $v_iv_j^-, v_iv_h^-, v_iv_h^+, v_j^-v_h^- \notin E(G)$. Since $G[\{v_h, v_h^+, v_h^-, v_i, v_j^-\}] \ncong K_{1,4}$ and $v_h^+v_h^- \notin E(G)$, we have $v_h^+v_j^- \in E(G)$. As $G[\{v_i, u, v_i^+, v_h\}] \ncong K_{1,3}$ and $uv_h, uv_i^+ \notin E(G)$, we have $v_i^+v_h \in E(G)$. Thus the cycle $v_ju\overleftarrow{C}[v_i, v_h^+]\overleftarrow{C}[v_j^-, v_i^+]\overleftarrow{C}[v_h, v_j]$ again extends C, a contradiction. So Claim 2 holds.

By Claim 2, $v_h \in \{v_j^+, v_i^-\}$. If $v_h = v_i^-$, then $v_j^-v_i^- \in E(G)$ since $G[\{v_j, u, v_j^-, v_i^-\}] \ncong K_{1,3}$. Thus the cycle $v_juC[v_i, v_j^-]\overleftarrow{C}[v_i^-, v_j]$ extends C, a contradiction. So $v_h = v_j^+$.

Claim 3 $v_i^+v_i^-, v_i^+v_j^+, v_i^+v_j, v_i^-v_j, v_i^-v_j^- \notin E(G)$. *Therefore,* $v_j^+v_i^- \in E(G)$, *and* $v_i \in A$.

We argue by contradiction using the following chart.

Case	Cycle C'
$v_i^+v_i^- \in E(G)$	$v_juv_iC[v_j^+, v_i^-]C[v_i^+, v_j]$
$v_i^+v_j^+ \in E(G)$	$v_ju\overleftarrow{C}[v_i, v_j^+]C[v_i^+, v_j]$
$v_i^+v_j \in E(G)$	$v_ju\overleftarrow{C}[v_i, v_j^+]\overleftarrow{C}[v_j^-, v_i^+]v_j$
$v_i^-v_j \in E(G)$	$v_juC[v_i, v_j^-]C[v_j^+, v_i^-]v_j$
$v_i^-v_j^- \in E(G)$	$v_juC[v_i, v_j^-]\overleftarrow{C}[v_i^-, v_j]$

In each case, the cycle C' extends C. So $v_i^+v_i^-, v_i^+v_j^+, v_i^+v_j, v_i^-v_j, v_i^-v_j^- \notin E(G)$. Therefore, $G[\{v_i, v_i^+, v_i^-, u\}] \cong K_{1,3}$ and $v_i \in A$. Since $G[\{v_i, v_i^+, v_i^-, u, v_j^+\}] \not\cong K_{1,4}$, we have $v_j^+v_i^- \in E(G)$ (see Figure 4).

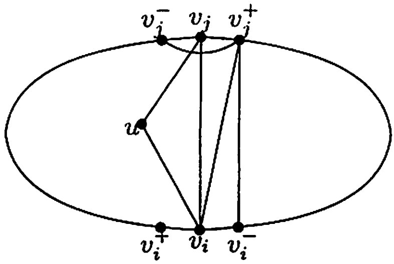


Figure 4

Claim 4 *There is no vertex in $N_G(v_i)$ that is adjacent to u and to one of v_i^+ and v_i^-.*

By contradiction. Suppose that there is a vertex $w \in N_G(v_i)$ that is adjacent to u and to one of v_i^+ and v_i^- (say v_i^+; the second case is similar). Clearly, $w \in V(C)$, $w \neq v_i^+$ and $w \notin A$. As $G[\{w, w^+, w^-, u\}] \not\cong K_{1,3}$, we have $w^+w^- \in E(G)$. Thus, the cycle $v_iuwC[v_i^+, w^-]C[w^+, v_i]$ extends C, a contradiction. So Claim 4 holds.

Note that $G[\{v_i, v_i^+, v_i^-, u\}] \cong K_{1,3}$ and $G[N_G(v_i)]$ is 2-dominated. By Claim 4, there is a vertex d in $N_G(v_i)$ dominating v_i^+ and v_i^-. Clearly, $du \notin E(G)$ and $d \in V(C)$. By Claim 3, $d \notin \{v_j^-, v_j, v_j^+\}$. So $d \in C[v_j^{++}, v_i^{--}] \cup C[v_i^{++}, v_j^{--}]$.

Claim 5 $d \in C[v_i^{++}, v_j^{--}]$

By contradiction. Suppose that $d \notin C[v_i^{++}, v_j^{--}]$. Then $d \in C[v_j^{++}, v_i^{--}]$ (see Figure 5). We consider the following cases.

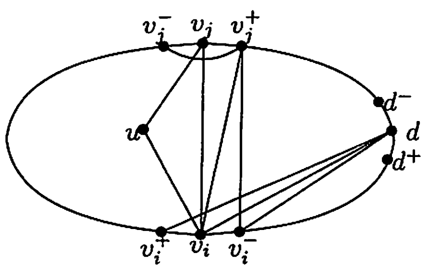


Figure 5

Case	Cycle C'
$d^+d^- \in E(G)$	$v_juv_iC[v_j^+, d^-]C[d^+, v_i^-]dC[v_i^+, v_j]$
$v_id^- \in E(G)$	$v_juv_i\overleftarrow{C}[d^-, v_j^+]\overleftarrow{C}[v_i^-, d]C[v_i^+, v_j]$
$v_id^+ \in E(G)$	$v_juv_iC[d^+, v_i^-]C[v_j^+, d]C[v_i^+, v_j]$

In each case, the cycle C' extends C. So $d^+d^-, v_id^+, v_id^- \notin E(G)$. Thus $G[\{d, d^+, d^-, v_i\}] \cong K_{1,3}$. But $v_i \in A$ and $v_id \in E(G)$, a contradiction. So $d \in C[v_i^{++}, v_j^{--}]$.

Claim 6 *$d^+d^-, d^-v_i^-, v_j^-v_j^{++} \notin E(G)$. Therefore, $v_i^-d^+ \in E(G)$, $v_j^{++} \neq v_i^-, v_i^+v_j^-, uv_j^{++}, v_iv_j^- \notin E(G)$ and $v_iv_j^{++} \in E(G)$.*

We argue by contradiction using the following chart.

Case	Cycle C'
$d^+d^- \in E(G)$	$v_juv_iC[v_j^+, v_i^-]dC[v_i^+, d^-]C[d^+, v_j]$
$d^-v_i^- \in E(G)$	$v_juv_iC[v_j^+, v_i^-]\overleftarrow{C}[d^-, v_i^+]C[d, v_j]$
$v_j^-v_j^{++} \in E(G)$	$v_juC[v_i, v_j^-]C[v_j^{++}, v_i^-]v_j^+v_j$

In each case, the cycle C' extends C. So $d^+d^-, d^-v_i^-, v_j^-v_j^{++} \notin E(G)$. As $G[\{d, d^-, d^+, v_i^-]\ncong K_{1,3}$, we have $v_i^-d^+ \in E(G)$. Thus $v_j^{++} \neq v_i^-$ (otherwise, the cycle $v_juC[v_i, d]v_i^-\ C[d^+, v_j^-]v_j^+v_j$ would extend C), $v_i^+v_j^- \notin E(G)$ (otherwise, the cycle $v_juv_i\overleftarrow{C}[d, v_i^+]\overleftarrow{C}[v_j^-, d^+]\overleftarrow{C}[v_i^-, v_j]$ would extend C), and $uv_j^{++} \notin E(G)$ (otherwise, the cycle $v_juC[v_j^{++}, v_i^-]v_j^+C[v_i, v_j]$ would again extend C). Noticing that $G[\{v_i, v_i^+, v_i^-, u, v_j^-\}]$ would be isomorphic to $K_{1,4}$ if $v_iv_j^- \in E(G)$, we have $v_iv_j^- \notin E(G)$. Consider $G[\{v_j^+, v_j^{++}, v_j^-, v_i\}]$. As $v_j^+ \notin A$, we have $v_iv_j^{++} \in E(G)$ (see Figure 6).

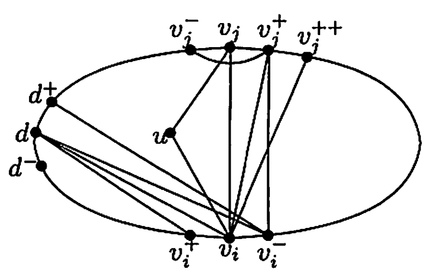


Figure 6

Claim 7 *u and v_j^{++} have no common neighbor in $N_G(v_i)$.*

Suppose, by contradiction, that u and v_j^{++} have a common neighbor t in $N_G(v_i)$. Clearly, $t \in V(C)$ (otherwise, the cycle $v_j^{++}tC[v_i, v_j^+]\overleftarrow{C}[v_i^-, v_j^{++}]$ extends C). If $t = v_j$, then the cycle $v_j uC[v_i, v_j^-]v_j^+\overleftarrow{C}[v_i^-, v_j^{++}]v_j$ extends C, and so $t \neq v_j$. As $uv_j^+ \notin E(G)$ and $uv_j^- \notin E(G)$, $t \notin \{v_j^+, v_j^-\}$. Since $v_i \in A$, we have $t \notin A$, which implies that $t^+t^- \in E(G)$. Thus the cycle $v_j utC[v_j^{++}, t^-]C[t^+, v_j^-]v_j^+v_j$ extends C, a contradiction. So Claim 7 holds.

Note that $G[N_G(x_i)]$ is 2-dominated and $d \in N_G(v_i)$ is a vertex dominating v_i^+ and v_i^-. By Claim 7, u and v_j^{++} cannot be dominated by a vertex in $N_G(v_i)$. Thus v_j^{++} must be dominated by d, and so $dv_j^{++} \in E(G)$. Therefore the cycle $v_j uC[v_i, d]C[v_j^{++}, v_i^-]C[d^+, v_j^-]v_j^+v_j$ extends C, a contradiction. This contradiction completes the proof.

References

[1] J. A. Bondy and U. S. R. Murty, "Graph Theory with Applications". American Elsevier (1976).

[2] L. Clark, *Hamiltonain properties of connected, locally connected graphs*, Congr. Numer. 32 (1981), 199-204.

[3] G. R. T. Hendry, *Extending cycles in graphs*, Discrete Math. 85 (1990), 59-72.

[4] D. Oberly and D. Summer, *Every connected, locally connected nontrivial graph with no induced claw is hamiltonian*, J. Graph Theory 3 (1979), 351-356.

[5] Z. Ryjáček, *Almost claw-free graphs*, J. Graph Theory 18(1994), 469-477.

[6] Y. Shao, *Claw-free graphs and line graphs*, Ph. D. Dissertation, West Virginia Univeristy, 2005.