Dominating Sets and Independent Sets in a Tree

Min-Jen Jou

Department of Insurance
Ling Tung University
Taichung, Taiwan 40852, R.O.C.

e-mail: mjjou@mail.ltu.edu.tw

Abstract

The domination number $\gamma(G)$ of a graph G is the minimum cardinality among all dominating sets of G, and the independence number $\alpha(G)$ of G is the maximum cardinality among all independent sets of G. For any graph G, it is easy to see that $\gamma(G) \leq \alpha(G)$. In this paper, we present a characterization of trees T with $\gamma(T) = \alpha(T)$.

1. Introduction

Let $G = (V(G), E(G))$ be a finite, undirected and simple graph with vertex set $V(G)$ and edge set $E(G)$. The cardinality of $V(G)$ is called the *order* of G, denoted by $|G|$. A set S of vertices in a graph G is a *dominating set* of G if each vertex not in S is adjacent to at least one vertex of S. The *domination number* $\gamma(G)$ of G is the minimum cardinality among all dominating sets of G. If S is a dominating set of G with cardinality $\gamma(G)$, we call S a *γ-set* of G. A set I of vertices in a graph G is an *independent set* of G if no two vertices of I are adjacent in G. The *independence number* $\alpha(G)$ of G is the maximum cardinality among all independent sets of G. If I is an independent set of G with cardinality $\alpha(G)$, we call I an *α-set* of G. A *tree* T is a connected graph with no cycles.

Over the past few years, several studies have been made on domination and independence [2, 3, 4, 5, 6]. The main purpose of this paper is to obtain a characterization of trees T with $\gamma(T) = \alpha(T)$.

2. Preliminary

For any vertex v, the set of neighbors of v in G is denoted by $N_G(v)$, and $N_G[v] = N_G(v) \cup \{v\}$. For any subset $A \subseteq V(G)$, denote $N_G(A) = \cup_{v \in A} N_G(v)$ and $N_G[A] = \cup_{v \in A} N_G[v]$. A vertex v of G is a *leaf* if $|N_G(v)| = 1$. A vertex v of G is a *support vertex* if it is adjacent to a leaf in G. We denote the set $L(G)$ the collection of all leaves of G, and the set $U(G)$ the collection of all support vertices of G. Two distinct vertices u and v are *duplicated* if $N_G(u) = N_G(v)$. For a subset $F \subseteq E(G)$, the *deletion of F from G* is the graph $G - F$ obtained from G by deleting all edges of F.

If u and v are duplicated vertices in a tree T, then both of them are leaves. The following lemma states that each α-set of a tree T contains all duplicated leaves in T.

Lemma 1 *Let u and v be two distinct duplicated leaves adjacent to y in a tree T. Then*
(1) both u and v lie in every α-set of T;
(2) the vertex y lies in every γ-set of T;
(3) $\gamma(T-u) = \gamma(T-v) = \gamma(T)$.

Proof. (1) Suppose to the contrary that there exists an α-set I of T such that $u \notin I$, then $y \in I$ and $v \notin I$. So $I' = (I - \{y\}) \cup \{u, v\}$ is an independent set of T with cardinality $|I'| = |I| + 1 > \alpha(T)$, this is a contradiction.

(2) Suppose to the contrary that there exists an γ-set S of T such that $y \notin I$, then $u \in S$ and $v \in S$. So $S' = (S - \{u, v\}) \cup \{y\}$ is a dominating set of T with cardinality $|S'| = |S| - 1 < \gamma(T)$, this is a contradiction.

(3) It follows by (2). □

Lemma 2 *Let T be a tree with duplicated leaves, and let T' be a maximal subtree of T with no duplicated leaves. Then $\gamma(T') = \gamma(T)$.*

Proof. It follows by Lemma 1 (3). □

Lemma 3 *If T is a tree such that $\gamma(T) = \alpha(T)$, then T has no duplicated leaves.*

Proof. Suppose to the contrary that there is a set $A = \{v_1, v_2, \cdots, v_k\}$ of duplicated leaves adjacent to y in T, where $k \geq 2$. Let I be an α-set of T. By Lemma 1, the independent set I contains each v_i for $i = 1, 2, \cdots, k$. So the set $I - A$ dominates all vertices of $T - N_G[A]$. Then $S = (I - A) \cup \{y\}$ is a dominating set of T with cardinality $|S| = (|I| - k) + 1 \leq |I| - 1 = \alpha(T) - 1 = \gamma(T) - 1$, this is a contradiction. □

First of all, we will focus our attention on the independence problem.

Lemma 4 *Let T be a tree of order $n \geq 2$ with no duplicated leaves, and let x be a leaf adjacent to y in T. Then there exists an α-set I of T such that $x \in I$.*

Proof. Suppose that I' is an α-set of T. If $x \in I'$, then we are done. So we assume that $x \notin I'$, this implies $y \in I'$. Therefore $I = (I' - \{y\}) \cup \{x\}$ is an independent set of T with cardinality $|I| = |I'| = \alpha(T)$ such that $x \in I$. □

Lemma 5 *If T is a tree of order $n \geq 1$, then $\alpha(T) \geq \frac{n}{2}$.*

Proof. Since T is a bipartite graph, it is possible to partition $V(T)$ into V_1 and V_2 such that every V_i is an independent set of T. It follows that $\alpha(T) \geq \max\{|V_1|, |V_2|\} \geq \frac{n}{2}$. □

Let us now shift the emphasis away from independence to domination.

Lemma 6 *Let T be a tree of order $n \geq 2$ with no duplicated vertices, and let x be a leaf adjacent to y in T. Then there exists an γ-set S of T such that $y \in S$.*

Proof. Suppose that S' is a γ-set of T. If $y \in S'$, then we are done. Hence we assume that $y \notin S'$, this implies $x \in S'$. So $S = (S' - \{x\}) \cup \{y\}$ is a dominating set of T with cardinality $|S| = |S'| = \gamma(T)$ such that $y \in S$. □

Lemma 7 *If G is a graph of order $n \geq 2$ without any isolated vertices, then $\gamma(G) \leq \frac{n}{2}$.*

Proof. Choose a maximum independent set I of G. Then I is a dominating set of G. As G has no isolated vertices, $V(G) - I$ is also a dominating set. One of the two dominating sets above gives that $\gamma(G) \leq \frac{n}{2}$. □

Lemma 8 *Let $e = uv$ be an edge of a tree T such that both u and v are not leaves of T. Suppose that the deletion $T - e$ is the union of trees T_1 and T_2, where $u \in V(T_1)$ and $v \in V(T_2)$. Then $\gamma(T) \leq \gamma(T_1) + \gamma(T_2)$.*

Proof. Let S_1 and S_2 be γ-sets of T_1 and T_2, respectively. Then $S = S_1 \cup S_2$ is a dominating set of T, this implies that $\gamma(T) \leq |S| = |S_1| + |S_2| = \gamma(T_1) + \gamma(T_2)$. □

3. Main Theorem

For a graph G, let $\widehat{G}$ be the graph with vertex set $V(\widehat{G}) = V(G) \cup \{\hat{x} : x \in V(G)\}$ and the edge set $E(\widehat{G}) = E(G) \cup \{x\hat{x} : x \in V(G)\}$.

Lemma 9 *For a graph G, $\gamma(\widehat{G}) = \alpha(\widehat{G}) = |V(G)|$.*

Proof. Note that for any vertex $x \in V(G)$, every dominating set of $\hat{G}$ contains at least one vertex in $\{x, \hat{x}\}$ and every independent set of $\hat{G}$ contains at most one vertex in $\{x, \hat{x}\}$. Hence,

$$|V(G)| \leq \gamma(\widehat{G}) \leq \alpha(\widehat{G}) \leq |V(G)|$$

and so $\gamma(\widehat{G}) = \alpha(\widehat{G}) = |V(G)|$. □

With this lemma in mind, we provide a characterization of trees T with $\gamma(T) = \alpha(T)$.

Theorem 1 *If T is a tree of order $n \geq 2$, then $\gamma(T) = \alpha(T)$ if and only if $T = \widehat{G}$ for some tree G of order $\frac{n}{2}$.*

Proof. We shall prove by induction on n that $\gamma(T) = \alpha(T)$ implies $T = \widehat{G}$ for some tree G of order $\frac{n}{2}$. The claim is true for the case of T is a star for which $\gamma(T) = \alpha(T)$ implies $T = K_2 = \widehat{K_1}$. Suppose now T is not a star. Choose a vertex y whose neighbors are all leaves, called $x_1, x_2, \ldots, x_k$, except one called z. Then $T' = T - \{y, x_1, x_2, \ldots, x_k\}$ is a tree of order $n' = n - 1 - k$. Since a dominating set of T', together with y, form a dominating set of T, we have $\gamma(T') + 1 \geq \gamma(T)$. Also an independent set in T', together with $x_1, x_2, \ldots, x_k$, form an independent set in T, we have $\alpha(T) \geq \alpha(T') + k$. Hence,

$$\alpha(T') + 1 \geq \gamma(T') + 1 \geq \gamma(T) = \alpha(T) \geq \alpha(T') + k$$

and so in fact $k = 1$ and all inequalities above are equalities. In particular, $\gamma(T') = \alpha(T')$. By the induction hypothesis, $T' = \widehat{G'}$ for some tree G' of order $\frac{n'}{2} = \frac{n}{2} - 1$.

If the non-leaf neighbor z of y is equal to x for some $x \in V(G')$, then $T = \widehat{G}$ where G is obtained from G' by adding a new vertex y adjacent to $x = z$. Now suppose $z = \hat{x}$ for some $x \in V(G')$. Then $(V(G') - \{x\}) \cup \{y\}$ is a dominating set of T of size $\frac{n}{2} - 1$, and $\{\hat{x} : x \in V(G')\} \cup \{x_1\}$ is an independent set of T of size $\frac{n}{2}$. These give that $\gamma(T) \leq \frac{n}{2} - 1 < \frac{n}{2} \leq \alpha(T)$, which is impossible. This completes the proof of the claim. □

Acknowledgements

The author is grateful to the referee for the helpful comments.

References

[1] J.A. Bondy, USR Murty, Graph Theory with Application, New York, 1976.

[2] M.J. Jou, G.J. Chang, The number of maximal independent sets in graphs, Taiwanese J. Math., 4 (2000) 685-695.

[3] M.J. Jou, G.J. Chang, Maximal independent sets in graphs with at most one cycle, Discrete Appl. Math. 79 (1997) 67-73.

[4] G.J. Chang, M.J. Jou, The number of maximal independent sets in connected triangle-free graphs, Discrete Math. 197/198 (1999) 169-178.

[5] O. Favaron, Least domination in a graph, Discrete Math. 150 (1996) 115-122.

[6] X. Lv, J. Mao, Total domination and least domination in a tree, Discrete Math. 265 (2003) 401-404.