GRACEFUL AND EDGE-ANTIMAGIC LABELINGS

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ABSTRACT

A graph labeling is an assignment of integers (*labels*) to the vertices and/or edges of a graph. Within vertex labelings, two main branches can be distinguish: difference vertex labelings that associate each edge of the graph with the difference of the labels of its endpoints. Graceful and edge-antimagic vertex labelings correspond to these branches, respectively. In this paper we study some connections between them. Indeed, we study the conditions that allow us to transform any α-labeling (an special case of graceful labeling) of a tree into an $(a, 1)$- and $(a, 2)$-edge antimagic vertex labeling.

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1. Introduction

Let G be a graph of order m and size n. An injective function $f : V(G) \to \{1, 2, ..., n+1\}$ is a *graceful labeling* of G if when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels (or *weights*) are distinct. A graph that admits a graceful labeling is said to be *graceful.* A graceful labeling f of a graph G is said to be an α*-labeling* if there exists an integer λ such that for each edge xy of G either $f(x) \le \lambda < f(y)$ or $f(y) \le \lambda < f(x)$. This number λ, is called the *boundary value* of f. A graph that admits an α-labeling is called an α*-graph.* These labelings were introduced by Rosa [5] in the mid sixties.

An α-graph is necessarily bipartite and for any of its α-labelings the vertices whose labels do not exceed the boundary value form one of the sets of the bipartition. When a positive constant is added to each label larger than the boundary value, each induced weight increases by the same constant. Clearly, if the constant is added to all the labels the set of induced weights remains the same. If f is a graceful labeling of a graph of size n, its *complementary labeling* $\overline{f}$, defined by $\overline{f}(v) = n+2-f(v)$ for all $v \in V(G)$ is also graceful. For more information about graceful and α-labelings the reader is referred to [3].

Simanjuntak, Bertault, and Miller [6] define an (a,d)-*edge-antimagic vertex labeling* for a graph G of order m and size n as an injective mapping $f : V(G) \to \{1,2,...,m\}$ such that the set $\{f(x)+f(y) : xy \in E(G)\}$ is $\{a, a+d, a+2d, ..., a+d(n-1)\}$ for two non-negative integers a and d. We use the notation (a,d)-EAV to refer to these labelings. A bijection $f : V(G) \cup E(G) \to \{1,2,...,m+n\}$ is called (a,d)-*edge-antimagic total* labeling of G if the set of *edge weights* $\{f(x)+f(y)+f(xy) : xy \in E(G)\}$ is $\{a, a+d, a+2d, ..., a+d(n-1)\}$ for two non-negative integers a and d. For this labeling we use the notation (a,d)-EAT. An (a,d)-EAT labeling f of a graph G is said to be *super* if the vertices of G receive the labels $1,2,...,m$. A graph that admits an (a,d)-EAV labeling or a super (a,d)-EAT labeling is called an (a,d)-EAV graph or a super (a,d)-EAT graph, respectively.

Sugeng, Miller, and Bača [7] proved that if G is a graph of order m and size n, that admits a super (a,d)-EAT labeling, then $d \leq \frac{2m+n-5}{n-1}$. From here, the following lemma can be proved.

LEMMA 1.1. *Let T be a tree of order at least 2. If T is super (a,d)-EAT, then $d \leq 3$.*

2. Connections Between Sum and Difference Labelings

LEMMA 2.1. *Let T be a tree of order m. If T admits an α-labeling, then T also admits an $(a,1)$-EAV labeling.*

Proof. Suppose that f is an α-labeling of T with boundary value λ. Let $\{A,B\}$ be the bipartition of $V(T)$; without loss of generality, we may assume that the vertex labeled λ belongs to A. Consider the following vertex labeling of T:

$$g(v) = \begin{cases} f(v), & \text{if } v \in A \\ m+1+\lambda-f(v), & \text{if } v \in B. \end{cases}$$

We claim that this is a $(\lambda+2,1)$-EAV labeling of T. In fact, the labels assigned by g to the vertices of A are $1,2,...,\lambda$, and those assigned to the vertices of B are $\lambda+1, \lambda+2, ..., m$. Thus, g is an injective function from

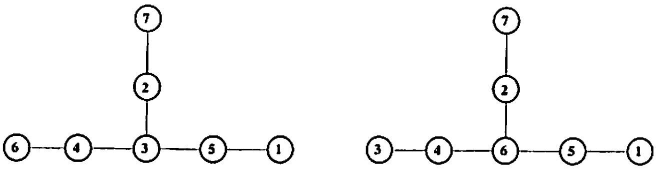

Fig. 1. $(a, 1)$-EAV labelings of a tree that is not an α-tree

$V(T)$ to $\{1, 2, ..., m\}$. Moreover, if uv is an edge of T, with $u \in A$ and $v \in B$, $g(u)+g(v) = m+1+\lambda-(f(v)-f(u))$. Since f is an α-labeling, $\{f(v)-f(u) : uv \in E(T)\}$ equals $\{1, 2, ..., m-1\}$, then $\lambda + 2 \leq g(u) + g(v) \leq \lambda + m$. In other terms, $\{g(u) + g(v) : uv \in E(T)\}$ equals $\{\lambda + 2, \lambda + 3, ..., \lambda + m\}$. Therefore, g is a $(\lambda + 2, 1)$-EAV labeling of T. ■

In general the converse of this lemma does not hold; there are $(a, 1)$-EAV trees that are not α-trees; in Figure 1 we show an example that support this statement.

The next lemma proves that under certain conditions an $(a, 1)$-EAV graph is also super $(a, 0)$- and super $(a, 2)$-EAT.

LEMMA 2.2. *Let G be a graph of order m and size n. If G admits an $(a, 1)$-EAV labeling, then G also admits a super $(m+n+a, 0)$-EAT labeling and a super $(m+1+a, 2)$-EAT labeling.*

Proof. Let g be an $(a, 1)$-EAV labeling of G. Now we transform g into a total labeling by defining it on the edges of G.

First, let $g(uv) = m+n+a-(g(u)+g(v))$ for each edge uv of G; since g is an $(a, 1)$-EAV labeling, $a \leq g(u)+g(v) \leq n-1+a$, thus $m+1 \leq g(uv) \leq m+n$. Each edge uv of G has weight $g(u)+g(v)+g(uv) = m+n+a$ and therefore g is a super $(m+n+a, 0)$-EAT labeling of G.

Consider now $g(uv) = m+1+g(u)+g(v)-a$, for each edge uv of G. Notice that $m+1 \leq g(uv) \leq m+n$. Thus each edge uv of G has weight $g(u)+g(v)+g(uv) = m+1+2(g(u)+g(v))-a$; therefore, the set of edge weights is $\{m+1+a, m+3+a, ..., m+2n-1+a\}$ and g is a super $(m+1+a, 2)$-EAT labeling of G. ■

As consequence of the last two lemmas we have that every α-tree also admits labelings of the types super $(a, 0)$- and super $(a, 2)$-EAT.

The following results are related to $(a, 2)$-EAV labelings of trees. Notice that if a tree of size n is $(a, 2)$-EAV, then $a = 3$. In fact, since the edge-weights are $a, a+2, ..., a+2(n-1)$, the inequality $a+2(n-1) \leq 2n+1$ holds, which implies that $a \leq 3$; therefore $a = 3$. In [1], Bača et al. proved (in Theorem 5) that if G is a graph of order m and size n such that G has an (a, d)-EAV labeling, then G also has super $(a+m+1, d+1)$- and super $(a+m+n, d-1)$-EAT labelings. As consequence of this theorem, we have the following corollaries.

COROLLARY 2.1. *Let T be a tree of order m. If T admits a (3, 2)-EAV labeling, then T also admits a super $(m+4,3)$-EAT labeling.*

COROLLARY 2.2. *Let T be a tree of order m. If T admits a (3, 2)-EAV labeling, then T also admits a super $(2m+2,1)$-EAT labeling.*

In our next results we establish a relationship between α-labelings and (3, 2)-EAV labelings of trees. As we mentioned before, any α-graph is bipartite; thus, when G is an α-graph we denote by $\{A, B\}$ the bipartition of its vertex set. Without loss of generality, we may assume that $|A| \geq |B|$. Note that if G is an α-graph of size n, there exists an α-labeling that assigns its boundary value to a vertex of A. In fact, if f is an α-labeling of G with boundary value λ and the vertex labeled λ is not in A, then its complementary labeling $\overline{f}$ assigns its boundary value $n+1+\lambda$ to a vertex of A.

LEMMA 2.3. *Let T be an α-tree. If $|A|-|B| \leq 1$, then T is (3, 2)-EAV.*

Proof. Let f be an α-labeling of a tree T of order m with boundary value λ. Suppose that the vertex labeled λ belongs to A. Consider the following labeling of the vertices of T:

$$g(v) = \begin{cases} 2f(v) - 1, & \text{if } v \in A \\ 2(m+1-f(v)), & \text{if } v \in B. \end{cases}$$

We claim that g is a (3, 2)-EAV labeling of T. In fact, notice that g is an injective function that assigns the labels $\{1,3,...,2\lambda-1\} \cup \{2,4,...,2(m-\lambda)\}$ to the vertices of T. Since $|A| - |B| \leq 1$, $\lambda = \lceil \frac{m}{2} \rceil$ and this union is $\{1,2,...,m\}$. Furthermore, $\{g(v)+g(u) : uv \in E(T)\} = \{2m+1-2(f(v)-f(u)) : uv \in E(T)\}$. Since f is an α-labeling, $\{f(v) - f(u) : uv \in E(T)\} = \{1,2,...,m-1\}$, we have that $\{g(v)+g(u) : uv \in E(T)\} = \{3,5,...,2m-1\}$. Thus, g is a (3, 2)-EAV labeling of T. ■

So far, we have proved that if T is a (3, 2)-EAV tree with $|A|-|B| \leq 1$, then T is super (a,d)-EAT for every $d \in \{1,3\}$. In the next two lemmas we prove that the converse of this statement also holds, which allow us to characterize (3, 2)-EAV trees.

LEMMA 2.4. *Let T be a tree of order m. If $|A| - |B| > 1$, then there is no (3, 2)-EAV labeling of T.*

Proof. By contradiction. Suppose that g is a (3, 2)-EAV labeling of T; so all edge weights induced by g are odd numbers. Thus, for each edge $uv \in E(T)$, $g(u)$ and $g(v)$ have different parity, which implies that the labels assigned to the vertices in A have the same parity. Since $|A| - |B| > 1$, we have that $|A| > \lceil \frac{m}{2} \rceil$, but in the set of labels $\{1, 2, ..., m\}$ there are $\lceil \frac{m}{2} \rceil$ odd numbers and $\lfloor \frac{m}{2} \rfloor$ even numbers, then not all vertex labels in A have the same parity, which is a contradiction. Hence, there is no (3, 2)-EAV labeling of T. ■

LEMMA 2.5. *Let T be a tree of order m. If T does not admit an α-labeling, then neither admits an (3, 2)-EAV labeling.*

Proof. By contradiction. Suppose that g is a (3, 2)-EAV labeling of T. Thus, if $v \in A$, $g(v) \in \{1, 3, ..., m\}$ when m is odd or $g(v) \in \{1, 3, ..., m-1\}$ when m is even, and if $v \in B$, $g(v) \in \{2, 4, ..., m-1\}$ when m is odd or $g(v) \in \{2, 4, ..., m\}$ when m is even.

Consider the labeling $f : V(T) \to \{0, 1, ..., m-1\}$ defined by

$$f(v) = \begin{cases} \frac{g(v)+1}{2}, & \text{if } v \in A \\ \frac{2m-g(v)+2}{2}, & \text{if } v \in B. \end{cases}$$

Thus, f assigns to the vertices of A the labels $\{1, 2, ..., \lceil \frac{m}{2} \rceil\}$ and to the vertices of B the labels $\{m, m-1, ..., \lceil \frac{m}{2} \rceil + 1\}$.

Let $x, y \in V(T)$ such that $x \in A$ and $y \in B$; thus, $f(y) - f(x) = \frac{2m+1-(g(x)+g(y))}{2}$. Since $\{g(x) + g(y) : xy \in E(T)\} = \{3, 5, ..., 2m-1\}$, we have $\{\frac{2m+1-(g(x)+g(y))}{2} : xy \in V(T)\} = \{m-1, m-2, ..., 1\}$. Then $\{f(y) - f(x) : xy \in E(T)\} = \{1, 2, ..., m-1\}$, that is, the weights induced by f on the edges of T are the first $m-1$ positive integers, which implies that f is a graceful labeling of T. Since the labels of the vertices in A are less than the labels of the vertices in B, f is an α-labeling of T which is a contradiction. Therefore, T does not admit a (3, 2)-EAV labeling. ■

Using these two lemmas the following theorem can be proved.

THEOREM 2.1. *A tree T is (3, 2)-EAV if and only if T is an α-tree and $||A| - |B|| \leq 1$, where $\{A, B\}$ is the bipartition of its vertex-set.*

REMARK 2.1. As a consequence of these results we have that any α-tree with $|A| - |B| \leq 1$ admits a super (a, d)-EAT labeling for every $d \in \{0, 1, 2, 3\}$.

3. Constructing Suitable α-Trees

Some methods for constructing α-trees are known; among them, there are two that produce trees that satisfy the conditions of Theorem 2.1. In this section we present these constructions extending the number of known trees that are super (a,d)-EAT.

In [4] Koh et al. provide a method for constructing bigger graceful trees from a given pair of graceful trees. Let T_1 and T_2 be two trees where $\{w_1, w_2, ..., w_m\}$ and $\{v_1, v_2, ..., v_n\}$ are their corresponding vertex sets. Let v^* be an arbitrary fixed vertex in T_2. Based upon the tree T_1, an isomorphic copy X_i of T_2 is adjoined to each vertex w_i $(i = 1, 2, ..., m)$ in such a way that v^* and w_i are identified. The m copies of T_2 just introduced are pairwise disjoint and no extra edges are added. Such a new tree was called $T_1 \triangle T_2$. They proved that if T_1 and T_2 are both graceful, then $T_1 \triangle T_2$ is also graceful (see Theorem 3 in [4]).

If T_1 is the path P_2, the labeling of $P_2 \triangle T_2$ is obtained by Koh et al., has the additional property required to be an α-labeling. In fact, let g and h be graceful labelings of P_2 and T_2, respectively. Consider the labeling $f : V(P_2 \triangle T_2) \to \{1, 2, ..., 2n\}$ defined as follows: for each v in X_i, $i = 1, 2$,

$$f(v) = \begin{cases} (g(w_i) - 1)n + h(v), \text{ if } d(v^*, v) \text{ is even} \\ (2 - g(w_i))n + h(v), \text{ if } d(v^*, v) \text{ is odd.} \end{cases}$$

Since we already know that f is a graceful labeling we just need to prove that it is an α-labeling.

Let $\{A_i, B_i\}$ be the bipartition of the vertex set of X_i. Then $A = A_1 \cup B_2$ and $B = B_1 \cup A_2$ form the bipartition of $V(P_2 \triangle T_2)$. Notice that $|A| = |B|$. Suppose that $\{A_0, B_0\}$ is the bipartition of $V(T_2)$, hence $\{1, 2, ..., n\} = \{h(v) : v \in A_0\} \cup \{h(v) : v \in B_0\}$. If $v^* \in B_0$, then $\{h(v) : v \in A_0\} = \{f(v) : v \in B_1\}$ and $\{h(v) : v \in B_0\} = \{f(v) : v \in A_2\}$. Either way, one of the partite sets of $P_2 \triangle T_2$ has assigned the labels $1, 2, ..., n$ and the other the labels $n+1, n+2, ..., 2n$. Thus, f is an α-labeling of $P_2 \triangle T_2$ whose boundary value is $\lambda = n$.

Since the partite sets of $V(P_2 \triangle T_2)$ have the same cardinality, the tree $T = P_2 \triangle T_2$ satisfies the conditions of Theorem 2.1, and the lemmas 2.1, 2.2, and 2.3. Thus, the following proposition can be proved.

PROPOSITION 3.1. *Every graceful tree produces a super (a,d)-EAT tree for every $d \in \{0, 1, 2, 3\}$.*

In Figure 1 we show an example of the α-labeling of $P_2 \triangle T_2$ obtained using the construction of Koh et al., we also exhibit the graceful labelings of P_2 and T_2. In Figure 2 we show the super (36, 0)- and (24, 2)-EAT labelings of $P_2 \triangle T_2$ obtained using Lemma 2.2, when the edge labels are eliminated we have the (9, 1)-EAV labeling obtained using Lemma 2.1. In Figure 4 we have the super (30, 1)- and (18, 3)-EAT labelings obtained using Corollary 2.2 and Corollary 2.1, respectively; when the edge labels are eliminated we have the (3, 2)-EAV labeling obtained using Lemma 2.3.

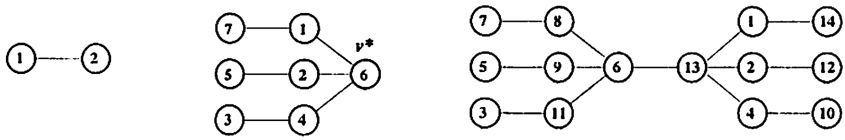

Fig. 1. α-labeling of $P_2 \triangle T_2$

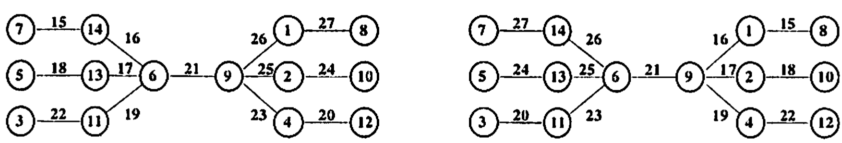

Fig. 2. Super (36, 0)- and (24, 2)-EAT labelings of $P_2 \triangle T_2$

Since the selection of the vertex $v^* \in T_2$ in the previous construction is arbitrary, each graceful tree T_2 of order m produces m α-labeled trees of the form $P_2 \triangle T_2$. In other words, every graceful tree produces at least one super (a, d)-EAT tree for every $d \in \{0, 1, 2, 3\}$.

In [2] we prove that given to α-graphs G_1 and G_2, there exists an α-graph G that results of the identification of suitable vertices $u \in V(G_1)$ and $v \in V(G_2)$. Some of the α-trees produced using this idea satisfy the conditions of Theorem 2.1 and the previous lemmas. In the next proposition we study the case where G_1 and G_2 are isomorphic to an α-tree.

PROPOSITION 3.2. *Every α-tree produces a super (a, d)-EAT tree for every $d \in \{0, 1, 2, 3\}$.*

Proof. Let T_0 be an α-tree of size n with bipartition $\{A, B\}$. Let f be an α-labeling of T_0 that assigns its boundary value λ to a vertex in A. For $i = 1, 2$, X_i is a copy of T_0; we define a labeling g of the vertices of $X_1 \cup X_2$ as follows:

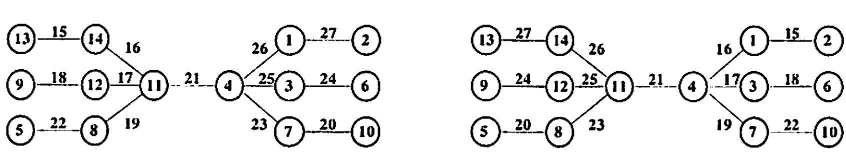

Fig. 3. Super (30, 1)- and (18, 3)-EAT labelings of $P_2 \triangle T_2$

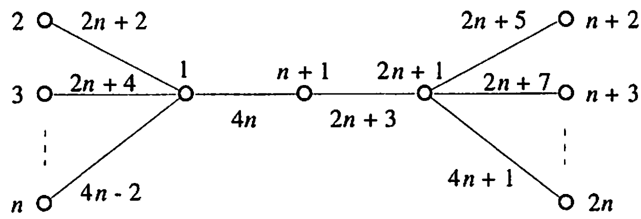

Fig. 4. Super $(2n+5,3)$-EAT labeling of $S_{n,2,n}$

$$g(v) = \begin{cases} f(v), & \text{if } v \in A_1 \\ n + f(v), & \text{if } v \in B_1 \\ n + \lambda + 1 - f(v), & \text{if } v \in V(X_2). \end{cases}$$

The labeling g assigns the labels $\{1, 2, ..., \lambda\} \cup \{n+\lambda+1, n+\lambda+2, ..., 2n+1\}$ to the vertices of X_1. The induced weights are $\{n+1, n+2, ..., 2n\}$. The labeling g assigns the labels $\{\lambda+1,\ \lambda+2, ..., \lambda+n\}$ to the vertices of X_2. Since g restricted to X_2 is just a translation of the complementary labeling $\overline{f}$ of f, we have that the induced weights are $\{1, 2, ..., n\}$. Both X_1 and X_2 have a vertex labeled λ. In X_1, λ is assigned to a vertex in A_1; in X_2, λ is assigned to a vertex in B_2. Thus, identifying both vertices labeled λ we have a tree T with an α-labeling of boundary value n. Since the cardinalities of the bipartite sets of T differ by one, we have that T satisfies the conditions of Theorem 2.1 and lemmas 2.1 and 2.3 and therefore T admits labelings of the kind super (a, d)-EAT for every $d \in \{0, 1, 2, 3\}$. ■

To conclude this work, in Figure 4 we present a super $(a$, 3)-EAT labeling of a family of caterpillars that do not satisfy the conditions of Theorem 2.1, which provides a counterexample for a conjecture posed by Sugeng et al. In [8] Sugeng et al. studied (a, d)-EAT labelings of caterpillars, they use the symbol $S_{n_1,n_2,...,n_r}$ to represent the caterpillar of diameter $r+2$ whose spine's vertices have degrees $n_1, n_2, ..., n_r$, respectively. They conjecture that if $||A| - |B|| > 1$ there is no super $(a$, 3)-EAT labeling of $S_{n_1,n_2,...,n_r}$ where $\{A, B\}$ is the bipartition of the vertex set of the caterpillar. In Figure 5 we exhibit a super (a, 3)-EAT labeling of the caterpillar $S_{n,2,n}$ for $n \geq 3$. Notice that in this case $|A| = 2$, $|B| = 2n-1$ and $||A| - |B|| = 2n-3 \geq 3$.

So far, this is the only counterexample that we have found.

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