Extraconnectivity of Cartesian product graphs of paths*

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Abstract. Given a graph G and a non-negative integer g, the g-extra-connectivity of G (written $\kappa_g(G)$)is the minimum cardinality of a set of vertices of G, if any, whose deletion disconnects G, and every remaining component has more than g vertices. The usual connectivity and superconnectivity of G correspond to $\kappa_0(G)$ and $\kappa_1(G)$, respectively. In this paper, we determine $\kappa_g(P_{n_1} \times P_{n_2} \times \cdots \times P_{n_s})$ for $0 \leq g \leq s$, where $\times$ denotes the Cartesian product of graphs. We generalize $\kappa_g(Q_n)$ for $0 \leq g \leq n$, $n \geq 4$, where Q_n denotes the n-cube.

Key words: Network; Extraconnectivity; Cartesian product

1 Introduction

The topology of interconnected network is often modeled by a connected graph of communication links. In the network the connectivity $\kappa(G)$ is an important factor determining the reliability and fault tolerance of the network. Here, we consider the extraconnectivity which corresponds to a kind of conditional connectivity introduced by Harary [3].

Let G be a connected undirected graph, and $\mathcal{P}$ be a graph-theoretic property, Harary [3] defined the conditional connectivity $\kappa(G;\mathcal{P})$ as the minimum cardinality of a set of vertices of G, if any, whose deletion disconnects G and every remaining component has property $\mathcal{P}$. Let g be a non-negative integer and let $\mathcal{P}_g$ be the property of having more than g vertices. Fàbrega and Fiol [2] defined the *g-extraconnectivity* $\kappa_g(G)$ of G as $\kappa(G;\mathcal{P}_g)$. Given a graph G and a non-negative integer g, the g-extraconnectivity of G(written $\kappa_g(G)$) is the minimum cardinality of a set of vertices of G, if any, whose deletion disconnects G, and every remaining component has more than g vertices.

The *Cartesian product* of two graphs G and H, denoted by $G \times H$, is the graph with vertex set $V(G) \times V(H)$ such that two vertices (u_1, u_2) and (v_1, v_2) are adjacent iff either $u_1 = v_1$ with $v_1v_2 \in E(H)$ or $u_2 = v_2$ with $u_1v_1 \in E(G)$. Let

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$G = P_{n_1} \times P_{n_2} \times \cdots \times P_{n_s}$, where P_{n_i} denotes the path with n_i vertices. If $n_i = 2$ for all i, then G is s-cube, denoted by Q_s, which has been studied by [4,5,6,7,9]. We assign the vertices of each path P_{n_i} a natural ordering by $\{1, 2, \cdots, n_i\}$. Thus we can use the $s-$dimensional array $x_1x_2\cdots x_s$ to denote the vertex of G, where $1 \leq x_i \leq n_i, i = 1, 2, \cdots, s$. Clearly, any two vertices $x = x_1x_2\cdots x_s$ and $y = y_1y_2\cdots y_s$ are adjacent iff there is exactly an integer i such that $x_i = y_i + 1$ or $y_i - 1$ and $x_j = y_j$ for $j \neq i$. We use G_i^j to denote the subgraph induced by $\{x_1\cdots x_{i-1}jx_{i+1}\cdots x_s | 1 \leq x_t \leq n_t, 1 \leq t \leq s, t \neq i\}$. By definition, we see that $G_i^1 \cong G_i^2 \cong \cdots \cong G_i^{n_i} \cong P_{n_1} \times \cdots \times P_{n_{i-1}} \times P_{n_{i+1}} \times \cdots \times P_{n_s}$. Sometimes we express G as $G_i^1 \bigodot G_i^2 \bigodot \cdots \bigodot G_i^{n_i}$. If $v = x_1x_2\cdots x_{i-1}1x_{i+1}\cdots x_s = v_i^1 \in V(G_i^1)$, we use v_i^j denote vertex $x_1x_2\cdots x_{i-1}jx_{i+1}\cdots x_{n_i}$. Clearly, $v_i^1\cdots v_i^{n_i} \cong P_{n_i}$. Let $B \subset G_i^j$, we use B_i^t denote the subgraph of G_i^t which use t to instead of j of the ith coordinate of the vertices of B. Clearly, $B \cong B_i^t$. It is well known that $\kappa(G \times H) \geq \kappa(G) + \kappa(H)$, we have $\kappa(P_{n_1} \times P_{n_2} \times \cdots \times P_{n_s}) = s - 1$, and $\kappa_1(P_{n_1} \times P_{n_2} \times \cdots \times P_{n_s}) = 2s - 1$ for $n_i \geq 3, i = 1, 2, \cdots, s$(see [8]). In this paper, we derive $\kappa_g(P_{n_1} \times P_{n_2} \times \cdots \times P_{n_s})$ when $g \leq s$ for $n_i \geq 2, i = 1, 2, \cdots, s$. We abbreviate $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_s}$ as G in the following sections. We use $N_G(v)$ to denote the set of the neighbors of v in G, $N_G(A)$ to denote the set $(\bigcup_{v \in V(A)} N_G(v)) \setminus V(A)$, $C_G(A)$ to denote the set $N_G(A) \cup V(A)$. We follow Bondy [1] for terminologies not given here.

2 Preliminaries

Before discussing the $\kappa_g(G)$, we give the following Lemmas.

Lemma 2.1. Assume $n_i \geq 3, i = 1, 2, \cdots, s$, $G = P_{n_1} \times P_{n_2} \times \cdots \times P_{n_s}$, $A \subset G$. If $|V(A)| = g + 1, g \leq s$, then $|N_G(A)| \geq s + (s+1)g - 2g - \binom{g}{2}$.

Proof. By induction on $|V(A)|$. Clearly, the result holds for $|V(A)| = 1$. Assume that the result holds for all A with $|V(A)| \leq h$. Next we show that the result is true for A with $|V(A)| = h + 1$. We directly use $g + 1$ instead of $h + 1$ in the following.

Let $A_j = G_i^j \cap A$. We first show that G can be decomposed into $G_i^1 \bigodot G_i^2 \bigodot \cdots \bigodot G_i^{n_i}$ for some i such that at least two subgraphs of $\{A_j, j = 1, 2, \cdots, n_i\}$ are nonempty. Note that $|V(A)| \geq 2$, let $x = x_1x_2\cdots x_s$ and $y = y_1y_2\cdots y_s$ be two distinct vertices of A. Without loss of generality, we assume $x_s \neq y_s$, and $x_s = 1, y_s = 2$, thus $x \in V(G_s^1), y \in V(G_s^2)$, that is $V(A_1) \neq \emptyset, V(A_2) \neq \emptyset$.

Without loss of generality, we consider $G = G_s^1 \bigodot G_s^2 \bigodot \cdots \bigodot G_s^{n_s}$ in the following. Assume all the nonempty subgraphs of $\{A_j | A_j = G_s^j \cap A\}$ are $A_1, A_2, \cdots, A_m$, and $|V(A_i)| = N_i$. Clearly, $N_i < |V(A)|$, by induction, $|N_{G_s^i}(A_i)| \geq s - 1 + s(N_i - 1) - 2(N_i - 1) - \binom{N_i - 1}{2}$. Since $N_{G_s^i}(A_i) \cap N_{G_s^j}(A_j) = \emptyset$ if $i \neq j$, it not difficult to see that $|N_G(A)| \geq \sum_{i=1}^{m} |N_{G_s^i}(A_i)| \geq s + (s+1)g - 2g - \binom{g}{2}$ for $m \geq 3$, or $m = 2$ and $N_i \geq 2, i = 1, 2$. Next we assume that $m = 2$ and $N_i = 1$(or $N_j = 1, j \neq i$). Without loss of generality, we consider $N_1 = 1$(or $N_2 = 1$). If $N_2 = 1$, then $|N_G(A)| \geq 2s - 1 = s + (s+1)g - 2g - \binom{g}{2}$. Assume that $N_2 \geq 2$. It is easy to see that $|N_{G_s^1}(A_1)| + |N_{G_s^2}(A_2)| \geq s + (s+1)g - 2g - \binom{g}{2} - 1$(since

$|N_{G_s^i}(A_i)| \geq s-1+s(N_i-1)-2(N_i-1)-\binom{N_i-1}{2})$. Note that $(A_2)_s^3 = N_{G_s^3}(A_2)$. Since $V((A_2)_s^3) \subset N_G(A)$ and $N_2 \geq 2$, we have that $|N_G(A)| \geq |N_{G_s^1}(A_1)| + |N_{G_s^2}(A_2)|+|V((A_2)_s^3)| \geq s+(s+1)g-2g-\binom{g}{2}-1+2 > s+(s+1)g-2g-\binom{g}{2}$(if $N_2 = 1$, we have that $|N_G(A)| \geq |N_{G_s^1}(A_1)| + |N_{G_s^2}(A_2)| + |V((A_2)_s^3)| \geq s+(s+1)g-2g-\binom{g}{2}-1+1 = s+(s+1)g-2g-\binom{g}{2}$). □

Corollary 2.2. Assume $G = P_{n_1} \times P_{n_2} \times \cdots \times P_{n_s}$, $n_i \geq 2$, $A \subset G$, $|V(A)| = g+1, g \leq s$. Let k be the number of K_2 of $\{P_{n_i}, i = 1, 2, \cdots, s\}$, where K_2 is the complete graph of two vertices, then $|N_G(A)| \geq s+(s+1)g-2g-\binom{g}{2}-min\{g,k\}$.

Proof. Combining the structure of G and a similar argument of Lemma 2.1, the result holds. □

Remark 2.3. Let $h_s(g) = s+(s+1)g-2g-\binom{g}{2}$, $h_s(g)$ is increasing when $0 \leq g \leq s-1$, the maximum of $h_s(g)$ is $h_s(s-1) = h_s(s) = \frac{s(s+1)}{2}$. By the proof of Lemma 2.1, we have $h_{s-1}(g_1)+\cdots h_{s-1}(g_t) > h_s(g)$ when $0 \leq g_1, \cdots, g_t \leq s, t \geq 3$ and $(g_1+1)+\cdots(g_t+1) \geq g+1$. In particular, if $t = 2$, then $h_{s-1}(g_1)+h_{s-1}(g_2) > h_s(g)$ for $g_i \geq 2, i = 1, 2$ and $(g_1+1)+(g_2+1) \geq g+1$ (see [9] for the detail).

Lemma 2.4. Let $G = G_s^1 \bigodot \cdots \bigodot G_s^{n_s}, B \subseteq G$, $|V(B)| \geq s$, $n_i \geq 3$, $i = 1, 2, \cdots, s$. If there exists a subgraph G_s^m such that $V(B \cap G_s^m) = \emptyset$, then $|N_G(B)| \geq \frac{s(s+1)}{2}$.

Proof. Let $B_i = B \cap G_s^i, i = 1, 2, \cdots, n_s$, without loss of generality, assume $B \cap G_s^{n_s} = \emptyset$. We verify the result by considering two cases.

Case 1. There exists a B_i such that $|V(B_i)| \geq s$.

Let T_{B_i} be a subgraph of B_i such that $|V(T_{B_i})| = s$. By Lemma 2.1, we have $|N_{G_s^i}(B_i)| \geq (s-1)+s(s-1)-2(s-1)-\binom{s-1}{2}$, that is $|C_{G_s^i}(B_i)| \geq (s-1)+s(s-1)-2(s-1)-\binom{s-1}{2}+s = \frac{s(s+1)}{2}$. Clearly, there are at least $\frac{s(s+1)}{2}$ internally disjoint paths between $C_{G_s^i}(B_i)$ and $G_s^{n_s}$. Therefore $|N_G(B)| \geq \frac{s(s+1)}{2}$.

Case 2. All B_i satisfy $|V(B_i)| < s$.

Assume that all $B_1, B_2, \cdots, B_m$ are nonempty. If $m \geq 2$, $|V(B_i)| \geq 2, i = 1, 2$, by Remark 2.3, we have $\sum_{i=1}^{n_s} |V(B_i)| \geq s$. It is sufficient to show that $|N_G(B)| \geq \frac{s(s+1)}{2}$ if there exists a B_i such that $|V(B_i)| = 1$. By a similar argument of the last part of Lemma 2.1, we complete this proof. □

Corollary 2.5. Let $G = G_s^1 \bigodot \cdots \bigodot G_s^{n_s}, B \subseteq G$, $|V(B)| \geq s$, $n_i \geq 2$, $i = 1, 2, \cdots, s$. If there exists a subgraph G_s^m such that $V(B \cap G_m) = \emptyset$, and the number of K_2 of $\{P_{n_1}, P_{n_2}, \cdots, P_{n_s}\}$ is k, then $|N_G(B)| \geq \frac{s(s+1)}{2} - k$.

Proof. By a similar argument of Lemma 2.4 and Corollary 2.2, the result holds. □

Lemma 2.6. Let $B \subseteq G$, $|V(B)| \geq s+1, n_i \geq 3, i = 1, 2, \cdots, s$. If $|V(G) \setminus$

$V(C_G(B))| \geq s+1$, then $|N_G(B)| \geq \frac{s(s+1)}{2}$.

Proof. By induction on s. In particular, $s=2$, then G is a grid, it is easy to see that $|N_G(B)| \geq \kappa_1(G) = 2s-1 = 3 = \frac{s(s+1)}{2}$. Assume that the result is true for $s < h$. Next we show that it holds for $s = h$. We directly use s instead of h in the following.

Let $G = G_s^1 \bigodot \cdots \bigodot G_s^{n_s}$, we verify the result by considering two cases.

Case 1. There exists a G_s^i such that $V(B \cap G_s^i) = \emptyset$.

By the Lemma 2.4, we have $|N_G(B)| \geq \frac{s(s+1)}{2}$.

Case 2. All $V(B_i) \neq \emptyset$.

If there exists a G_s^i such that $|V(B_i)| \geq s-1$ and $|V(G_s^i) \backslash V(C_{G_s^i}(B_i))| \geq s-1$, by induction, we have $|N_{G_s^i}(B_i)| \geq \frac{s(s-1)}{2}$. Since $n_s \geq 3$, and $\kappa(G_s^j) = s-1, j \neq i$, we have $|N_G(B)| \geq |N_{G_s^i}(B_i)| + \sum_{i \neq j} \kappa(G_j) \geq \frac{s(s-1)}{2} + 2(s-1) > \frac{s(s+1)}{2}$ for $s \geq 3$. If each pair $\{|V(B_i)|, |V(G_s^i) \setminus V(C_{G_i}(B_i))|\}$ has one element less than $s-1$, without loss of generality, we assume $|V(B_i)| < s-1$. Since $|V(B)| \geq s$, by Remark 2.3, we have $|N_G(B)| \geq \sum_{i=1}^{n_s} |N_{G_s^i}(B_i)| \geq \frac{s(s+1)}{2}$. □

Corollary 2.7. Let $B \subseteq G$, $|V(B)| \geq s+1, n_i \geq 2, i = 1, 2, \cdots, s$, and the number of K_2 of $\{P_{n_1}, P_{n_2}, \cdots, P_{n_s}\}$ is k. If $|V(G) \setminus V(C_G(B))| \geq s+1$, then $|N_G(B)| \geq \frac{s(s+1)}{2} - k$.

Lemma 2.8. Any two vertices x, y of $G = P_{n_1} \times P_{n_2} \times \cdots \times P_{n_s}$ have at most two common neighbors for $s \geq 2$ if they have any. Furthermore, if x and y exactly have two distinct coordinates, then x and y exactly have two common neighbors if they have any.

Proof. Let $x = x_1 x_2 \cdots x_s, y = y_1 y_2 \cdots y_s$ be two vertices of G. By definition, if x and y have common neighbor, then their coordinate have one or two is distinct. Clearly, they have one or two common neighbors. Furthermore, if x and y exactly have two distinct coordinates, say i and j, then $y_i = x_i + 1$ or $x_i - 1$, $y_j = x_j + 1$ or $x_j - 1$. Without loss of generality, we assume that $y_i = x_i + 1, y_j = x_j + 1$, then $x_1 \cdots x_{i-1} y_i x_{i+1} \cdots x_s$ and $x_1 \cdots x_{j-1} y_j x_{j+1} \cdots x_s$ are two common neighbors of x and y. □

Lemma 2.9. Let $n_i \geq 3, i = 1, 2, \cdots, s$, $G = P_{n_1} \times P_{n_2} \times \cdots \times P_{n_s} = G_s^1 \bigodot \cdots \bigodot G_s^{n_s}$ and $u = 11 \cdots 1 \in V(G_s^1)$, $u_i = 11 \cdots 2 \cdots 1 \in N(u), A = G[u, u_1, u_2, \cdots, u_g], 1 \leq i \leq g \leq s$, where u_i denotes the neighbor of v which the ith coordinate is different from u's. Then $|N_G(A)| = s + (s+1)g - 2g - \binom{g}{2}$ and $G - C_G(A)$ is a connected subgraph of G with property $\mathcal{P}_g$.

Proof. By lemma 2.8, it is easy to see that $|N_G(A)| = s + (s+1)g - 2g - \binom{g}{2}$.

Next we show $G - C_G(A)$ is connected. By induction on s. If $s = 2$, G is a grid, the result is clear. Assume that it is true for $s - 1 \geq 2$. Next we verify the Lemma for s.

Clearly, $A \subset G_s^1 \cup G_s^2$ and $|V(C_G(A) \cap G_s^2)| \leq 1$. By induction, $G_s^i - C_G(A)$ is connected for all i. That is, if $x, y \in V(G_s^i - C_G(A))$, then x connects to y. Next we assume that $x_s^i \in V(G_s^i), y_s^j \in V(G_s^j), i < j$. Since $n_s \geq 3$, then $x_s^i x_s^{i+1} \cdots x_s^{n_s}$ is a path of $G - C_G(A)$ from x_s^i to $x_s^{n_s} \in V(G_s^{n_s})$. Similarly, $y_s^j y_s^{j+1} \cdots y_s^{n_s}$ is a path of $G - C_G(A)$ from y_s^j to $y_s^{n_s} \in V(G_s^{n_s})$. Since $x_s^{n_s}$ connects $y_s^{n_s}$, we have x_s^i connects to y_s^j. □

Corollary 2.10. Let $n_i \geq 2, i = 1, 2, \cdots, s$, $G = P_{n_1} \times P_{n_2} \times \cdots \times P_{n_s} = G_s^1 \odot \cdots \odot G_s^{n_s}$ and $u = 11 \cdots 1 \in V(G_s^1)$, $u_i = 11 \cdots 2 \cdots 1 \in N(u), A = G[u, u_1, u_2, \cdots, u_g], 1 \leq i \leq g$, where u_i denotes the neighbor of u which the ith coordinate is different from u's. The number of K_2 of $\{P_{n_1}, P_{n_2}, \cdots, P_{n_s}\}$ is k. Then $|N_G(A)| \geq s + (s+1)g - 2g - \binom{g}{2} - min\{g, k\}$, for $0 \leq g \leq s$ and $G - C_G(A)$.

Proof. By a similar argument of Lemma 2.9. □

3 Main results

Now we come to our main results.

Theorem 3.1. Let $G = P_{n_1} \times P_{n_2} \times \cdots \times P_{n_s}, n_i \geq 3, i = 1, 2, \cdots, s, 0 \leq g \leq s$, then $\kappa_g(G) = s + (s+1)g - 2g - \binom{g}{2}$.

Proof. Let $u = (1, 1, \cdots, 1)$, $u_i = (1, 1, \cdots, 2, \cdots, 1)(1 \leq i \leq g)$ be the neighbors of u, where u_i denotes the neighbor of u which the ith coordinate is different from u's. Then $G[u, u_1, u_2, \cdots, u_g] \cong K_{1,g}$. By Lemma 2.9, we have $|N_G(A)| = s + (s+1)g - 2g - \binom{g}{2}$ and $G - C_G(A)$ has property $\mathcal{P}_g$. Thus we have $\kappa_g(G) \leq s + (s+1)g - 2g - \binom{g}{2}$.

Next we show that $\kappa_g(G) \geq s + (s+1)g - 2g - \binom{g}{2}$. Suppose F is a vertex cut set of G such that every component of $G - F$ has property $\mathcal{P}_g$ and $|F| \leq s + (s+1)g - 2g - \binom{g}{2} - 1$. We will show that it is impossible. If there exists a component A of $G - F$ such that $|V(A)| \leq s$, we have $g + 1 \leq |V(A)| \leq s$. By Lemma 2.1, we have $|F| \geq s + (s+1)g - 2g - \binom{g}{2}$, a contradiction. If every component of $G - F$ has size more than s, we derive a contradiction by Lemma 2.6. □

Corollary 3.2. Let $G = P_{n_1} \times P_{n_2} \times \cdots \times P_{n_s}, n_i \geq 2(i = 1, 2, \cdots, s)$. The number of K_2 of $\{P_{n_1}, P_{n_2}, \cdots, P_{n_s}\}$ is k.

(i) If $0 < k < \frac{s}{2}$, then $\kappa_g(G) \geq s + (s+1)g - 2g - \binom{g}{2} - min\{g, k\}$, for $0 \leq g \leq s-2$; $\kappa_g(G) = \frac{s(s+1)}{2} - k$, for $s - 1 \leq g \leq s$.

(ii) If $\frac{s}{2} \leq k < s$, then $\kappa_g(G) \geq s + (s+1)g - 2g - \binom{g}{2} - min\{g, k\}$, for $0 \leq g \leq s-3$; $\kappa_g(G) = \frac{s(s+1)}{2} - k$, for $s - 2 \leq g \leq s$.

(iii) If $k = s$, then $\kappa_g(G) \geq s + (s+1)g - 2g - \binom{g}{2} - min\{g, k\}$, for $0 \leq g \leq s-4$; $\kappa_g(G) = \frac{s(s+1)}{2} - s$, for $s - 3 \leq g \leq s$.

Proof. We only give the proof of (iii). Assume that A is the same subgraph as Theorem 3.1. By Corollary 2.10, we have that $|N_G(A)| = \frac{s(s+1)}{2} - s$ if $g = s$, and by Corollary 2.7, we have that $|N_G(B)| \geq \frac{s(s+1)}{2} - s$ for any subgraph B with order more than s. Thus $\kappa_g(G) \leq \frac{s(s+1)}{2} - s$ for $g \leq s$.

Let $f(g) = s + (s+1)g - 2g - \binom{g}{2} - min\{g, k\}$. It is easy to see that $f(1) < f(2) < \cdots < f(s-4) < f(s-3) = f(s) < f(s-2) = f(s-1)$. By Corollary 2.2, we have $\kappa_g(G) = f(g)$ for $g \leq s-4$ and $\kappa_g(G) = f(s-3) = f(s) = \frac{s(s+1)}{2} - s$. □

Corollary 3.3. $\kappa_g(Q_n) = (g+1)n - 2g - \binom{g}{2}$ for $0 \leq g \leq n-4$ and $\kappa_g(Q_n) = \frac{n(n+1)}{2} - n$ for $n-3 \leq g \leq n$, where $Q_n = P_2 \times P_2 \times \cdots \times P_2$ is a well known networks (see [9] for a detail proof).

Proof. By (iii) of Corollary 3.2, Corollary 3.3 is clearly true. □

Remark 3.4. There are also some other known results can be obtained directly by Theorem 3.1 and Corollary 3.3, such as the results about hypercube of [6] and the results of [8].

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References

[1] F.Haray, Conditional connectivity. Networks, 13 (1983) 346-357.

[2] J. Fàbrega and M.A. Fiol, On the extraconnectivity of graphs. Discrete Mathematics, 155 (1996) 49-57.

[3] J. Meng and Y. Ji, On a kind of restricted edge connectivity of graphs. Discrete Appl. Math, 117 (2002) 183-193.

[4] S. Latifi, M. Hegde and M. Naraghi-Pour, Conditional connectivity measures for large multiprocessor systems. IEEE. Trans. on Computers, 43(2) (2002), 218-222.

[5] Y. Saad and M. Schultz, Topological properties of hypercubes. IEEE Trans. on Compters, 37(7) (1988) 867-872.

[6] J. Xu, Q. Zhu, X. Hou and T. Zhou, On restricted connectivity and extra connectivity of hypercubes and foled hypercubes. J. of Shanghai Jiaotong Univ. 10(2) (2005) 203-207.

[7] W. Yang and J. Meng, Extraconnectivity of Hypercubes. Applied Mathematics Letters, 6(22) (2009) 887-891.

[8] M. Fu, J. Meng and X. Liang, Super connectivity of Cartesian product of paths. Infomation Processing Letters, in press.

[9] J.A. Bondy and U.S.R. Murty, Graph theory with application. Macmillan, London, 1976.