THE MODIFIED ZAGREB INDICES ABOUT DISJUNCTION AND SYMMETRIC DIFFERENCE OF GRAPHS

Jianxiu Hao

Institute of Mathematics, Physics and Information Sciences,

Zhejiang Normal University, P. O. Box: 321004,

Jinhua, Zhejiang, P.R. China; e-mail: sx35@zjnu.cn

Abstract: The modified Zagreb indices are important topological indices in mathematical chemistry. In this paper we study the modified Zagreb indices of disjunctions and symmetric differences.

INTRODUCTION

The *first Zagreb index* $M_1(G)$ and the *second Zagreb index* $M_2(G)$ are defined as follows [1, 2, 3, 4, 5]: for a simple connected graph G, let $M_1(G) = \sum_{v\in V(G)}(d(v))^2$, $M_2(G) = \sum_{uv\in E(G)}d(u)d(v)$, where d(u) and d(v) are the degrees of vertices u and v respectively.

However, both the first Zagreb index and the second Zagreb index give greater weights to the inner vertices and edges, and smaller weights to outer vertices and edges which opposes intuitive reasoning. Hence, they are amended as follows [6]: for a simple connected graph G, let ${}^mM_1(G) = \sum_{v\in V(G)}(d(v))^{-2}$, ${}^mM_2(G) = \sum_{uv\in E(G)}(d(u)d(v))^{-1}$, we call ${}^mM_1(G)$ and ${}^mM_2(G)$ as *the first modified Zagreb index* and *the second modified Zagreb index respectively.*

PRELIMINARIES

Definition 2.1[7, 8, 9]. The zeroth-order general Randic index ${}^0R_t(G) = \sum_{v\in V(G)}d(v)^t$ for general real number t, where d(v) is the degree of v. When $t = -0.5$, ${}^0R_{-0.5}(G)$ is the famous zeroth-order Randic index $R^0(G)$. Randic index of graph G, denotes $\chi(G)$, is defined as follows: $\chi(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d(u)d(v)}}$.

Definition 2.2[11]. The disjunction $G\vee H$ of graphs G and H is the graph with vertex set $V(G)\times V(H)$ and (u_1, v_1) is adjacent with (u_2, v_2) whenever $u_1u_2\in E(G)$ or $v_1v_2\in E(H)$.

Lemma 2.3[12, 13]. $d_{G\vee H}((a, b)) = |V(H)|d_G(a) + |V(G)|d_H(b) - d_G(a)d_H(b)$, $|E(G\vee H)| = |E(G)||V(H)|^2 + |E(H)||V(G)|^2 - 2|E(G)||E(H)|$.

Lemma 2.4[10]. Every nontrivial tree has at least two vertices of degree one.

Definition 2.5[11]. The symmetric difference $G\oplus H$ of graphs G and H is the graph with vertex set $V(G)\times V(H)$ and $E(G\oplus H) = \{(u_1, u_2)(v_1, v_2)|\ u_1v_1\in E(G)$

or $u_2v_2 \in E(H)$ but not both}.

Lemma 2.6[12, 13]. $d_{G\oplus H}((a, b)) = |V(H)|d_G(a) + |V(G)|d_H(b) - 2d_G(a)d_H(b)$, $|E(G\oplus H)| = |E(G)||V(H)|^2 + |E(H)||V(G)|^2 - 4|E(G)||E(H)|$.

MAIN RESULTS ABOUT DISJUNCTIONS

Theorem 3.1. ${}^mM_1(P_m \vee P_n) =$

$$\frac{4}{(m+n-1)^2}+\frac{2(n-2)}{(2m+n-2)^2}+\frac{2(m-2)}{(m+2n-2)^2}+\frac{(m-2)(n-2)}{4(m+n-2)^2},$$ where $m, n \geq 2$.

Theorem 3.2. ${}^mM_1(K_m \vee K_n) = \frac{mn}{(mn-1)^2}$, where $m, n \geq 2$.

Theorem 3.3. ${}^mM_1(C_m \vee C_n) = \frac{mn}{4(m+n-2)^2}$, where $m, n \geq 3$.

Theorem 3.4. ${}^mM_1(K_{1,\ m-1} \vee K_{1,\ n-1}) = \frac{(m-1)(n-1)}{(m+n-1)^2} +$

$$\frac{m-1}{(mn-m+1)^2}+\frac{n-1}{(mn-n+1)^2}+\frac{1}{(mn-1)^2},$$ where $m, n \geq 2$.

Theorem 3.5. ${}^mM_1(P_m \vee C_n) = \frac{2n}{(2m+n-2)^2}+\frac{n(m-2)}{4(m+n-2)^2}$, where $m \geq 2, n \geq 3$.

Theorem 3.6. ${}^mM_1(P_m \vee K_{1,\ n-1}) =$

$$\frac{2(n-1)}{(m+n-1)^2}+\frac{2}{(mn-m+1)^2}+\frac{(m-2)(n-1)}{(m+2n-2)^2}+\frac{m-2}{(mn-m+2)^2},$$ where $m, n \geq 2$.

Theorem 3.7. ${}^mM_1(C_m \vee K_{1,\ n-1}) = \frac{m(n-1)}{(m+2n-2)^2}+\frac{m}{(mn-m+2)^2}$, where $m \geq 3, n \geq 2$.

Theorem 3.8. ${}^mM_1(P_m \vee K_n) = \frac{2n}{(mn-m+1)^2}+\frac{n(m-2)}{(mn-m+2)^2}$, where $m, n \geq 2$.

Theorem 3.9. ${}^mM_1(C_m \vee K_n) = \frac{mn}{(mn-m+2)^2}$, where $m \geq 3, n \geq 2$.

Theorem 3.10. ${}^mM_1(K_m \vee K_{1,n-1}) = \frac{m(n-1)}{(mn-n+1)^2}+\frac{m}{(mn-1)^2}$, where $m, n \geq 2$.

Theorem 3.11. Let G and H be simple connected graphs, we have

$$\frac{mn}{(mn-1)^2} \leq {}^mM_1(G \vee H) \leq \min\left\{\frac{{}^0R_{-1}(G)\,{}^0R_{-1}(H)}{mn},\ \frac{(m-1)(n-1)}{(m+n-1)^2}+\right.$$

$$\left.\frac{m-1}{(mn-m+1)^2}+\frac{n-1}{(mn-n+1)^2}+\frac{1}{(mn-1)^2}\right\},$$ where ${}^0R_{-1}(G)$ is defined in

Definition 2.1, $m = |V(G)| \geq 2$, $n = |V(H)| \geq 2$.

Proof. Since $d_G(a)<m$, $d_H(b)<n$, $nd_G(a) + md_H(b) \geq 2(mnd_G(a)d_H(b))^{0.5}$, by the definition of ${}^{m}M_1$ we have ${}^{m}M_1(G \vee H) = \sum_{(a,b)\in V(G\vee H)} \frac{1}{[d_{G\vee H}(a,b)]^2}$

$$= \sum_{(a,b)\in V(G\vee H)} \frac{1}{[nd_G(a)+md_H(b)-d_G(a)d_H(b)]^2}$$

$$< \sum_{(a,b)\in V(G\vee H)} \frac{1}{[2\sqrt{nd_G(a)md_H(b)}-\sqrt{md_G(a)nd_H(b)}]^2} =$$

$$\sum_{(a,b)\in V(G\vee H)} \frac{1}{mnd_G(a)d_H(b)} = \frac{{}^{0}R_{-1}(G)\,{}^{0}R_{-1}(H)}{mn}$$. Since $G \vee H$ is a subgraph of $K_m \vee K_n = K_{mn}$, by Theorem 3.2 we have ${}^{m}M_1(G \vee H) \geq \frac{mn}{(mn-1)^2}$.

Claim 1: Let A be a spanning tree of G, B be a spanning tree of H, we have

$$d_{A\vee B}((a, b)) \leq d_{G\vee H}((a, b)).$$

In fact, let $d_G(a) = d_A(a) + t$, $d_H(b) = d_B(b) + s$, where $t, s \geq 0$. Since $m > d_G(a)$, we have $m - d_A(a) > t$. By Lemma 2.3 we have

$$d_{A\vee B}((a, b)) = nd_A(a) + md_B(b) - d_A(a)d_B(b),$$

$$d_{G\vee H}((a, b)) = nd_G(a) + md_H(b) - d_G(a)d_H(b)$$

$$= n(d_A(a) + t) + m(d_B(b) + s) - (d_A(a) + t)(d_B(b) + s)$$

$= d_{A\vee B}((a, b)) + (m - d_A(a))s + (n - d_B(b))t - st \geq d_{A\vee B}((a, b))$. Claim 1 follows.

By Claim 1 we have

Claim 2: Let A be a spanning tree of G, B be a spanning tree of H, we have

$${}^{m}M_1(G \vee H) \leq {}^{m}M_1(A \vee B).$$

Claim 3: Let G be a tree with $|V(G)| \geq 4$, $G \neq K_{1,m-1}$, by Lemma 2.4 there exists $c \in V(G)$ such that $d(c) = 1$, $ac \in E(G)$. Let $d \in V(G)$ such that $d(d) = \Delta(G)$, where $\Delta(G)$ is the maximum degree of G. We construct a new graph D from G as follows: $D = G - ac + dc$. We have ${}^{m}M_1(G \vee H) \leq {}^{m}M_1(D \vee H)$.

In fact, by Lemma 2.3 we have

$$d_{G\vee H}((a, b)) = nd_G(a) + md_H(b) - d_G(a)d_H(b),$$

$$d_{D\vee H}((a, b)) = n(d_G(a) - 1) + md_H(b) - (d_G(a) - 1)d_H(b)$$

$$= d_{G\vee H}((a, b)) - (n - d_H(b)).$$

$$d_{G\vee H}((d, b)) = n\Delta(G) + md_H(b) - \Delta(G)d_H(b),$$

$$d_{D\vee H}((d, b)) = n(\Delta(G) + 1) + md_H(b) - (\Delta(G) + 1)d_H(b)$$

$= d_{G\vee H}((d, b)) + (n - d_H(b))$.

Let $f(x) = \frac{1}{x^2} - \frac{1}{(x+p)^2}$, where p >0. We have f'(x) < 0. Hence, f(x) is a decreasing function. Let $\Delta(G) = d_G(a) + r$, where $r \geq 0$, we can prove $d_{G\vee H}((a, b)) - (n - d_H(b)) < d_{G\vee H}((d, b))$ easily. Hence,

$$\frac{1}{[d_{D\vee H}((a,b))]^2} - \frac{1}{[d_{G\vee H}((a,b))]^2} > \frac{1}{[d_{G\vee H}((d,b))]^2} - \frac{1}{[d_{D\vee H}((d,b))]^2}.$$

Claim 3 follows. Similarly, we can discuss H. By Claim 3 many times we have Claim 4: ${}^mM_1(G\vee H) \leq {}^mM_1(K_{1,m-1} \vee K_{1,n-1})$.

By Claim 4 and Theorem 3.4 we have ${}^mM_1(G\vee H) \leq \frac{(m-1)(n-1)}{(m+n-1)^2} + \frac{m-1}{(mn-m+1)^2} + \frac{n-1}{(mn-n+1)^2} + \frac{1}{(mn-1)^2}$. The theorem follows.

Similarly, we have

Theorem 3.12. ${}^mM_2(P_m \vee P_n) =$

$$\frac{1}{2}\{\frac{4}{m+n-1}[\frac{2}{m+2n-2} + \frac{2}{2m+n-2} + \frac{m+n-5}{2(m+n-2)}] +$$

$$\frac{1}{2m+n-2}[\frac{4(m+n-5)}{m+2n-2} + \frac{n^2+2mn-6m-10n+22}{m+n-2} + \frac{8}{m+n-1} + \frac{8n-24}{2m+n-2}] +$$

$$\frac{1}{m+2n-2}[\frac{8}{m+n-1} + \frac{4m+4n-20}{2m+n-2} + \frac{8m-24}{m+2n-2} + \frac{m^2+2mn-10m-6n+22}{m+n-2}]$$

$$+ \frac{1}{m+n-2}[\frac{2m+2n-10}{m+n-1}$$

$$+ \frac{m^2+2mn-10m-6n+22}{m+2n-2} + \frac{n^2+2mn-6m-10n+22}{2m+n-2} +$$

$$\frac{m^2n+mn^2-10mn-3m^2-3n^2+22m+22n-42}{2(m+n-2)}]\}, \text{ where } m, n \geq 3.$$

Theorem 3.13. ${}^mM_2(K_m \vee K_n) = \frac{mn}{2(mn-1)}$, where $m, n \geq 2$.

Theorem 3.14. ${}^mM_2(C_m \vee C_n) = \frac{mn}{4(m+n-2)}$, where $m, n \geq 3$.

Theorem 3.15. ${}^mM_2(K_{1,m-1} \vee K_{1,n-1}) = \frac{(m-1)(n-1)^2}{(m+n-1)(mn-n+1)} +$

$$\frac{(m-1)^2(n-1)}{(m+n-1)(mn-m+1)} + \frac{(m-1)(n-1)}{(m+n-1)(mn-1)} + \frac{(m-1)(n-1)}{(mn-n+1)(mn-m+1)}$$

$$+\frac{n-1}{(mn-n+1)(mn-1)} + \frac{m-1}{(mn-m+1)(mn-1)}$$, where m, n $\geq$ 2.

Theorem 3.16. ${}^{m}M_2(P_m \vee C_n) = \frac{n^2+2mn-6n}{(m+n-2)(2m+n-2)} +$

$$\frac{m^2n+mn^2-6mn-3n^2+10n}{4(m+n-2)^2} + \frac{4n}{(2m+n-2)^2}$$, where m, n $\geq$ 3.

Theorem 3.17. ${}^{m}M_2(P_m \vee K_{1,\ n-1}) = \frac{2n-2}{(m+n-1)^2} +$

$$\frac{mn^2-2mn-3n^2+6n+m-3}{(m+2n-2)^2} +$$

$$\frac{m-2}{(mn-m+2)^2} + \frac{n^2+mn-m-4n+3}{(m+n-1)(m+2n-2)} + \frac{mn-m-n+1}{(m+n-1)(mn-m+2)} +$$

$$\frac{2n-2}{(m+n-1)(mn-m+1)} + \frac{1}{(mn-m+1)(mn-m+2)} +$$

$$\frac{2mn-2m-4n+4}{(mn-m+1)(m+2n-2)} + \frac{m^2n-4mn-m^2+4m+4n-4}{(m+2n-2)(mn-m+2)}$$, where m, n $\geq$ 5.

Theorem 3.18. ${}^{m}M_2(C_m \vee K_{1,\ n-1}) = \frac{m(n-1)^2}{(m+2n-2)^2} + \frac{m}{(mn-m+2)^2} +$

$$\frac{m^2(n-1)}{(mn-m+2)(m+2n-2)}$$, where m $\geq$ 3, n $\geq$ 2.

Theorem 3.19. ${}^{m}M_2(P_m \vee K_n) = \frac{2n^2-2n}{(mn-m+1)^2} +$

$$\frac{m^2n^2-4mn^2+6mn+4n^2-m^2n-10n}{2(mn-m+2)^2} + \frac{2n(mn-m-2n+3)}{(mn-m+1)(mn-m+2)}$$, where m $\geq$ 3, n $\geq$ 2.

Theorem 3.20. ${}^{m}M_2(C_m \vee K_n) = \frac{mn}{2(mn-m+2)}$, where m $\geq$ 3, n $\geq$ 2.

Theorem 3.21. ${}^{m}M_2(K_m \vee K_{1,n-1}) = \frac{m(m-1)(n-1)^2}{2(mn-n+1)^2} + \frac{m(m-1)}{2(mn-1)^2}$

$$+\frac{m^2(n-1)}{(mn-1)(mn-n+1)}$$, where m, n $\geq$ 2.

Theorem 3.22. Let G and H be simple connected graphs, we have ${}^{m}M_2(G \vee H)$

$$< \frac{1}{mn}\left[\frac{n^2\chi(G)}{\delta(H)}+\frac{(m^2-2|E(G)|)\chi(H)}{\delta(G)}\right]$$, where $\chi(G)$ is defined in Definition 2.1, m = |V(G)| ≥ 2, n = |V(H)| ≥ 2,

$|E(G)|\chi(H)\delta(H) \geq |E(H)|\chi(G)\delta(G)$, $\delta(G)$ is the minimum degree of G.

Proof. By Lemma 2.3 we have

$$^{m}M_2(G \vee H) = \sum_{(a,b)(x,y)\in E(G\vee H)} \frac{1}{d_{G\vee H}((a,b))d_{G\vee H}((x,y))} =$$

$$\sum_{(a,b)(x,y)\in E(G\vee H)} \frac{1}{[nd_G(a)+md_H(b)-d_G(a)d_H(b)]}\frac{1}{[nd_G(x)+md_H(y)-d_G(x)d_H(y)]}.$$

Since $p + q \geq 2\sqrt{pq}$, where p and q are positive numbers, we have $M_2(G \vee H)$

$$\leq \sum_{(a,b)(x,y)\in E(G\vee H)} \frac{1}{2\sqrt{mnd_G(a)d_H(b)}-d_G(a)d_H(b)}\frac{1}{2\sqrt{mnd_G(x)d_H(y)}-d_G(x)d_H(y)}$$

$$< \sum_{(a,b)(x,y)\in E(G\vee H)} \frac{1}{2\sqrt{mnd_G(a)d_H(b)}-\sqrt{mnd_G(a)d_H(b)}}\frac{1}{2\sqrt{mnd_G(x)d_H(y)}-\sqrt{mnd_G(x)d_H(y)}}$$

$$= \frac{1}{mn}\sum_{(a,b)(x,y)\in E(G\vee H)} \frac{1}{\sqrt{d_G(a)d_G(x)}}\frac{1}{\sqrt{d_H(b)d_H(y)}}$$

$$= \frac{1}{mn}\Big[\sum_{a\in V(G)}\sum_{x\in V(G)}\sum_{by\in E(H)} \frac{1}{\sqrt{d_G(a)d_G(x)}}\frac{1}{\sqrt{d_H(b)d_H(y)}}$$

$$+\sum_{b\in V(H)}\sum_{y\in V(H)}\sum_{ax\in E(G)} \frac{1}{\sqrt{d_G(a)d_G(x)}}\frac{1}{\sqrt{d_H(b)d_H(y)}}$$

$$-2\sum_{ax\in E(G)}\sum_{by\in E(H)} \frac{1}{\sqrt{d_G(a)d_G(x)}}\frac{1}{\sqrt{d_H(b)d_H(y)}}\Big]$$

$$= \frac{1}{mn}\left[\frac{n^2\chi(G)}{\delta(H)}+\frac{(m^2-2|E(G)|)\chi(H)}{\delta(G)}\right]$$. The theorem follows.

MAIN RESULTS ABOUT SYMMETRIC DIFFERENCES

Theorem 4.1. $^{m}M_1(P_m \oplus P_n) =$

$$\frac{4}{(m+n-2)^2}+\frac{2(n-2)}{(2m+n-4)^2}+\frac{2(m-2)}{(m+2n-4)^2}+\frac{(m-2)(n-2)}{(2m+2n-8)^2}$$, where m, n ≥ 2.

Theorem 4.2. $^{m}M_1(K_m \oplus K_n) = \frac{mn}{(m+n-2)^2}$, where m, n ≥ 2.

Theorem 4.3. ${}^{m}M_1(C_m \oplus C_n) = \frac{mn}{(2m+2n-8)^2}$, where m, n ≥ 3.

Theorem 4.4. ${}^{m}M_1(K_{1,m-1} \oplus K_{1,n-1}) = \frac{mn-m-n+2}{(m+n-2)^2} + \frac{m+n-2}{(mn-m-n+2)^2}$, where m, n ≥ 2.

Theorem 4.5. ${}^{m}M_1(P_m \oplus C_n) = m \geq 2, n \geq 3$. where m, n ≥ 3.

Theorem 4.6. ${}^{m}M_1(P_m \oplus K_{1,n-1}) = \frac{2n-2}{(m+n-2)^2} + \frac{2}{(mn-m-n+2)^2}$

$+ \frac{(m-2)(n-1)}{(m+2n-4)^2} + \frac{m-2}{(mn-m-2n+4)^2}$, where m, n ≥ 2.

Theorem 4.7. ${}^{m}M_1(C_m \oplus K_{1,n-1}) = \frac{m(n-1)}{(m+2n-4)^2} + \frac{m}{(mn-m-2n+4)^2}$, where m ≥ 3, n ≥ 2.

Theorem 4.8. ${}^{m}M_1(P_m \oplus K_n) = \frac{2n}{(mn-m-n+2)^2} + \frac{n(m-2)}{(mn-m-2n+4)^2}$, where m , n ≥ 2.

Theorem 4.9. ${}^{m}M_1(C_m \oplus K_n) = \frac{mn}{(mn-m-2n+4)^2}$, where m ≥ 3, n ≥ 2.

Theorem 4.10. ${}^{m}M_1(K_m \oplus K_{1,n-1}) = \frac{m(n-1)}{(mn-m-n+2)^2} + \frac{m}{(m+n-2)^2}$, where m , n ≥ 2.

Theorem 4.11. Let G and H be simple connected graphs, we have

$\max\{\frac{mn}{(mn-1)^2}, \frac{mn}{[n\Delta(G)+m\Delta(H)-2\delta(G)\delta(H)]^2}\} \leq {}^{m}M_1(G \oplus H)$

$\leq \min\{0.25\,{}^{0}R_{-1}(G)\,{}^{0}R_{-1}(H), \frac{mn}{(m+n-2)^2}\}$, where ${}^{0}R_{-1}(G)$ is defined in Definition 2.1, m = |V(G)| ≥ 2, n = |V(H)| ≥ 2.

Proof. By Lemma 2.6 we have $d_{G\oplus H}((a, b)) \leq n\Delta(G)+m\Delta(H)-2\delta(G)\delta(H)$,

thus, ${}^{m}M_1(G \oplus H) \geq \frac{mn}{[n\Delta(G)+m\Delta(H)-2\delta(G)\delta(H)]^2}$. ${}^{m}M_1(G \oplus H) =$

$\sum_{(a,b)\in V(G\oplus H)} \frac{1}{[d_{G\oplus H}((a,b))]^2} = \sum_{(a,b)\in V(G\oplus H)} \frac{1}{[nd_G(a)+md_H(b)-2d_G(a)d_H(b)]^2} >$

$\sum_{(a,b)\in V(G\oplus H)} \frac{1}{[nd_G(a)+md_H(b)-d_G(a)d_H(b)]^2} = {}^{m}M_1(G \vee H)$. By Theorem 3.11

we have ${}^{m}M_1(G \oplus H) > \dfrac{mn}{(mn-1)^2}$.

$${}^{m}M_1(G \oplus H) = \sum_{(a,b)\in V(G\oplus H)} \frac{1}{[d_{G\oplus H}((a,b))]^2} =$$

$$\sum_{(a,b)\in V(G\oplus H)} \frac{1}{[nd_G(a)+md_H(b)-2d_G(a)d_H(b)]^2} \le$$

$$\sum_{(a,b)\in V(G\oplus H)} \frac{1}{[(d_H(b)+1)d_G(a)+(d_G(a)+1)d_H(b)-2d_G(a)d_H(b)]^2} =$$

$$\sum_{(a,b)\in V(G\oplus H)} \frac{1}{[d_G(a)+d_H(b)]^2} \le \sum_{(a,b)\in V(G\oplus H)} \frac{1}{4d_G(a)d_H(b)} = 0.25\,{}^{0}R_1(G)\,{}^{0}R_1(H).$$

Let $V(G) = \{u_1, u_2, \ldots, u_m\}$, $V(H) = \{v_1, v_2, \ldots, v_n\}$. Since G and H are connected graphs, for $u_r \in V(G)$ there exists $u_s \in V(G)$ such that $u_ru_s \in E(G)$. For $(u_r, v_i) \in V(G \oplus H)$, if there exists $v_j \in V(H)$ such that v_iv_j does not belong to E(H), by Definition 2.5 $(u_r, v_i)(u_r, v_j)$ does not belong to $E(G \oplus H)$, $(u_r, v_i)(u_s, v_j) \in E(G \oplus H)$. Similarly, if there exists $u_t \in V(G)$ such that u_ru_t does not belong to E(G), there exists $v_k \in V(H)$ such that $v_iv_k \in E(H)$, by Definition 2.5 $(u_r, v_i)(u_t, v_i)$ does not belong to $E(G \oplus H)$, $(u_r, v_i)(u_t, v_k) \in E(G \oplus H)$. Hence, we have $d_{G\oplus H}((u_r, v_i)) \ge m + n - 2$. By the definition of ${}^{m}M_1(G \oplus H)$ we have

${}^{m}M_1(G \oplus H) \le \dfrac{mn}{(m+n-2)^2}$. The theorem follows.

Similarly, we have

Theorem 4.12. ${}^{m}M_2(P_m \oplus P_n) =$

$$\frac{2}{m+n-2}\left[\frac{2}{m+2n-4}+\frac{2}{2m+n-4}+\frac{m+n-6}{2m+2n-8}\right]+$$

$$\frac{1}{2m+n-4}\left[\frac{4}{m+n-2}+\frac{2m+2n-12}{m+2n-4}+\frac{4n-12}{2m+n-4}+\frac{2mn+n^2-6m-12n+28}{2m+2n-8}\right]$$

$$+\frac{1}{m+2n-4}\left[\frac{4}{m+n-2}+\frac{4m-12}{m+2n-4}+\frac{2m+2n-12}{2m+n-4}+\right.$$

$$\left.\frac{2mn+m^2-12m-6n+28}{2m+2n-8}\right]+$$

$$\frac{1}{2m+2n-8}\left[\frac{2m+2n-12}{m+n-2}+\frac{n^2+2mn-6m-12n+28}{2m+n-4}+\right.$$

$$\left.\frac{m^2+2mn-12m-6n+28}{m+2n-4}+\frac{m^2n+mn^2-12mn-3m^2-3n^2+28m+28n-60}{2m+2n-8}\right],$$

where m, n ≥ 3.

Theorem 4.13. $^{m}M_2(K_m \oplus K_n) = \dfrac{mn}{2(m+n-2)}$, where m, n ≥ 2.

Theorem 4.14. $^{m}M_2(C_m \oplus C_n) = \dfrac{mn}{4(m+n-4)}$, where m, n ≥ 3.

Theorem 4.15. $^{m}M_2(K_{1,\,m-1} \oplus K_{1,\,n-1}) = 1$, where m, n ≥ 2.

Theorem 4.16. $^{m}M_2(P_m \oplus C_n) = \dfrac{4n}{(2m+n-4)^2} + \dfrac{n(2m+n-8)}{(2m+n-4)(m+n-4)} + \dfrac{m^2n+mn^2-3n^2-8mn+16n}{(2m+2n-8)^2}$, where m, n ≥ 3.

Theorem 4.17. $^{m}M_2(P_m \oplus K_{1,\,n-1}) =$

$$\frac{n-1}{m+n-2}\left[\frac{n-1}{m+2n-4}+\frac{2}{mn-m-n+2}+\frac{m-3}{mn-m-2n+4}\right]$$

$$+\frac{1}{mn-m-n+2}\left[\frac{2n-2}{m+n-2}+\frac{mn-m-3n+3}{m+2n-4}+\frac{1}{mn-m-2n+4}\right]+$$

$$\frac{1}{2(m+2n-4)}\left[\frac{2n^2-4n+2}{m+n-2}+\frac{2mn^2-4mn-6n^2+2m+12n-6}{m+2n-4}+\right.$$

$$\left.\frac{2mn-2m-6n+6}{mn-m-n+2}+\frac{m^2n-m^2-6mn+6m+10n-10}{mn-m-2n+4}\right]+\frac{1}{2(mn-m-2n+4)}\left[\right.$$

$$\frac{2mn-2m-6n+6}{m+n-2}+\frac{m^2n-m^2-6mn+6m+10n-10}{m+2n-4}+\frac{2}{mn-m-n+2}+$$

$$\left.\frac{2m-6}{mn-m-2n+4}\right]\text{, where } m \geq 3,\ n \geq 2.$$

Theorem 4.18. $^{m}M_2(C_m \oplus K_{1,\,n-1}) = \dfrac{m(n-1)^2}{(m+2n-4)^2} +$

$$\frac{m(m-2)(n-1)}{(m+2n-4)(mn-m-2n+4)}+\frac{m}{(mn-m-2n+4)^2}\text{, where } m \geq 3,\ n \geq 2.$$

Theorem 4.19. $^{m}M_2(P_m \oplus K_n) = \dfrac{2n(n-1)}{(mn-m-n+2)^2} +$

$$\frac{2n(mn-m-3n+4)}{(mn-m-n+2)(mn-m-2n+4)}+\frac{n(m^2n-m^2-6mn+8m+10n-16)}{2(mn-m-2n+4)^2},$$

where m ≥ 3, n ≥ 2.

Theorem 4.20. $^{m}M_2(C_m \oplus K_n) = \dfrac{mn}{2(mn-m-2n+4)}$, where m ≥ 3, n ≥ 2.

Theorem 4.21. ${}^{m}M_2(K_m \oplus K_{1,n-1}) = \frac{m(n-1)}{2(mn-m-n+2)}\left[\frac{(m-1)(n-1)}{mn-m-n+2} + \frac{1}{m+n-2}\right] + \frac{m}{2(m+n-2)}\left[\frac{n-1}{mn-m-n+2} + \frac{m-1}{m+n-2}\right]$, where m , n ≥ 2.

Theorem 4.22. Let G and H be simple connected graphs, we have ${}^{m}M_2(G \oplus H) < \frac{1}{4}\left[\frac{n^2\chi(G)}{\delta(H)} + \frac{m^2\chi(H)}{\delta(G)}\right] - \chi(G)\chi(H)$, where $\chi(G)$ is defined in Definition 2.1, m = |V(G)| ≥ 2, n = |V(H)| ≥ 2, $\delta(G)$ is the minimum degree of G.

Proof. By Lemma 2.6 we have ${}^{m}M_2(G \oplus H) =$

$$\sum_{(a,b)(x,y)\in E(G\oplus H)} \frac{1}{d_{G\oplus H}((a,b))d_{G\oplus H}((x,y))} =$$

$$\sum_{(a,b)(x,y)\in E(G\oplus H)} \frac{1}{[nd_G(a)+md_H(b)-2d_G(a)d_H(b)]}\frac{1}{[nd_G(x)+md_H(y)-2d_G(x)d_H(y)]} \leq$$

$$\sum_{(a,b)(x,y)\in E(G\oplus H)} \frac{1}{[(d_H(b)+1)d_G(a)+(d_G(a)+1)d_H(b)-2d_G(a)d_H(b)]}$$

$$\times\frac{1}{[(d_H(y)+1)d_G(x)+(d_G(x)+1)d_H(y)-2d_G(x)d_H(y)]}$$

$$= \sum_{(a,b)(x,y)\in E(G\oplus H)} \frac{1}{[d_G(a)+d_H(b)]}\frac{1}{[d_G(x)+d_H(y)]}$$

$$\leq \sum_{(a,b)(x,y)\in E(G\oplus H)} \frac{1}{2\sqrt{d_G(a)d_H(b)}}\frac{1}{2\sqrt{d_G(x)d_H(y)}}$$

$$= \frac{1}{4}\sum_{(a,b)(x,y)\in E(G\oplus H)} \frac{1}{\sqrt{d_G(a)d_G(x)}}\frac{1}{\sqrt{d_H(b)d_H(y)}}$$

$$=\frac{1}{4}\Big[\sum_{a\in V(G)}\sum_{x\in V(G)}\sum_{by\in E(H)} \frac{1}{\sqrt{d_G(a)d_G(x)}}\frac{1}{\sqrt{d_H(b)d_H(y)}}$$

$$+\sum_{b\in V(H)}\sum_{y\in V(H)}\sum_{ax\in E(G)} \frac{1}{\sqrt{d_G(a)d_G(x)}}\frac{1}{\sqrt{d_H(b)d_H(y)}}$$

$$-4\sum_{ax\in E(G)}\sum_{by\in E(H)} \frac{1}{\sqrt{d_G(a)d_G(x)}}\frac{1}{\sqrt{d_H(b)d_H(y)}}\Big]$$

$\leq \frac{1}{4}\left[\frac{n^2\chi(G)}{\delta(H)} + \frac{m^2\chi(H)}{\delta(G)}\right] - \chi(G)\chi(H)$. The theorem follows.

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