Genus polynomials for three types of graphs*

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Abstract On the basis of joint trees introduced by Yanpei Liu, by choosing different spanning trees and classifying the associated surfaces, we obtain the explicit expressions of genus polynomials for three types of graphs, namely K_5^n, W_6^n and $K_{3,3}^n$, which are different from the graphs whose embedding distributions by genus have been got. And K_5^n and $K_{3,3}^n$ are non-planar.

Keywords: surface, joint tree, orientable embedding, genus distribution

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1 Introduction

In this paper we consider only connected graphs. A *linear order* X is a sequence of letters such that if $X = abc\dots z$, then it is indicated that $a \prec b \dots \prec z$. A *reverse order* $\hat{X}$ of S is the linear order such that $\hat{X} = z\dots ba$. A linear order Y is called a *suborder* of X if and only if each letter on Y is also on X and if $a \prec b$ in X, then $a \prec b$ in Y, denoted by $Y \subseteq X$. Let $\langle a_1a_2\dots a_t\rangle$ mean a set of such orders as $a_{i_1}a_{i_2}\dots a_{i_t}$, where $i_1i_2\dots i_t$ is a random permutation on $\{1, 2, \dots, t\}$ and $A \subseteq \langle a_1a_2\dots a_t\rangle$ mean that A is a suborder of some order in set $\langle a_1a_2\dots a_t\rangle$. A *supplementary order* $\bar{Y}$ of Y corresponding to X is a suborder of X such that $a \in \bar{Y}$ for each $a \notin Y$ and that $a \notin \bar{Y}$ for each $a \in Y$.

A *surface* is a compact 2-dimensional manifold without boundary. Because it can be obtained by identifying each of pairs of edges along a given direction on a polygon with even number of edges, an *orientable surface* corresponds to a cyclic order S of letters satisfying the following conditions[9]:

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• If $a \in S$, then $a^- \in S$.
• For each letter a on S, both a and a^- occur once on S.

Then the canonical forms of surfaces are $a_0a_0^-$ and $\sum_{k=1}^{i} a_kb_ka_k^-b_k^-$ which are the sphere and orientable surfaces with genus i, where $i \geq 1$ and a_k, b_k are distinct letters for $k \geq 0$.

Let $o(S)$ be the genus of surface S and $\mathbf{S}$ be the set of surfaces. In order to determine $o(S)$, an *equivalent relation* $\sim$[7] on $\mathbf{S}$ can be obtained by the following operations: for $A, B, C, D, E \in \mathbf{S}$ and $a, b \notin ABCDE$,

Op 1. $AB \sim (Aa)(a^-A) \qquad AB \neq \phi$
Op 2. $Aaa^- \sim A \qquad A \neq \phi$
Op 3. $Aa_1a_2Ba_2^-a_1^- \sim AaBa^- = Aa^-Ba$

Lemma 1.1[9] *$AaBbCa^-Db^-E \sim ADCBEaba^-b^-$, where $A, B, C, D, E \in S$ and $a, b \notin ABCDE$.*

Let $P_p = \begin{cases} aa^-, & p = 0; \\ \prod\limits_{1\leq i\leq p} a_ib_ia_i^-b_i^-, & p \geq 1. \end{cases}$, where $a_1, b_1, \ldots, a_p, b_p$ are distinct letters. Let $\mathbf{P}$ be the set of all orientable surfaces.

Lemma 1.2[9] $\forall S \in \mathbf{P}, \exists p \geq 0, S \sim P_p$.

For convenience, suppose that there are n sets of linear orders, say, $A_1, A_2, \ldots, A_n$. Let $aX_1^na^-X_2^nP_k$ and $X_1^nX_2^nP_l$ be surfaces, where $X_1^n = Z_1Z_2\ldots Z_n$, $X_2^n = \hat{\hat{Z}}_n\ldots\hat{\hat{Z}}_2\hat{\hat{Z}}_1$, $Z_l \subseteq A_l$ for $1 \leq l \leq n$. By $A_{(n,\ k)}$, we mean a set constituted by such elements as $aX_1^na^-X_2^nP_k$, taken over all $Z_l \subseteq A_l$ for $1 \leq l \leq n$. Use $B_{(n,\ l)}$ to denote a set as $\{X_1^nX_2^nP_l\}$. And the letters on P_k or P_l do not appear on X_1^n and X_2^n. Note that $A_{(n,\ k)}$ as a form is meant the different set when A_l varies for $1 \leq l \leq n$. So is $B_{(n,\ l)}$.

Lemma 1.3 *Let $S \in A_{(n,\ 0)}$ and S^0 be the surface obtained by deleting a and a^- from S, then*

$$o(S) = \begin{cases} o(S^0), & \text{if } S \in A_{(n-1,k)}; \\ o(S^0) + 1, & \text{if } S \in B_{(n-1,l)}. \end{cases}$$

where k and l are positive integers or zero.

Proof Let $S = aX_1^na^-X_2^n = aX_1^{n-1}Z_na^-\hat{\hat{Z}}_nX_2^{n-1}$, where $X_1^n = Z_1Z_2\ldots Z_n$, $X_2^n = \hat{\hat{Z}}_n\ldots\hat{\hat{Z}}_2\hat{\hat{Z}}_1$, and Z_i has the same form for $1 \leq i \leq n$. If

$S \in B_{(n,\ l)}$, then it means that there is some $x \in Z_n$ such that $axa^- x^- \in P_l$, further, $aX_1^n a^- X_2^n \sim X_1^{n-1} X_2^{n-1} axa^- x^- P_{l-1}$. By deleting a and a^- from S, $X_1^n X_2^n \sim X_1^{n-1} X_2^{n-1} P_{l-1}$. Then $o(S) = o(S^0) + 1$. If $S \in A_{(n,\ k)}$, then $S = aX_1^n a^- X_2^n \sim aX_1^{n-1} a^- X_2^{n-1} P_k$. According to the properties of Z_i, $o(S) = o(S^0)$ even after deleting a and a^- from S. □

An *embedding* (or cellular embedding in early references) of a graph G into a surface S is a homeomorphism τ: $G \to S$, and each component of $S - \tau(G)$ is homeomorphic to an open disc. The embedding is called *orientable* if S is orientable. Throughout this article, whenever we use the term embedding, we are referring to an orientable embedding.

For a graph G, a *rotation* is a cyclic order of all semiedges at each vertex of G. Let T be a spanning tree of G. A *joint tree*[7] $\tilde{T}$ can be got by splitting every cotree edge into two semiedges denoted by a same letter with a choice of indices: +(always omitted) or −. On $\tilde{T}$, the surface determined by the boundary of the infinite face on the planar embedding with label indices is said to be an *associated surface*[8]. It is seen that the corresponding relation is established between the joint trees and the embeddings. Based on joint trees, the topological problem for enumerating non-homeomorphic embeddings of a graph is transformed into a combinatorial problem for counting distinct associated surfaces in each equivalent class.

For example, for K_4, the spanning tree is presented with thick lines as shown in Fig.1.1 and a joint tree in Fig.1.2. Denote cotree edge v_2v_4 by a_1, v_2v_3 by a_2, v_3v_4 by a_3. Let each vertex of joint trees for K_4 have an anticlockwise rotation. Then the associated surface of the joint tree is shown as $S = a_1 a_2 a_2^- a_3 a_1^- a_3^-$.

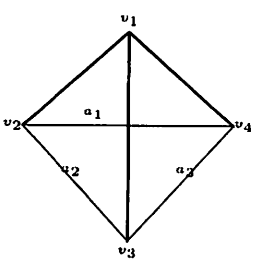

Fig.1.1 K_4

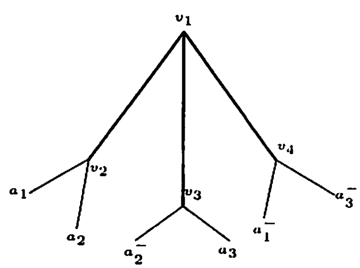

Fig.1.2 A joint tree of K_4

The number of the orientable embeddings for G is $\sum_{u \in V(G)} (\rho_u - 1)!$, where ρ_u is the degree of u. The number of embeddings of G into an orientable surface S_m with genus m is denoted by $g_m(G)$. By the *genus* *polynomial* of G, we mean the polynomial

$$f_G(x) = \sum_{i=0}^{\infty} g_i(G)x^i.$$

For convenience, throughout this article, we write $g_i(n)$ instead of $g_i(G)$, where n is a variant of G.

Since Gross and Furst[3] introduced the embedding distributions of graphs by genus, the genus polynomials of bouquets of circles[5], closed-end ladders and cobblestone paths[2], necklaces[3], *etc.* had been computed. In 1979, Liu[7] initiated joint trees and established a set of complete theory in 2003. On the basis of joint trees, the genus polynomials of more complicated graphs can be computed. Li and Liu[6] got the genus distribution of embeddings for a type of 3-regular graphs, Zhao and Liu[11] for tree-like graphs, Wan and Liu[10] for another type of 3-regular graphs, Chen and Liu[1] obtained the total embedding distributions for two classes of graphs. In this paper, on the basis of joint trees and by classifying the associated surfaces, we obtain the explicit expressions of genus polynomials for three types of graphs, namely K_5^n, W_6^n and $K_{3,3}^n$. Each of them is the repetition of a subgraph along a cycle and different from the graphs whose embedding distributions by genus have been got. And K_5^n and $K_{3,3}^n$ are non-planar.

2 Main results

Suppose vertices $u_1^1,\ u_2^1,\dots,u_p^1,\ u_1^2,\ u_2^2,\dots,u_p^2,\ \dots,\ u_1^n,\ u_2^n,\dots,u_p^n$ are on cycle C in such a sequence and $v_1^1,\ v_2^1,\dots,v_q^1,\ v_1^2,\ v_2^2,\dots,v_q^2,\ \dots,\ v_1^n,\ v_2^n,\dots,v_q^n$ not on C. When $p=4$ and $q=1$, connect $u_i^k v_1^k$ $(1 \le i \le 4,\ 1 \le k \le n)$, $u_1^k u_3^k$ and $u_2^k u_4^k$ to obtain a new graph K_5^n. A new graph W_6^n is obtained by connecting $u_i^k v_1^k$ $(1 \le i \le 6,\ 1 \le k \le n)$ when $p=6$ and $q=1$. A new graph $K_{3,3}^n$, by connecting $u_i^k v_i^k$, $u_1^k v_3^k$, $u_3^k v_1^k$, $v_1^k v_2^k$, $v_2^k v_3^k$ for $1 \le i \le 3$ and $1 \le k \le n$. Note that K_5^n and $K_{3,3}^n$ are non-planar(see the following figures).

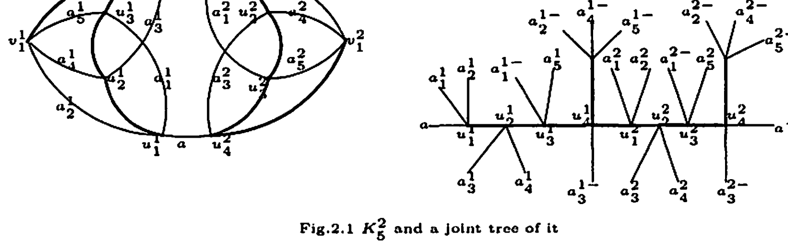

Fig.2.1 K_5^2 and a joint tree of it

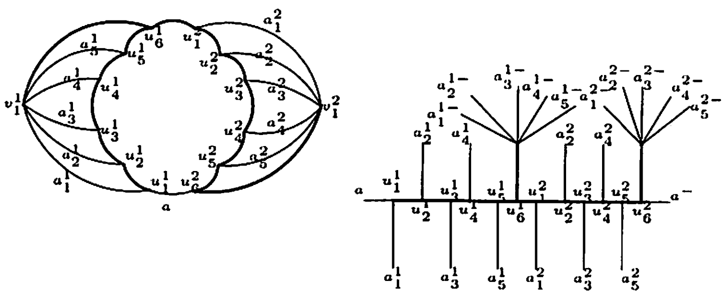

Fig.2.2 W_6^2 and a joint tree of it

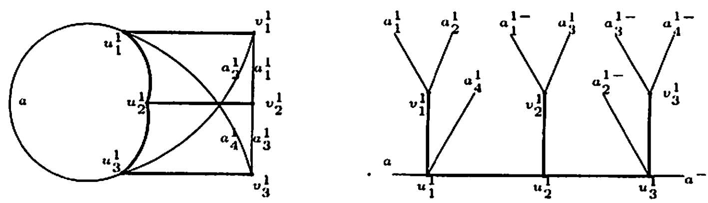

Fig.2.3 $K_{3,3}^1$ and a joint tree of it

Theorem 2.1 $f_{K_5^n}(x) = \sum_{i=0}^{\infty} g_i(n)x^i$,

where

$$g_i(n) = \begin{cases} 0, & n < i - [\frac{i}{2}]; \\ \frac{[(2n-i+1)3558-(i-n)444]n!}{(i-n)!(2n-i+1)!}. & \\ 444^{2n-i}3558^{i-n-1} + g_{i-1}^0(n), & i - [\frac{i}{2}] \le n < i; \\ 444^i + g_{i-1}^0(i), & n = i; \\ g_{i-1}^0(n), & n > i. \end{cases}$$

$$g_i^0(n) = \sum_{k=0}^{[\frac{i}{2}]} \frac{5900^k 1858^{i-2k} 18^{n-i+k} n!}{k!(n-i+k)!(i-2k)!}.$$

Proof Firstly, choose all edges on C except $u_1^1 u_4^n$ denoted by a and $v_1^k u_4^k$ $(1 \le k \le n)$ as a spanning tree as shown with thick lines in Fig.2.1. Other edges are cotree edges. And denote $v_1^k u_1^k$ by a_2^k, $u_1^k u_3^k$ by a_1^k, $u_2^k u_4^k$ by a_3^k, $v_1^k u_2^k$ by a_4^k, $v_1^k u_3^k$ by a_5^k for $1 \le k \le n$, where all letters are distinct. Let each vertex of joint trees for K_5^n have a clockwise rotation.

Let $A_1^k \subseteq \langle a_1^k a_2^k \rangle$, $A_2^k \subseteq \langle a_3^k a_4^k \rangle$, $A_3^k \subseteq \langle a_1^{k-} a_5^k \rangle$,

$$A_4^k \subseteq \langle a_2^{k-} a_4^{k-} a_5^{k-} \rangle, \quad A_5^k \subseteq \langle A_4^k a_3^{k-} \rangle, \quad B_k = A_1^k A_2^k A_3^k A_5^k,$$

for $k = 1, 2, \ldots, n$, and the number of letters in A_4^k is 3 or 0. Let

$$X_1^n = B_1 B_2 \ldots B_n, \qquad X_2^n = \hat{\hat{B}}_n \ldots \hat{\hat{B}}_2 \hat{\hat{B}}_1.$$

So the set of associated surfaces for K_5^n is $A_{(n,\ 0)}$. The set can be classified into such sets as $A_{(n-1,\ k)}$ and $B_{(n-1,\ l)}$ with different k and l. By deleting a and a^- from these sets, we get the classifying sets of $B_{(n,\ 0)}$. Of course $g_i(n)$ is equal to the number of associated surfaces with genus i in $A_{(n,\ 0)}$. And we use $g_i^0(n)$ to denote the number of surfaces with genus i in $B_{(n,\ 0)}$. According to Lemma 1.1, Lemma1.2 and Lemma1.3, we can get the following equations:

$$\begin{cases} g_i(n) = 444 g_{i-1}(n-1) + 3558 g_{i-2}(n-1) + 18 g_{i-1}^0(n-1) \\ \qquad + 1414 g_{i-2}^0(n-1) + 2342 g_{i-3}^0(n-1) & (2.1) \\ g_i^0(n) = 18 g_i^0(n-1) + 1858 g_{i-1}^0(n-1) + 5900 g_{i-2}^0(n-1) & (2.2) \\ g_0(0) = 1 & (2.3) \\ g_0^0(0) = 1 & (2.4) \\ g_i^0(0) = 0 \quad i \neq 0 & (2.5) \end{cases}$$

From (2.2, 2.4, 2.5), we obtain that

$$g_i^0(n) = \sum_{k=0}^{[\frac{i}{2}]} \frac{5900^k 1858^{i-2k} 18^{n-i+k} n!}{k!(n-i+k)!(i-2k)!}.$$

Then

$$g_i(n) = \begin{cases} 0, & n < i - [\frac{i}{2}]; \\ \frac{[(2n-i+1)3558 - (i-n)444]n!}{(i-n)!(2n-i+1)!} \cdot & \\ 444^{2n-i} 3558^{i-n-1} + g_{i-1}^0(n), & i - [\frac{i}{2}] \leq n < i; \\ 444^i + g_{i-1}^0(i), & n = i; \\ g_{i-1}^0(n), & n > i. \end{cases}$$

This completes the proof. □

For example,

$$_{K_5^1}(x) = 462x + 4972x^2 + 2342x^3.$$

$$_{K_5^2}(x) = 324x + 264024x^2 + 6626932x^3 + 31424260x^4 + 22150636x^5.$$

$$K_5^3(x) = 5832x + 1805976x^2 + 279680040x^3 + 9614739592x^4 + 77741085984x^5 + 222210684264x^6 + 160336982888x^7.$$

Using the same method as Theorem 2.1, Theorem 2.2 and Theorem 2.3 can be derived. And we list only the equations that $g_i(n)$ and $g_i^0(n)$ satisfied in the course of proofs.

Theorem 2.2 $f_{W_6^n}(x) = \sum_{i=0}^{\infty} g_i(n)x^i$,

where

$$g_i(n) = \frac{3216^{\frac{i}{2}} 2^{n-\frac{i}{2}} n!}{(\frac{i}{2})!(n-\frac{i}{2})!} l_i + \sum_{k=0}^{[\frac{i-1}{2}]} a(n,i,k),$$

$$a(n,i,k) = [\binom{n}{i-k}\binom{i-k}{k} 488^{i-2k} 3216^k 2^{n-i+k} + \binom{n}{i-1-k}\binom{i-1-k}{k} 2240^{i-1-2k} 5376^k 64^{n-i+k+1} + \binom{n}{i-1-k}\binom{i-1-k}{k} 488^{i-1-2k} 3216^k 2^{n-i+k+1}],$$

$$l_i = \begin{cases} 0, & i \text{ is odd number;} \\ 1, & i \text{ is even number.} \end{cases}, \text{ and } \binom{n}{t} = 0 \text{ when } t < 0.$$

Proof $g_i(n)$ satisfies the following equations:

$$\begin{cases} g_i(n) = 2g_i(n-1) + 488g_{i-1}(n-1) + 3216g_{i-2}(n-1) + \\ \qquad 62g_{i-1}^0(n-1) + 1752g_{i-2}^0(n-1) + 2160g_{i-3}^0(n-1) \\ g_i^0(n) = 64g_i^0(n-1) + 2240g_{i-1}^0(n-1) + 5376g_{i-2}^0(n-1) \\ g_0(0) = 1 \\ g_0^0(0) = 1 \\ g_i^0(0) = 0, \quad i \neq 0 \end{cases}$$

□

For example,

$$W_6^1(x) = 2 + 550x + 4968x^2 + 2160x^3.$$

$$W_6^2(x) = 4 + 6044x + 535776x^2 + 8593536x^3 + 31288320x^4 + 18558720x^5.$$

$$W_6^3(x) = 8 + 267992x + 28986720x^2 + 1163019200x^3 + 18088266240x^4 + 99254896128x^5 + 212337580032x^6 + 122111815680x^7.$$

Theorem 2.3 $f_{K_{3,3}^n}(x) = \sum_{i=0}^{\infty} g_i(n)x^i$,

where

$$g_i(n) = \begin{cases} \binom{n}{i-n}132^{i+n}216^{i-n} + \binom{n}{i-n-1}216^{i-n-1}\cdot \\ (360^{2n-i+1} - 132^{2n-i+1}), & i - [\frac{i}{2}] \leq n < i; \\ 0, & otherwise. \end{cases}$$

and $\binom{n}{t} = 0$ *when* $t < 0$.

Proof $g_i(n)$ satisfies the following equations:

$$\begin{cases} g_i(n) = 132g_{i-1}(n-1) + 216g_{i-2}(n-1) + 228g_{i-2}^0(n-1) \\ g_i^0(n) = 360g_{i-1}^0(n-1) + 216g_{i-2}^0(n-1) \\ g_0(0) = 1 \\ g_0^0(0) = 1 \\ g_i^0(0) = 0, \quad i \neq 0 \end{cases}$$

□

For example,

$$K_{3,3}^1(x) = 132x + 444x^2.$$

$$K_{3,3}^2(x) = 17424x^2 + 169200x^3 + 145152x^4.$$

$$K_{3,3}^3(x) = 2299968x^3 + 55646784x^4 + 91165824x^5 + 41990400x^6.$$

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