Capacitated Domination

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ABSTRACT. We define an r-capacitated dominating set of a graph $G = (V, E)$ as a set $\{v_1, \dots, v_k\} \subseteq V$ such that there is a partition $(V_1, \dots, V_k)$ of V where for all i, $v_i \in V_i$, v_i is adjacent to all of $V_i - \{v_i\}$, and $|V_i| \leq r + 1$. $\urcorner_r(G)$ is the minimum cardinality of an r-capacitated dominating set. We show properties of $\urcorner_r$, especially as regards the trivial lower bound $|V|/(r+1)$. We calculate the value of the parameter in several graph families, and show that it is related to codes and polyominoes. The parameter is NP-complete in general to compute, but a greedy approach provides a linear-time algorithm for trees.

1 Introduction

A ***dominating set*** $S \subseteq V$ of a graph $G = (V, E)$ is a set of vertices such that every vertex in $V - S$ is adjacent to, or is dominated by, at least one vertex in S. The minimum cardinality of a dominating set in G is called the ***domination number*** of G and is denoted $\gamma(G)$. For more information on dominating sets in graphs, see [6].

Gunther et al. [3] considered $[r, s]$***-dominating sets*** S, where every vertex in S is assigned to dominate ***exactly*** r of its neighbors (adjacent vertices), and every vertex in V must be dominated by ***exactly*** s vertices of S. (This can be thought of as an orientation of a spanning subgraph G^+ of G such that every vertex has in-degree s and out-degree either 0 or r.) A similar type of dominating set, called a k***-bounded star packing***, is studied in [4]. In this case a vertex can be assigned to dominate ***at most*** k vertices, while every vertex in V must be dominated by a vertex other than itself.

In this paper we consider a similar type of dominating set, motivated by a design constraint of computer networks. One wishes to place a minimum number of servers in a network so that every node in the network either has a server or is adjacent to a node with a server. However, the number of neighbors that a server can serve is limited due to its bandwidth capacity.

Formally, an ***r-capacitated dominating set*** (rCDS) of a graph G is a set $S = \{v_1, v_2, \ldots, v_k\} \subseteq V$ for which there exists a partition $\Pi = \{V_1, V_2, \ldots, V_k\}$ of V satisfying the following three conditions for every i, $1 \leq i \leq k$:

(i) $v_i \in V_i$,
(ii) $|V_i| \leq r + 1$, and
(iii) v_i is adjacent to all other vertices in V_i.

In an rCDS, each vertex v_i is assigned to, or serves, at most r other vertices. The minimum cardinality of such a set S is called the ***r-capacitated domination number*** of G and is denoted $\daleth_r(G)$. Such a set S is called a ***$\daleth_r$-set*** of G.

Consider the special case $r = 1$. A 1CDS is equivalent to a partition of V into K_1s and K_2s. Further, finding a smallest $\daleth_1$-set is equivalent to finding the maximum number of vertex-disjoint K_2s, that is, finding a maximum matching. We denote the matching number by $\beta_1(G)$ and the edge cover number by $\alpha_1(G)$. Gallai showed that $\alpha_1(G) + \beta_1(G) = |V|$. It follows that

For any graph G, $\daleth_1(G) = \alpha_1(G)$.

Although capacitated domination is similar to several other models, it appears to be new. Chen et al. [1] and Lu et al. [9] considered this question in the context of directed graphs, specifically tournaments. The latter paper called it ***bounded domination***.

In this paper we study the basic properties of capacitated dominating sets, determine exact values of $\daleth_r(G)$ for various classes of graphs, and bounds on $\daleth_r(G)$ for cubic graphs G. While the parameter is NP-complete in general, we exhibit a greedy algorithm for determining $\daleth_r(T)$ for any tree T.

2 Values

There is a trivial lower bound that is useful:

Observation 1 *For any graph G with n vertices, $\daleth_r(G) \geq \lceil n/(r+1) \rceil$.*

Clearly one obtains equality in the bound for complete graphs. Also, for the case $r = 2$ there is equality if the graph has a hamiltonian path; for example, $\daleth_2 = \lceil n/3 \rceil$ for the path and cycle of order n.

2.1 Complete bipartite graphs

An interesting case is the determination of $\daleth_r(G)$ for complete bipartite graphs.

Theorem 1 *Let $r \geq 1$ be an integer and consider the complete bipartite graph $K(a,b)$ with $a \leq b$.*
(i) If $b \geq ra$, then $\daleth_r(K(a,b)) = b - a(r-1)$.
(ii) If $b \leq ra$, then $(a+b)/(r+1) \leq \daleth_r(K(a,b)) < (a+b)/(r+1) + 2$. To be specific, define $x = \lfloor (rb-a)/(r^2-1) \rfloor$ and $y = \lfloor (ra-b)/(r^2-1) \rfloor$. Further, define ε as 0 if $x + yr = a$ and $xr + y = b$; else 1 if $x + yr \geq a - 1$ or $xr + y \geq b - 1$; and 2 otherwise. Then $\daleth_r = x + y + \varepsilon$.

PROOF. Let $K(a,b)$ have bipartition (A,B) with $|A| = a$. Suppose a set S has α vertices from A and β vertices from B. Then it can handle at most $\alpha + \beta r$ vertices of A and at most $\alpha r + \beta$ vertices of B. Indeed, it follows that S is an rCDS if and only if $\alpha + \beta r \geq a$ and $\alpha r + \beta \geq b$.

It follows that $\daleth_r(K(a,b))$ is equal to the solution θ of the ***integer program***:

> $\min \alpha + \beta$, such that $\alpha + \beta r \geq a$, $\alpha r + \beta \geq b$, $0 \leq \alpha \leq a$, and $0 \leq \beta \leq b$.

If $b \geq ra$, then the integer program has the solution $\alpha = a$ and $\beta = b - ra$.

So we consider the case where $b < ra$. Relaxed to a linear program, the minimum is achieved at $\alpha^* = (rb-a)/(r^2-1)$ and $\beta^* = (ra-b)/(r^2-1)$. If α^* and β^* are integral, we are done. So assume that at least one is not integral.

We set $x = \lfloor \alpha^* \rfloor$ and $y = \lfloor \beta^* \rfloor$. Since $\theta \geq \alpha^* + \beta^*$ and is integral, it follows that $\theta \geq x + y + 1$. On the other hand, the point $\alpha = x + 1$ and $\beta = y + 1$ is feasible, and so $\theta \leq x + y + 2$.

Now, if either $(x, y+1)$ or $(x+1, y)$ lies in the feasible region, then $\theta = x + y + 1$. But, if neither $(x, y+1)$ nor $(x+1, y)$ lies in the feasible region, then there cannot be an integral point of the line $\alpha + \beta = x + y + 1$ that is feasible (see Figure 1), and so $\theta = x + y + 2$. QED

2.2 The hypercube

The domination numbers of the cubes are in general not known. So we consider the case for small r. Since all cubes have a hamiltonian path, $\daleth_1(Q_d) = 2^{d-1}$ and $\daleth_2(Q_d) = \lceil 2^d/3 \rceil$.

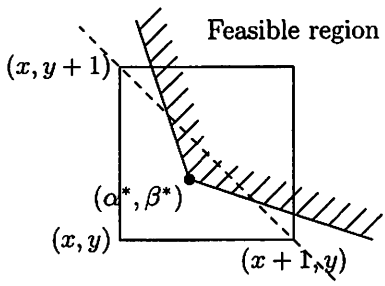


Figure 1: Rounding the solution to the linear program

Theorem 2 *For $d \geq 2$, $\rceil_3(Q_d) = 2^{d-2}$.*

PROOF. The lower bound is the trivial one (Observation 1).

There are two ways to see the upper bound. The first is that the cube Q_d contains the prism $C_{2^{d-1}} \square K_2$. A 3CDS of the prism is obtained by taking every fourth vertex on the one cycle and every fourth vertex on the other cycle, offset by 2. The second way is that Q_3 contains a perfect dominating set—so that $\rceil_3(Q_3) = 2$—and 2^{d-3} copies of Q_3 form a spanning subgraph of Q_d. QED

The second proof also shows that, for example,

$$\rceil_7(Q_d) = 2^{d-3}$$

for $d \geq 7$. And in general we get exact results whenever r is 1 less than a power of 2, because then Q_r has a perfect dominating set (e.g., Hamming's [5] perfect 1-error-correcting code).

This leaves open the question of what, for example, is the asymptotics for $\rceil_4(Q_d)$. Is it possible to mostly achieve the trivial lower bound rounded up?

2.3 Cubic graphs

Let G be a cubic graph. As noted before, $\rceil_1(G)$ is the edge cover number while $\rceil_3(G)$ is the ordinary domination number. In this respect Reed [10] showed that $\gamma(G) \leq 3n/8$; recently Kostochka and Stodolsky [8] showed that there are connected cubic graphs G with $\gamma(G) \geq 8/23 - o(1)$.

So consider the case $r = 2$. The following simple lemma is surely known:

Lemma 1 *The vertex set of a regular graph can be partitioned into sets such that each set is spanned by either P_2 or P_3.*

PROOF. Say graph G is r-regular. Start with a maximum matching M covering vertices S. Then $V - S$ is an independent set and so each vertex in $V - S$ has r edges to S; on the other hand, every vertex in S has at most $r - 1$ edges to $V - S$. By Hall's theorem it follows that there is a matching N from $V - S$ into S. An edge of M cannot be incident with two edges of N as this contradicts the maximality of M. It follows that each edge of M is incident with at most one edge of N and hence lies in a P_2 or P_3. This establishes the result. QED

Corollary 1 *For any regular graph G, $\daleth_r(G) \leq \beta_1(G)$.*

It follows that $\daleth_2(G) \leq n/2$ for any regular graph of order n. Indeed, it is not hard to show that $\daleth_2(G) < n/2$ for any connected cubic graph G other than K_4. And further, that there is some small constant $\varepsilon > 0$ such that $\daleth_2(G) \leq (1/2 - \varepsilon)n$ for such cubic graphs. But the true maximum value of $\daleth_2$ remains wide open.

2.4 Grids

Define the grid $G_{s,t}$ as the grid with s rows and t columns (that is, $P_s \square P_t$). The grids all have hamiltonian paths: It follows that $\daleth_1(G_{s,t}) = \lceil st/2 \rceil$ and $\daleth_2(G_{s,t}) = \lceil st/3 \rceil$. Further, $\daleth_4(G_{s,t})$ is just the domination number $\gamma(G_{s,t})$, which remains an unsolved problem. So the new question is $r = 3$.

2.4.1 $\daleth_3$ in grids where both sides are even

The grid of height 2 has maximum degree 3, and so $\daleth_3(G_{2,t}) = \lfloor t/2 \rfloor + 1$, since this is just the domination number (found by Jacobson and Kinch [7]).

Theorem 3 *If both s and t are multiples of 4, then $\daleth_3(G_{s,t}) = rs/4$.*

PROOF. The lower bound is just the trivial lower bound (Observation 1). The upper bound follows since $G_{4,4}$ can be partitioned into four $K_{1,3}$s (see Figure 2). QED

It is to be noted that the parameter $\daleth_3$ in a grid is related to packing rectangles with what is called a ***T-tetromino***: this is a tile formed by sticking together four unit squares to form a T shape. Walkup [11] showed that a rectangle is packable by a T-tetromino iff both sides of the rectangle are a multiple of 4.

Theorem 4 *If both s and t are even, but at least one is not a multiple of 4, then $\daleth_3(G_{s,t}) = st/4 + 1$.*

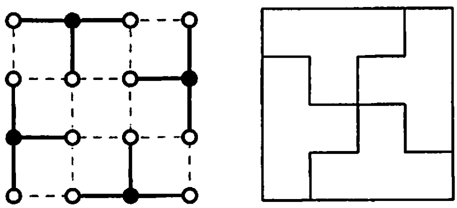


Figure 2: An efficient 3CDS of $G_{4,4}$ and the associated T-tetromino packing

PROOF. The lower bound is from Walkup. For the upper bound, one can tile the rectangle whose sides are a multiple of 4. So that leaves either a rectangular strip or an L-shaped strip which is 2 units wide. The value for the rectangular strip was calculated above. By starting in the elbow of the L, the L-shaped strip can be tiled with T-tetrominoes to leave at most 4 holes, which can be handled by two dominators. See Figure 3 for an example. QED

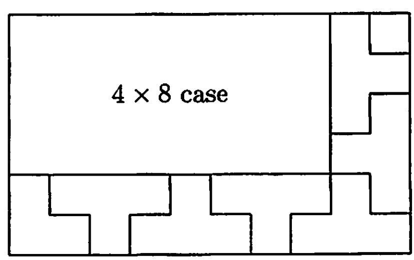


Figure 3: An almost T-tetromino packing of $G_{6,10}$

2.4.2 $\urcorner_3$ in grids where one side is 3

Theorem 5 *For* $t \geq 1$, $\urcorner_3(G_{3,t}) = \lceil 4t/5 \rceil$.

PROOF. Here is the upper bound. The picture of Figure 4 shows that $\urcorner_3(G_{3,5}) \leq 4$. Say $a = \lfloor t/5 \rfloor$. Then repeat the picture of Figure 4 a times, and take the middle vertex in the remaining $t - 5a$ columns.

The lower bound is more complex. This is the outline: *We define a "near-set" as a 3CDS of a grid restricted to some initial segment of the columns: this is essentially a 3CDS except for possibly the final column. We then show that one can partition the collection of near-sets into a finite number of classes, based on (the properties of) their final column; this then yields a table that gives the class of near-set as one adds each column.*

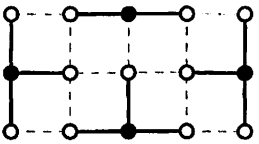


Figure 4: $\urcorner_3$-set of $G_{3,5}$

Finally, by simultaneous induction, we calculate the minimum cardinality of each class of near-set. The proof was generated with help from a computer (but can be checked by hand).

The collection of near-sets can be partitioned into classes, where the class captures what one needs to know about the last column: what it needs and what it offers. The table given in Figure 5 shows the claimed 17 classes and a sample configuration for each (dark vertices).

Lemma 2 *There are 17 classes of near-sets. An example of each is given in the table in Figure 5. Further, that table gives the correct transitions for adding columns.*

PROOF. We first need to show that the partition is correct.

There are 8 classes for the near-set S' where the last column X is empty: these correspond to which subset of X is dominated. If two or more vertices in X are in S, then the last column is dominated, and these vertices can handle their neighbors-to-come. If the only vertex v in $X \cap S'$ is the top or bottom vertex, then there are two possibilities, depending on whether all the vertices of X are dominated or not (v has only three neighbors and so can handle its neighbor-to-come). Finally, if the only vertex v in $X \cap S'$ is the middle vertex, then there are at least two possibilities: v can handle its neighbor-to-come or it cannot. Note that if both the top and bottom vertex of X are undominated by any other vertex, we might as well use v to dominate them, since otherwise there must be a vertex $w \in S$ in the next column on the edge that handles one of these, and w can handle the middle vertex of that column.

The verification of the entries of the table is left to the reader. (It was computer generated.) QED

Finally, one lets $f_i(s)$ denote the minimum cardinality of a near-set in class i. It can then be calculated that the values of f_i are given in the following table. The notation $\sim$ means that no such near-set exists; for $s \geq 4$ the values are obtained by adding $\lceil 4s/5 \rceil$ throughout.

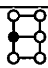
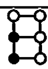
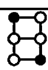
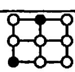
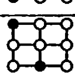
										valid at end
	1	–	–	–	–	–	–	–	8	false
	2	–	–	–	–	9	6	7	8	true
	3	10	2	3	4	5	6	7	8	true
	4	11	2	3	4	9	6	7	8	true
	5	–	12	–	4	–	6	–	8	false
	6	13	12	3	4	9	6	7	8	true
	7	14	12	3	4	5	6	7	8	true
	8	15	12	3	4	9	6	7	8	true
	9	16	12	3	4	5	6	7	8	true
	10	–	–	–	–	–	6	–	8	false
	11	–	–	–	–	5	6	7	8	false
	12	17	2	3	4	9	6	7	8	true
	13	–	–	15	4	–	–	7	8	false
	14	–	2	–	4	–	6	–	8	false
	15	1	2	3	4	5	6	7	8	true
	16	–	–	–	4	–	–	–	8	false
	17	–	–	–	–	–	–	7	8	false

Figure 5: The table for extending near-sets in grids of height 3

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$s=1$	0	1	1	2	1	2	2	3	~	~	~	~	~	~	~	~	~
$s=2$	~	2	2	3	2	3	3	3	2	1	2	2	2	2	3	~	~
$s=3$	3	3	3	4	3	3	4	4	3	2	3	3	3	3	3	2	2
$s \geq 4$: (add $\lceil 4s/5 \rceil$)																	
$s \bmod 5 = 0$	0	1	1	1	1	1	1	2	1	0	0	1	0	0	0	0	0
$s \bmod 5 = 1$	−1	0	0	1	0	1	1	2	1	0	0	1	0	0	0	0	0
$s \bmod 5 = 2$	−1	0	0	1	0	1	1	1	0	−1	0	0	0	0	0	0	0
$s \bmod 5 = 3$	−1	0	0	1	0	0	1	1	0	−1	0	0	0	0	0	−1	−1
$s \bmod 5 = 4$	−1	0	0	0	0	0	0	1	0	−1	0	0	−1	0	0	−1	−1

Now, only certain classes of near-sets correspond to 3CDSs (viz. 2, 3, 4, 6, 7, 8, 9, 12, 15). If one takes the minimum of $f_i(s)$ over these classes, one obtains $\lceil 4s/5 \rceil$, as required. QED

2.4.3 $\daleth_3$ in grids where one side is odd

We believe that if one side is odd, then $\daleth_3(G_{s,t})$ is at least a linear factor (in $\max(s,t)$) more than $st/4$. In particular, we believe that, if s is odd, then there exists a constant $\varepsilon_s > 0$ such that $\daleth_3(P_s \Box P_t) \geq (1+\varepsilon_s)st/4 - O(1)$. For $s = 3$, we saw above that it is $\varepsilon_3 = 1/15$.

With the help of a computer we can show that such a constant exists for $s = 5, 7, 9$ and 11, as follows. For an $s \times t$ rectangle, we define a ***sloppy*** tiling with T-tetrominoes as a tiling of the rectangle where one is allowed to slop over the left and right ends.

Theorem 6 *There is no sloppy tiling for odd* $G_{5,7}$, $G_{7,9}$, $G_{9,17}$ *and* $G_{11,43}$.

PROOF. Exhaustive computer search. QED

For example, since one needs at least an extra 1/4 for every 5×7 rectangle, this shows that $\daleth_3(G_{5,t}) \geq 9t/7$.

3 Algorithmics

It can be shown that computing the parameter $\daleth_r$ is NP-hard. For example, testing whether $\daleth_r(G) = n/(r+1)$ is equivalent to asking whether G has a spanning subgraph consisting of stars $K_{1,r}$. This is covered by problem GT12 in [2].

Also, there is a realization algorithm. Given a set S of vertices, one wants to know whether S is an rCDS. Consider the subgraph H induced by the edges joining S and $V-S$; if one allows each vertex in S a capacity of

r, the question is whether there is what is called a b-matching that saturates $V - S$. (Equivalently, one can in H replicate each vertex of S so that there are r copies of it, and then ask for a matching that covers $V - S$.)

3.1 Tree Algorithm

In this section, we present a linear-time algorithm for finding $\daleth_r(T)$ for any tree T and value of $r \geq 1$. The tree algorithm is unusually simple: a greedy algorithm suffices.

The algorithm starts at the leaves and moves up to the root, marking vertices as either dominated or in the rCDS as it moves up. For each vertex v, there is a counter c that is initialized to the value of r. This counter is used to determine whether or not a vertex can dominate any other vertices.

By ordering the vertices with a pre-order traversal, all the children of a node are visited before it. When a vertex v is visited, the action depends on the value of the counter. Suppose first that $c(v) < r$. Then mark v as being in the dominating set S; further, if $c(v) > 0$ then mark the vertex's parent p as covered. Suppose second that $c(v) = r$. Then, if v has been marked as covered, we are done. Otherwise it must be dominated by itself or its parent. So check the parent's counter: If $c(p) > 0$, then decrement it, mark vertex v as covered and proceed; If $c(p) = 0$, then p cannot dominate any more vertices, so mark v as in the set S.

```
Pseudocode for tree algorithm
for(vertices v in postorder) :
  if( c(v)=0 ) {
      mark v as in S
  } else if( 0<c(v)<r ) {
      mark v as in S
      mark parent(v) as covered
  } else if( c(v)=r and v not covered {
          if( c(parent(v))>0 {
              mark v as covered
              c(parent(v)) --
          } else
              mark v as in S
  }
```

The running time is linear. The correctness of this algorithm can be seen by viewing any typical vertex v in the tree T. If $c(v) < r$, then at least one of v's children is relying on v to dominate it. Also, if $0 < c(v) < r$, then v has to be in the dominating set and can still dominate other vertices. If

$c(v) = r$ then no child of v needs to be dominated and v can dominate itself, or if its parent can still dominate more vertices, v can be dominated by its parent. Formally, the algorithm finds a minimum rCDS such that no other minimum set is "closer" to the root.

4 Open Questions

Apart from the questions about cubic graphs and hypercubes already mentioned, one can also look at the upper parameter. Define the maximum cardinality of a minimal rCDS. It seems that the parameter for $r = 1$ is $3n/5 \pm O(1)$ for the path P_n. In particular, this means that the case $r = 1$ is not related to the minimum size of maximal matching, unlike the relationship between $\urcorner_1$ and the matching number.

5 Thanks

We would like to thank Doug Weakley for information about domination in hypercubes and Mike Henning for information about domination in cubic graphs.

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