Randomly r-Orthogonal $(0, f)$-Factorizations of Bipartite $(0, mf-(m-1)r)$-Graphs

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Abstract

Let $G=(X,Y,E(G))$ be a bipartite graph with vertex set $V(G)=X$! Y and edge set $E(G)$, and let g, f be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) \le f(x)$ for each $x \in V(G)$. A (g, f)-factor of G is a spanning subgraph F of G such that $g(x) \le d_F(x) \le f(x)$ for each $x \in V(F)$; a (g, f)-factorization of G is a partition of $E(G)$ into edge-disjoint (g, f)-factors. Let $F=\{F_1, F_2, " , F_m\}$ be a factorization of G and H be a subgraph of G with mr edges. If $F_i, 1 \le i \le m$, has exactly r edges in common with H, we say that F is r-orthogonal to H. In this paper it is proved that every bipartite $(0, mf-(m-1)r)$-graph has $(0, f)$-factorizations randomly r-orthogonal to any given subgraph with mr edges if $2r \le f(x)$ for any $x \in V(G)$.

Keywords: bipartite graph, subgraph, $(0, f)$-factor, orthogonal factorization.

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1 Introduction

Orthogonal Factorizations in the graphs are very useful in combinatorial design, network design, circuit layout and so on [1]. Graphs considered in this paper will be finite undirected simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex x is denoted by $d_G(x)$. Let g and f be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) \le f(x)$ for each $x \in V(G)$. Then a (g, f)-factor of G is a spanning subgraph F of G with $g(x) \le d_F(x) \le f(x)$ for each $x \in V(F)$. In particular, G is called a (g, f)-graph if G itself is a (g, f)-factor. A subgraph H of G is called an m-subgraph if H has

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m edges in total. A (g,f)-factorization $F=\{F_1,F_2,\ldots,F_m\}$ of G is a partition of $E(G)$ into edge-disjoint (g,f)-factors $F_1,F_2,\ldots,F_m$. Let a and b be two nonnegative integers with $a\le b$. If $g(x)=a$ and $f(x)=b$ for each $x\in V(G)$, then a (g,f)-factor is called an $[a,b]$-factor. Let H be a mr-subgraph of a graph G. A (g,f)-factorization $F=\{F_1,F_2,\ldots,F_m\}$ is r-orthogonal to H if $|E(H)\cap E(F_i)|=r$ for $1\le i\le m$. If for any partition $\{A_1,A_2,\ldots,A_m\}$ of $E(H)$ with $|A_i|=r$ there is a (g,f)-factorization $F=\{F_1,F_2,\ldots,F_m\}$ of G such that $A_i\subseteq E(F_i), 1\le i\le m$, then we say that G has (g,f)-factorizations randomly r-orthogonal to H. Other definitions and terminologies can be found in [2].

A graph denoted by $G=(X,Y,E(G))$ is a bipartite graph with vertex bipartition (X,Y) and edge set $E(G)$. Alspach et al. [1] posed the following problem: given a subgraph H, does there exist a factorization F of G with some fixed type orthogonal to H? Li and Liu [3] gave a sufficient condition for a graph to have a (g,f)-factorization orthogonal to any given m-subgraph. Lam et al. [4] studied orthogonal factorizations of graphs. Anstee and Caccetta [5] discussed orthogonal matchings. Li et al. [6] studied orthogonal (g,f)-factorizations of $(mg+k,mf-k)$-graphs. Feng [7] proved that every $(0,mf-m+1)$-graph has a $(0,f)$-factorization orthogonal to any given m-subgraph. Liu and Zhu [8] proved that every bipartite $(mg+m-1, mf-m+1)$-graph has the randomly k-orthogonal (g,f)-factorizations. The purpose of this paper is to solve some problems on orthogonal factorizations for bipartite $(0,mf-(m-1)r)$-graphs. It is shown that a bipartite $(0,mf-(m-1)r)$-graph G has $(0,f)$-factorizations randomly r-orthogonal to any given mr-subgraph if $2r\le f(x)$ for any $x\in V(G)$.

2 Preliminary results

Let G be a graph, and let S and T be two disjoint subsets of $V(G)$. We denoted by $E_G(S,T)$ the set of edges with one end in S and the other in T, and by $e_G(S,T)$ the cardinality of $E_G(S,T)$. For $S\subset V(G)$ and $A\subset E(G)$, $G-S$ is a subgraph obtained from G by deleting the vertices in S together with the edges to which the vertices in S incident, and $G-A$ is a subgraph obtained from G by deleting the edges in A, and $G[S]$ (rep. $G[A]$) is a subgraph of G induced by S (rep. A). For a subset X of $V(G)$, we write $f(X)=\sum_{x\in X} f(x)$ for any function f defined on $V(G)$, and define $f(\phi)=0$. Specially, $d_G(X)=\sum_{x\in X} d_G(x)$.

Folkman and Fulkerson [9] obtained the following necessary and sufficient condition for the existence of a (g,f)-factor in a bipartite graph (see Theorem 6.8 in [9]).

Lemma 1 Let $G=(X,Y,E(G))$ be a bipartite graph and let g and

f be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) \le f(x)$ for each $x \in V(G)$. Then G has a (g,f)-factor if and only if for all $S \subseteq X, T \subseteq Y$,

$$r_{1G}(S,T,g,f) = f(S) - g(T) + d_{G-S}(T) \ge 0$$

and

$$r_{2G}(S,T,g,f) = f(T) - g(S) + d_{G-T}(S) \ge 0$$

Note that $d_{G-S}(T) = e_G(T, X \setminus S)$ and $d_{G-T}(S) = e_G(S, Y \setminus T)$. Let E_1 and E_2 be two disjoint subsets of $E(G)$ and let $S \subseteq X, T \subseteq Y$ .Set

$$E_{iS} = E_i \ ! \ E_G(S, Y \setminus T), \quad E_{iT} = E_i \ ! \ E_G(T, X \setminus S) \qquad \text{for } i = 1,2$$

and set

$$\alpha_S = |E_{1S}| \ , \ \alpha_T = |E_{1T}|, \ \beta_S = |E_{2S}| \ , \ \beta_T = |E_{2T}|$$

It is easily seen that $\alpha_S \le d_{G-T}(S)$, $\alpha_T \le d_{G-S}(T)$, $\beta_T \le d_{G-S}(T)$, $\beta_S \le d_{G-T}(S)$.

Liu and Zhu [8] give a necessary and sufficient condition for a bipartite graph to admit a (g,f)-factor containing E_1 and excluding E_2.

Lemma 2 (Liu and Zhu [8]) Let $G = (X, Y, E(G))$ be a bipartite graph and let g and f be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) \le f(x)$ for each $x \in V(G)$, and let E_1 and E_2 be two disjoint subsets of $E(G)$. Then G has a (g,f)-factor F such that $E_1 \subseteq E(F)$ and $E_2 \ ! \ E(F) = \phi$ if and only if for all $S \subseteq X$, $T \subseteq Y$,

$$r_{1G}(S,T,g,f) \ge \alpha_S + \beta_T$$

and

$$r_{2G}(S,T,g,f) \ge \alpha_T + \beta_S$$

In the following , we always assume that G is a bipartite $(0, mf - (m-1)r)$-graph , where $m \ge 1$ and $r \ge 1$ is integers. Define

$$g(x) = \max\{0, d_G(x) - ((m-1)f(x) - (m-2)r)\}$$

$$\Delta_1(x) = \frac{1}{m} d_G(x) - g(x)$$

$$\Delta_2(x) = f(x) - \frac{1}{m} d_G(x)$$

By the definitions of $g(x)$, $\Delta_1(x)$ and $\Delta_2(x)$,We have the following Lemma.

Lemma 3. For all $x \in V(G)$,the following inequalities holds:

(1) If $m \geq 2$, then $0 \leq g(x) < f(x)$

(2) If $g(x) = d_G(x) - ((m-1)f(x) - (m-2)r)$, then

$$\Delta_1(x) \geq \frac{r}{m}$$

(3) $\Delta_2(x) \geq \frac{(m-1)r}{m}$

Proof. (1) Note that G is a bipartite $(0, mf-(m-1)r)$-graph, where $m \geq 2$ is an integer. Since $0 \leq mf-(m-1)r$, we have

$$f(x) \geq \frac{(m-1)r}{m}.$$

Note that $f(x)$ is nonnegative integer-valued function . Then $f(x) \geq 1$.

If $g(x) = 0$, then $0 \leq g(x) < f(x)$.

If $g(x) = d_G(x) - ((m-1)f(x) - (m-2)r)$, then

$$\begin{aligned} f(x) - g(x) &= f(x) - d_G(x) + (m-1)f(x) - (m-2)r \\ &= mf(x) - (m-2)r - d_G(x) \\ &\geq mf(x) - (m-2)r - (mf(x) - (m-1)r) = r \geq 1 \end{aligned}$$

Hence we get that

$$0 \leq g(x) < f(x)$$

(2) If $g(x) = d_G(x) - ((m-1)f(x) - (m-2)r)$, then

$$\Delta_1(x) = \frac{1}{m}d_G(x) - g(x)$$

$$= \frac{1}{m}d_G(x) - [d_G(x) - ((m-1)f(x) - (m-2)r)]$$

$$= \frac{1-m}{m}d_G(x) + (m-1)f(x) - (m-2)r$$

$$\geq \frac{1-m}{m}(mf(x)-(m-1)r)+(m-1)f(x)-(m-2)r$$

$$=(1-m)f(x)+(m-1)r-\frac{(m-1)r}{m}+(m-1)f(x)-(m-2)r$$

$$=\frac{r}{m}$$

(3) $\Delta_2(x)=f(x)-\frac{1}{m}d_G(x)\geq f(x)-\frac{1}{m}(mf(x)-(m-1)r)$

$$=f(x)-f(x)+\frac{(m-1)r}{m}=\frac{(m-1)r}{m}$$

completing the proof.

Lemma 4. For any $S\subseteq X$ and $T\subseteq Y$,

$$r_{1G}(S,T,g,f)=\Delta_1(T)+\Delta_2(S)+\frac{m-1}{m}d_{G-S}(T)+\frac{1}{m}d_{G-T}(S)$$

and

$$r_{2G}(S,T,g,f)=\Delta_1(S)+\Delta_2(T)+\frac{m-1}{m}d_{G-T}(S)+\frac{1}{m}d_{G-S}(T)$$

Proof. We prove only the first equality. The second one can be verified similarly. According to the definition of r_{1G}, we have

$$\begin{aligned} r_{1G}(S,T,g,f) &= d_{G-S}(T)-g(T)+f(S) \\ &= d_G(T)-e_G(S,T)-g(T)+f(S) \\ &= (\frac{1}{m}d_G(T)-g(T))+(f(S)-\frac{1}{m}d_G(S)) \\ &\quad +\frac{m-1}{m}d_{G-S}(T)+\frac{1}{m}d_{G-T}(S) \end{aligned}$$

$$= \Delta_1(T) + \Delta_2(S) + \frac{m-1}{m} d_{G-S}(T) + \frac{1}{m} d_{G-T}(S)$$

Completing the proof.

Lemma 5 (Feng [7]) Let G be a $(0, mf - m + 1)$-graph, let f be one integer-valued function defined on $V(G)$ such that $f(x) \geq 0$, and let H be an m-subgraph of G. Then G has a $(0, f)$-factorization orthogonal to H.

3 Main result and proof

In this section, we are going to state our main theorem ,and present a proof of it.

Let G be a bipartite graph, let H be an mr-subgraph of G, and let E_1 be an arbitrary subset of $E(H)$ with $|E_1| = r$. Put $E_2 = E(H) \setminus E_1$. Then $|E_2| = (m-1)r$. For any two subsets $S \subseteq X$ and $T \subseteq Y$,let $g(x)$, E_{is}, E_{iT} for $i = 1, 2$, $\alpha_S, \alpha_T, \beta_S$ and β_T be defined as in Section 2 and $T_0 = \{x | x \in T, g(x) = 0\}$, $T_1 = T \setminus T_0$. It is easily seen that $T = T_0 \,!\, T_1$ and $T_0 \,"\, T_1 = \phi$. It follows instantly from the definition of $\alpha_S, \alpha_T, \beta_S$ and β_T that

$$\alpha_T = \alpha_{T_0} + \alpha_{T_1}, \quad \beta_T = \beta_{T_0} + \beta_{T_1}, \quad \beta_{T_0} \leq d_{G-S}(T_0)$$

$$\alpha_S \leq r, \quad \alpha_T \leq r, \quad \beta_s \leq (m-1)r, \quad \beta_T \leq (m-1)r$$

$$\alpha_{T_1} \leq r, \quad \beta_{T_1} \leq (m-1)r$$

The proof of theorem relies heavily on the following lemma.

Lemma 6 Let $G = (X, Y, E(G))$ be a bipartite $(0, mf - (m-1)r)$-graph with $m \geq 2$ and $f(x) \geq 2r$ with $r \geq 2$,Then G admits a (g, f)-factor F_1 such that $E_1 \subseteq E(F_1)$ and $E_2 \,"\, E(F_1) = \phi$.

Proof . by Lemma 2, it suffices to show that for any two subsets $S \subseteq X$ and $T \subseteq Y$, We have

$$r_{1G}(S, T, g, f) \geq \alpha_S + \beta_T$$

and

$$r_{2G}(S, T, g, f) \geq \alpha_T + \beta_S$$

We prove only the first inequality . The second one can be justified similarly. For S and T, we get that

$$r_{1G}(S, T, g, f) = d_{G-S}(T) - g(T) + f(S)$$
$$= d_{G-S}(T_1) - g(T_1) + f(S) + d_{G-S}(T_0) - g(T_0)$$
$$\geq d_{G-S}(T_1) - g(T_1) + f(S) + \beta_{T_0}$$

$$= \frac{1}{m} d_G(T_1) - g(T_1) + f(S) - \frac{1}{m} d_G(S) + \frac{m-1}{m} d_{G-S}(T_1)$$

$$+ \frac{1}{m} d_{G-T_1}(S) + \beta_{T_0}$$

By Lemma 4 and Lemma 3, we have

$$r_{1G}(S,T,g,f) = \Delta_1(T_1) + \Delta_2(S) + \frac{m-1}{m} d_{G-S}(T_1) + \frac{1}{m} d_{G-T_1}(S) + \beta_{T_0}$$

$$\geq \frac{r|T_1|}{m} + \frac{(m-1)r}{m}|S| + \frac{m-1}{m} d_{G-S}(T_1) + \frac{1}{m} d_{G-T_1}(S) + \beta_{T_0} \qquad (1)$$

Now let us distinguish among four cases.

Case 1. If $S = \phi, T_1 = \phi$, then $\alpha_S = 0, \beta_{T_1} = 0$

In view of (1), we have

$$r_{1G}(S,T,g,f) \geq \frac{r|T_1|}{m} + \frac{(m-1)r}{m}|S|$$

$$+ \frac{m-1}{m} d_{G-S}(T_1) + \frac{1}{m} d_{G-T_1}(S) + \beta_{T_0}$$

$$= \beta_{T_0} = \alpha_S + \beta_{T_1} + \beta_{T_0} = \alpha_S + \beta_T$$

Case 2. $S = \phi, T_1 \neq \phi$. Then $\alpha_S = 0$.

According to the definition of T_1, It is easy to see that

$$g(x) \geq 1, \quad \forall x \in T_1$$

Note that $g(x) = \max\{0, d_G(x) - ((m-1)f(x) - (m-2)r)\}$. For $\forall x \in T_1$. We have

$$g(x) = d_G(x) - ((m-1)f(x) - (m-2)r) \geq 1$$

i.e.

$$\begin{aligned} d_G(x) &\geq (m-1)f(x) - (m-2)r + 1 \\ &\geq 2r(m-1) - (m-2)r + 1 \\ &= mr + 1 \qquad (x \in T_1) \end{aligned} \qquad (2)$$

From (1) and (2), we get that

$$r_{1G}(S,T,g,f) \geq \frac{m-1}{m} d_G(T_1) + \beta_{T_0} \geq \frac{m-1}{m} d_G(x) + \beta_{T_0} \qquad (x \in T_1)$$

$$\geq \frac{m-1}{m}(mr+1) + \beta_{T_0} = (m-1)r + \frac{m-1}{m} + \beta_{T_0}$$

$$\geq (m-1)r + \beta_{T_0} \geq \beta_{T_1} + \beta_{T_0} = \beta_T = \alpha_S + \beta_T$$

Case 3 If $S \neq \phi, T_1 = \phi$, then $\beta_{T_1} = 0$

Thus, we have

$$\begin{aligned} r_{1G}(S,T,g,f) &= d_{G-S}(T) - g(T) + f(S) \\ &= d_{G-S}(T_1) - g(T_1) + f(S) + d_{G-S}(T_0) - g(T_0) \\ &\geq d_{G-S}(T_1) - g(T_1) + f(S) + \beta_{T_0} \\ &= f(S) + \beta_{T_0} \geq 2r|S| + \beta_{T_0} \geq r + \beta_{T_0} \\ &\geq \alpha_S + \beta_{T_0} = \alpha_S + \beta_{T_1} + \beta_{T_0} = \alpha_S + \beta_T \end{aligned}$$

Case 4. $S \neq \phi, T_1 \neq \phi$.

Note that $d_{G-T_1}(S) \geq \alpha_S$. According to (1) and (2), we get that

$$r_{1G}(S,T,g,f) = \Delta_1(T_1) + \Delta_2(S) + \frac{m-1}{m} d_{G-S}(T_1) + \frac{1}{m} d_{G-T_1}(S) + \beta_{T_0}$$

$$\geq \frac{r|T_1|}{m} + \frac{(m-1)r}{m}|S| + \frac{m-1}{m} d_{G-S}(T_1) + \frac{1}{m} d_{G-T_1}(S) + \beta_{T_0}$$

$$= \frac{r|T_1|}{m} + \frac{(m-1)(r-1)}{m}|S| + \frac{m-1}{m}(d_{G-S}(T_1) + |S|) + \frac{1}{m} d_{G-T_1}(S) + \beta_{T_0}$$

$$\geq \frac{m-1}{m} d_G(x) + \frac{1}{m} d_{G-T_1}(S) + \frac{r}{m} + \frac{(m-1)(r-1)}{m} + \beta_{T_0} \qquad (x \in T_1)$$

$$\geq \frac{m-1}{m}(mr+1) + \frac{1}{m}\alpha_S + \frac{r}{m} + \frac{(m-1)(r-1)}{m} + \beta_{T_0}$$

$$\geq (m-1)r + \frac{1}{m}\alpha_S + \frac{r}{m} + \frac{(m-1)r}{m} + \beta_{T_0}$$

$$\geq \beta_{T_1} + \beta_{T_0} + \frac{1}{m}\alpha_S + \frac{m-1}{m}\alpha_S = \alpha_S + \beta_T$$

Completing the proof.

Now we are ready to prove the theorem.

Theorem1. Let G be a bipartite $(0, mf-(m-1)r)$-graph . Let f be one integer-valued function defined on $V(G)$ such that $f(x) \geq 2r$ for each $x \in V(G)$, and H be an mr-subgraph of G . Then G has a $(0, f)$-factorization randomly r-orthogonal to H .

Proof. According to Lemma 5, the theorem is trivial for $r=1$. In the following ,we consider $r \geq 2$. Let $\{A_1, A_2, \ldots, A_m\}$ be any partition of $E(H)$ with $|A_i| = r, 1 \leq i \leq m$. We prove that there a (g, f)-factorization $F = \{F_1, F_2, \ldots, F_m\}$ of G such that $A_i \subseteq E(F_i)$ for all $1 \leq i \leq m$. We apply induction on m . The assertion is trivial for $m=1$ .Suppose the result holds for $m-1$,let us proceed to the induction step.

Let $E_2 = E(H) \setminus A_1$. By Lemma 6, G has a (g, f)-factor F_1 such that $A_1 \subseteq E(F_1)$ and $E_2 \cap E(F_1) = \phi$. Clearly, F_1 is also a $(0, f)$-factor of G . Set $G' = G - E(F_1)$. It follows from the definition of $g(x)$ that

$$\begin{aligned} 0 \leq d_{G'}(x) = d_G(x) - d_{F_1}(x) &\leq d_G(x) - g(x) \\ &\leq d_G(x) - [d_G(x) - ((m-1)f(x) - (m-2)r)] \\ &= (m-1)f(x) - (m-2)r \end{aligned}$$

Hence G' is a bipartite $(0, (m-1)f-(m-2)r)$-graph. Let $H' = G[E_2]$. Then the induction hypothesis guarantees the existence of a $(0, f)$-factorization $F' = \{F_2, \ldots, F_m\}$ in G' which satisfies $A_i \subseteq E(F_i), 2 \leq i \leq m$. Hence G has a $(0, f)$-factorization which is randomly r-orthogonal to H .

Completing the proof.

Remark 1. Apperently, the lower bound 0 in Theorem 1 is sharp in any sense . The upper bound $mf-(m-1)r$ is necessary in the proof of Lemma 3. In this sense , Theorem 1 is the best possible . In the proof of Lemma 6 , it is required that $f(x) \geq 2r$ for all $x \in V(G)$. We do not know whether the condition $f(x) \geq 2r$ can be improved .

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