Sufficient conditions for maximally edge-connected and super-edge-connected oriented graphs depending on the clique number

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Abstract

An orientation of a simple graph G is called an oriented graph. If D is an oriented graph, $\delta(D)$ its minimum degree and $\lambda(D)$ its edge-connectivity, then $\lambda(D) \leq \delta(D)$. The oriented graph is called maximally edge-connected if $\lambda(D) = \delta(D)$ and super-edge-connected, if every minimum edge-cut is trivial. In this paper we show that an oriented graph D of order n without any clique of order $p+1$ in its underlying graph is maximally edge-connected when

$$n \leq 4\left\lfloor \frac{p\delta(D)}{p-1} \right\rfloor - 1.$$

Some related conditions for oriented graphs to be super-edge-connected are also presented.

Keywords: *oriented graph, edge-connectivity, super-edge-connectivity, clique number*

1. Introduction and terminology

We consider finite digraphs without loops and multiple edges. A digraph without any directed cycle of length 2 is called an *oriented graph*. For a digraph D the vertex set is denoted by $V(D)$ and the edge set (or arc set) by $E(D)$. If xy is an arc, then we also write $x \to y$ and say x *dominates* y. We define the *order* of D by $n = n(D) = |V(D)|$ and the *size* by $|E(D)|$. For a vertex $v \in V(D)$ of a digraph D let $d^+(v) = d_D^+(v)$ its *out-degree* and $d^-(v) = d_D^-(v)$ its *in-degree*. The *minimum out-degree* and *minimum in-degree* of a digraph D are denoted by $\delta^+ = \delta^+(D)$ and $\delta^- = \delta^-(D)$ and $\delta = \delta(D) = \min\{\delta^+(D), \delta^-(D)\}$ is its *minimum degree*.

A digraph D is *strongly connected* or simply *strong* if for every pair u, v of vertices there exists a directed path from u to v in D. A digraph D is k-*edge-connected* if for any set S of at most $k-1$ edges the subdigraph $D - S$ is strong. The *edge-connectivity* $\lambda = \lambda(D)$ of a digraph D is defined as the largest value of k such that D is k-edge-connected. Because of $\lambda(D) \leq \delta(D)$, we call a digraph D *maximally edge-connected* if $\lambda(D) = \delta(D)$. A digraph is *super-edge-connected* or *super-*λ, if every minimum edge-cut is trivial, that means, that every minimum edge-cut consists of edges adjacent to or from a vertex of minimum degree.

For two disjoint vertex sets X and Y of a digraph D let (X, Y) be the set of edges from X to Y. If D is a digraph, then its *underlying graph* $G(D)$ is the graph obtained by replacing each arc of D by an undirected edge joining the same pair of vertices. If D is an oriented graph with the property that the underlying graph $G(D)$ contains no complete subgraph of order $p+1$, then we say that the *clique number* $\omega(D)$ is less or equal p. If D is an oriented graph with clique number $\omega(D) \leq p$, then the well-known Theorem of Turán [18] leads to the fundamental upper bound

$$|E(D)| \leq \frac{p-1}{2p}|V(D)|^2. \tag{1}$$

A *p-partite tournament* is an orientation of a complete p-partite graph. For other graph theory terminology we follow Bondy and Murty [4] or Chartrand and Lesniak [6].

Sufficient conditions for digraphs to be maximally edge-connected or super-λ were given by several authors, for example by Balbuena and Carmona [2], Balbuena, Carmona, Fàbrega and Fiol [3], Carmona and Fàbrega [5], Dankelmann and Volkmann [7], Fàbrega and Fiol [8], Fiol [9, 10], Geller and Harary [11], Hellwig and Volkmann [12, 13, 14], Imase, Soneoka and Okada [15], Jolivet [16], Soneoka [17], Volkmann [19] and Xu [20]. However, closely related conditions for maximally edge-connected and super-edge-connected oriented graphs have received little attention until recently. In this paper we will present some new sufficient conditions for oriented graphs to be maximally edge-connected and super-λ, respectively.

2. Maximally edge-connected oriented graphs

We start with a simple observation, which play an important role in our investigations.

Lemma 2.1 Let D be an oriented graph of edge-connectivity λ, $\delta^+ = \delta^+(D)$, $\delta^- = \delta^-(D)$ and $\delta = \delta(D) \geq 1$. If $\lambda < \delta$, then there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$ such that $|X| \geq 2\delta^+ + 1$ and $|Y| \geq 2\delta^- + 1$.

Proof. Let $X, Y \subset V(D)$ be two disjoint sets with $X \cup Y = V(D)$ such that $|(X,Y)| = \lambda$. By reason of symmetry we only prove $|X| \geq 2\delta^+ + 1$. If we suppose to the contrary that $|X| \leq 2\delta^+$, then we arrive at the contradiction

$$|X|\delta^+ \leq \sum_{x \in X} d^+(x) \leq \frac{|X|(|X|-1)}{2} + \lambda \leq \delta^+(|X|-1) + \delta^+ - 1. \; \Box$$

Corollary 2.2 Let D be an oriented graph of order n, $\lambda = \lambda(D)$, $\delta^+ = \delta^+(D)$, $\delta^- = \delta^-(D)$ and $\delta = \delta(D) \geq 1$. If $\delta^+ + \delta^- \geq \lceil (n-1)/2 \rceil$, then $\lambda = \delta$.

Corollary 2.3 (Ayoub, Frisch [1] 1970) If D is an oriented graph with minimum degree $\delta(D) \geq \lceil (n(G)-1)/4 \rceil$, then $\lambda(D) = \delta(D)$.

Using Turán's inequality (1), we will present some analogue results for oriented graphs D with clique number $\omega(D) \leq p$.

Theorem 2.4 Let $p \geq 2$ be an integer, and let D be an oriented graph with clique number $\omega(D) \leq p$, $\lambda = \lambda(D)$, $\delta^+ = \delta^+(D)$, $\delta^- = \delta^-(D)$ and $\delta = \delta(D) \geq 1$. If $\lambda < \delta$, then there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$ such that

$$|X| \geq 2\left\lfloor \frac{p\delta^+}{p-1} \right\rfloor \text{ and } |Y| \geq 2\left\lfloor \frac{p\delta^-}{p-1} \right\rfloor.$$

Proof. Let $X, Y \subset V(D)$ be two disjoint sets with $X \cup Y = V(D)$ such that $|(X,Y)| = \lambda$. By reason of symmetry we only prove the desired bound for the set X. If $\delta^+ = k(p-1) + r$ with integers $k \geq 0$ and r such that $0 \leq r \leq p-2$, then

$$\left\lfloor \frac{p\delta^+}{p-1} \right\rfloor = \left\lfloor \frac{(p-1)\delta^+ + \delta^+}{p-1} \right\rfloor = \delta^+ + k.$$

This shows that our statement is equivalent to $|X| \geq 2\delta^+ + 2k$. In view of Lemma 2.1, the desired bound is valid for $k = 0$. Thus we only consider the case that $k \geq 1$ in the following.

First assume that $|X| \leq 2\delta^+ + 2k - 2$. This assumption and Turán's inequality (1) imply

$$\begin{aligned} |X|\delta^+ &\leq \sum_{x \in X} d^+(x) \leq \frac{p-1}{2p}|X|^2 + \lambda \\ &\leq |X|\frac{p-1}{2p}(2\delta^+ + 2k - 2) + \delta^+ - 1 \\ &= |X|\frac{p-1}{p}(\delta^+ + k - 1) + \delta^+ - 1. \end{aligned}$$

It follows that

$$|X|(k(p-1)+r+k-1-p(k-1)) = |X|(\delta^+ + k - 1 - p(k-1)) \leq p(\delta^+ - 1),$$

and this leads to

$$|X| \leq \frac{p(\delta^+ - 1)}{p + r - 1} \leq \frac{p(\delta^+ - 1)}{p - 1}.$$

Because of $p(\delta^+ - 1)/(p-1) \leq 2\delta^+$, we obtain $|X| \leq 2\delta^+$, a contradiction to Lemma 2.1. Hence we have shown that $|X| \geq 2\delta^+ + 2k - 1$.

Second we assume that $|X| = 2\delta^+ + 2k - 1$. Again (1) yields

$$\begin{aligned} |X|\delta^+ &\leq \sum_{x \in X} d^+(x) \leq \frac{p-1}{2p}|X|^2 + \lambda \\ &\leq |X|\frac{p-1}{2p}(2\delta^+ + 2k - 1) + \delta^+ - 1. \end{aligned}$$

It follows that

$$|X|(2\delta^+ + 2k - 1 - 2kp + p)) \leq 2p(\delta^+ - 1),$$

and this leads to

$$|X| \leq \frac{2p(\delta^+ - 1)}{p + 2r - 1}.$$

Since $k, \delta \geq 1$, we observe that $2p(\delta^+ - 1)/(p + 2r - 1) \leq 2\delta^+ + 2k - 2$, and thus we arrive at the contradiction $2\delta^+ + 2k - 1 = |X| \leq 2\delta^+ + 2k - 2$. □

Corollary 2.5 Let $p \geq 2$ be an integer and let D be an oriented graph of order n with clique number $\omega(D) \leq p$, $\lambda = \lambda(D)$, $\delta^+ = \delta^+(D)$, $\delta^- = \delta^-(D)$ and $\delta = \delta(D) \geq 1$. If

$$n \leq 2\left\lfloor \frac{p\delta^+}{p-1} \right\rfloor + 2\left\lfloor \frac{p\delta^-}{p-1} \right\rfloor - 1,$$

then $\lambda = \delta$.

Corollary 2.6 Let $p \geq 2$ be an integer and let D be an oriented graph of order n with clique number $\omega(D) \leq p$, edge-connectivity λ and minimum degree $\delta \geq 1$. If

$$n \leq 4\left\lfloor \frac{p\delta}{p-1} \right\rfloor - 1,$$

then $\lambda = \delta$.

Corollary 2.7 Let D be a bipartite oriented graph of order n with edge-connectivity λ, minimum out-degree δ^+, minimum in-degree δ^- and minimum degree $\delta \geq 1$. If

$$\delta^+ + \delta^- \geq \left\lceil \frac{n+1}{4} \right\rceil,$$

then $\lambda = \delta$.

Let $p \geq 2$ be an integer, and let T be a regular p-partite tournament with the partite sets $V_1, V_2, \ldots, V_p$ such that $|V_1| = |V_2| = \ldots = |V_p| = 2r$ for an integer $r \geq 1$. If D consists of two disjoint copies of T, then $\omega(D) \leq p$, $n(D) = 4pr$, $\delta(D) = \delta^+(D) = \delta^-(D) = r(p-1)$ and $\lambda(D) = 0$. This family of examples show that Theorem 2.4 as well as Corollaries 2.5 - 2.7 are best possible.

3. Super-edge-connected oriented graphs

Theorem 3.4 in Fiol's article [9] states that the conditions in Corollary 2.7 is sufficient for a bipartite oriented graph to be super-λ. However, the next example will show that this is not valid in general.

Example 3.1 Let T be the bipartite oriented graph of order 14 with the partition sets

$$X = \{x_1, x_2, x_3, x_4, x_1', x_2', x_3'\} \text{ and } Y = \{y_1, y_2, y_3, y_1', y_2', y_3', y_4'\}$$

such that $y_1 \to x_1 \to y_2 \to x_2 \to y_3 \to x_3 \to y_2 \to x_2'$, $y_3 \to x_4 \to y_2$, $x_1 \to y_3$, $\{x_2, x_3, x_4\} \to y_1 \to x_1'$, $y_1' \to x_1' \to y_2' \to x_2' \to y_3' \to x_3' \to y_2'$, $x_2' \to y_4' \to x_3' \to y_1'$, $x_1' \to \{y_3', y_4'\}$, $x_2' \to y_1'$ and $y_i' \to x_j$ for $1 \leq i, j \leq 4$.

Now $n(T) = 14$, $\delta(T) = \delta^+(T) = \delta^-(T) = 2$, $4 = \delta^+(T) + \delta^-(T) = \lceil (n(T)+1)/4 \rceil$ and thus $\lambda(T) = \delta(T) = 2$ by Corollary 2.7. However, T is not super-λ, since $S = \{y_1x_1', y_2x_2'\}$ is a minimum edge-cut.

Corresponding examples also exist for every $\delta^+ = \delta^- \geq 3$. In this section, we will present (see Corallary 3.9 below) a correct sufficient condition for bipartite oriented graphs to be super-edge-connected.

Lemma 3.2 Let D be an oriented graph with $\lambda = \lambda(D)$, $\delta^+ = \delta^+(D)$, $\delta^- = \delta^-(D)$ and $\delta = \delta(D) \geq 2$. If D is not super-λ, then there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$ such that $|X| \geq 2\delta^+$ and $|Y| \geq 2\delta^-$.

Proof. Since D is not super-λ, there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$ such that $|X|, |Y| \geq 2$. We only prove the desired bound for the set $|X|$.

First suppose that $|X| \leq \delta^+$. It follows that

$$|X|\delta^+ \leq \sum_{x \in X} d^+(x) \leq \frac{|X|(|X|-1)}{2} + \lambda \leq \frac{\delta^+(|X|-1)}{2} + \delta^+,$$

and this implies the contradiction $\delta^+|X| \leq \delta^+$. Hence we have shown that $|X| \geq \delta^+ + 1$.

Second suppose that $|X| \leq 2\delta^+ - 1$. This leads to

$$|X|\delta^+ \leq \sum_{x \in X} d^+(x) \leq \frac{|X|(|X|-1)}{2} + \lambda \leq |X|(\delta^+ - 1) + \delta^+,$$

and we obtain the contradiction $|X| \leq \delta^+$. □.

Corollary 3.3 (Fiol [9] 1992) Let D be an oriented graph of order n, $\lambda = \lambda(D)$, $\delta^+ = \delta^+(D)$, $\delta^- = \delta^-(D)$ and $\delta = \delta(D)$. If $\delta^+ + \delta^- \geq \lceil (n+1)/2 \rceil$, then D is super-λ.

Theorem 3.4 Let $p \geq 2$ be an integer, and let D be an oriented graph with $\omega(D) \leq p$, $\lambda = \lambda(D)$, $\delta^+ = \delta^+(D)$, $\delta^- = \delta^-(D)$ and $\delta = \delta(D) \geq 2$. If D is not super-λ, then there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$ such that

$$|X| \geq 2\left\lfloor \frac{p\delta^+}{p-1} \right\rfloor - 2 \text{ and } |Y| \geq 2\left\lfloor \frac{p\delta^-}{p-1} \right\rfloor - 2.$$

Proof. Since D is not super-λ, there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$ such that $|X|, |Y| \geq 2$. We only prove the desired bound for the set X. If $\delta^+ = k(p-1) + r$ with integers $k \geq 0$ and r such that $0 \leq r \leq p-2$, then our statement is equivalent to $|X| \geq 2\delta^+ + 2k - 2$. In view of Lemma 3.2, the bound is valid for $k \leq 1$. Thus let $k \geq 2$ in the following. Assume that $|X| \leq 2\delta^+ + 2k - 3$. This assumption and inequality (1) imply

$$|X|\delta^+ \leq \sum_{x \in X} d^+(x) \leq \frac{p-1}{2p}|X|^2 + \lambda \leq |X|\frac{p-1}{2p}(2\delta^+ + 2k - 3) + \delta^+.$$

It follows that

$$|X| \le \frac{2p\delta^+}{3p+2r-3} \le \frac{2p\delta^+}{3p-3}.$$

Because of $2p\delta^+/(3p-3) \le 2\delta^+ - 1$, we obtain $|X| \le 2\delta^+ - 1$, a contradiction to Lemma 3.2. □

Corollary 3.5 Let $p \ge 2$ be an integer, and let D be an oriented graph with $\omega(D) \le p$, $\lambda = \lambda(D)$, $\delta^+ = \delta^+(D)$, $\delta^- = \delta^-(D)$ and $\delta = \delta(D) \ge 2$. Then D is super-λ when

$$n \le 2\left\lfloor \frac{p\delta^+}{p-1} \right\rfloor + 2\left\lfloor \frac{p\delta^-}{p-1} \right\rfloor - 5.$$

The next example will show that Theorem 3.4 and Corollary 3.5 are best possible for the case that $\delta^+ = \delta^- = \delta = p-1$.

Example 3.6 Let $p \ge 3$ be an integer, and let D_1' be a $(p-1)$-regular p-partite tournament with the partite sets $\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_p, v_p\}$ such that $\{u_2, u_3, \ldots, u_p\} \to u_1$. In addition, let D_2' be a $(p-1)$-regular p-partite tournament with the partite sets $\{x_1, y_1\}$, $\{x_2, y_2\}, \ldots, \{x_p, y_p\}$ such that $x_1 \to \{x_2, x_3, \ldots, x_p\}$. If $D_1 = D_1' - u_1$ and $D_2 = D_2' - x_1$, then let D be the p-partite tournament consisting of the disjoint union of D_1 and D_2 such that $\{v_1, y_1\}$ and $\{u_i, v_i, x_i, y_i\}$ for $2 \le i \le p$ are the partite sets of D together with edge set

$$S = \{u_2x_3, u_3x_4, \ldots, u_{p-1}x_p, u_px_2\}$$

and all further possible edges from D_2 to D_1. The resulting p-partite tournament D is of order $n(D) = 4p-2$ such that $\delta^+(D) = \delta^-(D) = \delta(D) = p-1$. According to Corollary 2.6, we deduce that $\lambda(D) = \delta(D) = p-1$. However, since S is a minimum edge-cut, D is not super-λ.

In the cases that $\delta \ne p-1$ or $\delta \ne t(p-1)$ for any integer $t \ge 1$, we are able to present better bounds.

Theorem 3.7 Let $p \ge 2$ be an integer, and let D be an oriented graph with $\omega(D) \le p$, $\lambda = \lambda(D)$, $\delta^+ = \delta^+(D)$, $\delta^- = \delta^-(D)$ and $\delta = \delta(D) \ge 2$. If D is not super-λ and $\delta^+ \ne p-1$ or $\delta^- \ne p-1$, then there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$ such that

$$|X| \ge 2\left\lfloor \frac{p\delta^+}{p-1} \right\rfloor - 1 \text{ or } |Y| \ge 2\left\lfloor \frac{p\delta^-}{p-1} \right\rfloor - 1.$$

Proof. Since D is not super-λ, there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$ such that $|X|, |Y| \ge 2$. We only prove the desired bound for the set X. If $\delta^+ = k(p-1)+r$ with integers $k \ge 0$ and r such that $0 \le r \le p-2$, then our statement is equivalent to $|X| \ge 2\delta^+ + 2k - 1$, where $r \ge 1$ when $k = 1$. In view of Lemma 3.2, the bound is valid for $k = 0$. Thus let in the following $k \ge 1$ and $r \ge 1$ when $k = 1$. Because of Theorem 3.4, we know that $|X| \ge 2\delta^+ + 2k - 2$. Assume that $|X| = 2\delta^+ + 2k - 2$. This assumption and inequality (1) imply

$$|X|\delta^+ \le \sum_{x \in X} d^+(x) \le \frac{p-1}{2p}|X|^2 + \lambda \le |X|\frac{p-1}{2p}(2\delta^+ + 2k - 2) + \delta^+.$$

It follows that

$$|X| \le \frac{p\delta^+}{p+r-1}. \tag{2}$$

If $k = 1$, then $r \ge 1$ and (2) leads to the $|X| \le \delta^+$, a contradiction to Lemma 3.2. If $k \ge 2$, then (2) yields

$$|X| \le \frac{p\delta^+}{p+r-1} \le \frac{p\delta^+}{p-1} \le 2\delta^+ + 2k - 3,$$

a contradiction to our assumption.

Corollary 3.8 Let $p \ge 2$ be an integer, and let D be an oriented graph with $\omega(D) \le p$, $\lambda = \lambda(D)$, $\delta^+ = \delta^+(D)$, $\delta^- = \delta^-(D)$ and $\delta = \delta(D) \ge 2$. If $\delta^+ \ne p-1$, $\delta^- \ne p-1$ and

$$n \le 2\left\lfloor \frac{p\delta^+}{p-1} \right\rfloor + 2\left\lfloor \frac{p\delta^-}{p-1} \right\rfloor - 3,$$

then D is super-λ.

Corollary 3.9 Let D be an oriented graph of clique number 2, order n, $\lambda = \lambda(D)$, $\delta^+ = \delta^+(D)$, $\delta^- = \delta^-(D)$ and $\delta = \delta(D) \ge 2$. Then D is super-λ when

$$\delta^+ + \delta^- \ge \left\lceil \frac{n+3}{4} \right\rceil.$$

For the case that $p \ge 3$ and $\delta^+ \ne t(p-1)$ and $\delta^- \ne t(p-1)$ for any integer $t \ge 1$, we can improve Theorem 2.4 as well as Corollary 2.5.

Theorem 3.10 Let $p \ge 3$ be an integer, and let D be an oriented graph with $\omega(D) \le p$, $\lambda = \lambda(D)$, $\delta^+ = \delta^+(D)$, $\delta^- = \delta^-(D)$ and $\delta = \delta(D) \ge 2$. If D is not super-λ and $\delta^+ \ne t(p-1)$ or $\delta^- \ne t(p-1)$ for an integer $t \ge 1$, then there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$ such that

$$|X| \ge 2\left\lfloor \frac{p\delta^+}{p-1} \right\rfloor \quad \text{or} \quad |Y| \ge 2\left\lfloor \frac{p\delta^-}{p-1} \right\rfloor.$$

Proof. Since D is not super-λ, there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$ such that $|X|, |Y| \geq 2$. We only prove the bound for the set X. If $\delta^+ = k(p-1)+r$ with integers $k \geq 0$ and r such that $1 \leq r \leq p-2$, then our statement is equivalent to $|X| \geq 2\delta^+ + 2k$. In view of Lemma 3.2, the bound is valid for $k = 0$. Thus let $k \geq 1$ in the following. Because of Theorem 3.7, we know that $|X| \geq 2\delta^+ + 2k - 1$. Assume that $|X| = 2\delta^+ + 2k - 1$. This assumption and inequality (1) imply

$$|X| \leq \frac{2p\delta^+}{p+2r-1} \leq \frac{2p\delta^+}{p+1}.$$

Because of $k \geq 1$, we obtain $2p\delta^+/(p+1) \leq 2\delta^+ + 2k - 2$, and thus we arrive at the contradiction $|X| \leq 2\delta^+ + 2k - 2$. □.

Corollary 3.11 Let $p \geq 3$ be an integer, and let D be an oriented graph with $\omega(D) \leq p$, $\lambda = \lambda(D)$, $\delta^+ = \delta^+(D)$, $\delta^- = \delta^-(D)$ and $\delta = \delta(D) \geq 2$. If $\delta^+ \neq t(p-1)$, $\delta^- \neq t(p-1)$ for an integer $t \geq 1$ and

$$n \leq 2\left\lfloor \frac{p\delta^+}{p-1} \right\rfloor + 2\left\lfloor \frac{p\delta^-}{p-1} \right\rfloor - 1,$$

then D is super-λ.

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