ON QUOTIENT CURVES OF THE SUZUKI CURVE

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ABSTRACT. Inspired by a recent paper by Giulietti, Korchmáros and Torres [3], we provide equations for some quotient curves of the Deligne-Lusztig curve associated to the Suzuki group $Sz(q)$.

1. INTRODUCTION

The Deligne-Lusztig curve of Suzuki type (shortly DLS-curve) is the (projective geometrically irreducible, non-singular) algebraic curve defined to be the non-singular model over the finite field $\mathbf{F}_q$ of the (absolutely irreducible) plane curve C of equation $X^{q_0}(X^q + X) = Y^q + Y$, where $q_0 = 2^s, s \geq 1$ and $q = 2q_0^2$. Several authors have studied the DLS-curve also in connection with coding theory, see [1], [2], [6], [7], [8], [9]. Here we only mention that the DLS-curve has genus $g = q_0(q-1)$ and that the number of its $\mathbf{F}_q$-rational points is q^2+1. Actually, the two latter properties characterize the DLS-curve, see [2]. The automorphism group of the DLS-curve is the Suzuki group $Sz(q)$.

In [3] the quotient curves of the DLS-curve arising from the subgroups of $Sz(q)$ are thoroughly investigated. For tame covering, that is for subgroups of odd order, the authors obtain an exhaustive list of such curves. A similar complete list for non-tame coverings cannot be produced because the Suzuki group contains a huge number of pairwise non-isomorphic subgroups of even order.

Our contribution here is to provide such a complete list for the cases $q = 8$ and $q = 32$. For all curves in the list, a plane equation is given as well.

A motivation for the present work comes from the current interest in curves over finite fields with many rational points, see van der Geer's survey [4]. Indeed, the number of $\mathbf{F}_q$-rational points of a curve of genus g which is $\mathbf{F}_q$-covered by the DLS-curve is $N = 1 + q + 2q_0g$ [3, Proposition 3.1] and this value is in the interval from which the entries of the tables of curves with many rational points are taken for $g \leq 50$, $q \leq 128$ in [5]. It should be

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noted that we provide here (see Theorem 3.9) plane equations for quotient curves of the DLS-curve defined over $\mathbf{F}_{32}$ with genus $g \in \{12, 28, 30\}$. By [5] these curves attain the largest value of $\mathbf{F}_q$-rational points for which an $\mathbf{F}_q$-rational curve of genus g is previously known to exist.

2. Preliminary results on the DLS-curve

Throughout the present chapter $\mathcal{X}$ will stand for the DLS-curve over $\mathbf{F}_q$. As we have mentioned in the Introduction $\mathcal{X}$ has genus $q_0(q-1)$ and contains exactly q^2+1 $\mathbf{F}_q$-rational points. The proposition below will be useful in the sequel.

Proposition 2.1. *For any $b \in \mathbf{F}_q$, $b \neq 0$, there are elements $x, y \in \mathbf{F}_q(\mathcal{X})$ such that*

$$\mathbf{F}_q(\mathcal{X}) = \mathbf{F}_q(x,y), \quad x^{2q_0}(x^q + x) = b(y^q + y).$$

Proof. We have $\mathbf{F}_q(\mathcal{X}) = \mathbf{F}_q(x,t)$ with $x^{q_0}(x^q+x) = t^q + t$. Let $y = b^{-1}(x^{2q_0+1} + t^{2q_0})$, that is $t^q = b^{q_0}y_0^q + x^{q+q_0}$. Then $\mathbf{F}_q(\mathcal{X}) = \mathbf{F}_q(x,y)$. Furthermore, $y^{q_0} = b^{-q_0}(x^{q+q_0} + t^q) = b^{-q_0}(x^{q_0+1} + t)$, and hence $y^q = b^{-1}(x^{q+q_0} + t^{2q_0})$. Now, since $y^q + y = b^{-1}(x^{q+q_0} + t^{2q_0} + x^{2q_0+1} + t^{2q_0}) = b^{-1}x^{2q_0}(x^q + x)$, the claim follows. □

Let C_b be the plane curve of equation $X^{2q_0}(X^q+X) = b(Y^q+Y)$. C_b has only one singular point, namely the infinite point Y_∞ of the Y-axis which point is a q_0-fold point. We know from [8] that $\bar{\mathbf{F}}_q(\mathcal{X})$ has just one place centered at Y_∞.

For $a, c, d \in \mathbf{F}_q$ with $d \neq 0$, we define the following automorphisms of $\mathbf{F}_q(\mathcal{X})$:

$$(2.1) \qquad \psi_{a,c} := \begin{cases} x \mapsto x + a, \\ y \mapsto a^{2q_0}x + y + c; \end{cases} \qquad \gamma_d := \begin{cases} x \mapsto dx, \\ y \mapsto d^{2q_0+1}y; \end{cases}$$

for $h := xy + x^{2q_0+2} + y^{2q_0}$,

$$(2.2) \qquad \varphi := \begin{cases} x \mapsto y/h, \\ y \mapsto x/h. \end{cases}$$

The automorphism group of $\bar{\mathbf{F}}_q(\mathcal{X})$ generated by $\psi_{a,c}$, γ_d and φ is the full automorphism group of $\bar{\mathbf{F}}_q(\mathcal{X})$, it is isomorphic to $Sz(q)$ and it acts on the set of places of $\mathcal{X}(\mathbf{F}_q)$ as $Sz(q)$ in its unique 2-transitive permutation representation [3, Proposition 3.5].

By Proposition 2.1 $\bar{\mathbf{F}}_q(\mathcal{X}) = \bar{\mathbf{F}}_q(x,y)$ with $x^{2q_0}(x^q+x) = y^q + y$. The extension $\bar{\mathbf{F}}_q(\mathcal{X})|\bar{\mathbf{F}}_q(x)$ is Galois of degree q, and x has a unique pole in $\mathbf{F}_q(\mathcal{X})$ that we denote by $\mathcal{P}_\infty$. Such a place is totally ramified in $\bar{\mathbf{F}}_q(\mathcal{X})$, while all the other rational places of $\bar{\mathbf{F}}_q(x)$ split completely in $\bar{\mathbf{F}}_q(\mathcal{X})|\bar{\mathbf{F}}_q(x)$. The Galois group of $\bar{\mathbf{F}}_q(\mathcal{X})|\bar{\mathbf{F}}_q(x)$ is $\bar{\mathbf{T}}_0 := \{\psi_{0,c} \mid c \in \mathbf{F}_q\}$. Note that $\bar{\mathbf{T}}_0$ comprises the identity and the elements of order 2 of the Sylow 2-subgroup $\bar{\mathbf{T}} = \{\psi_{a,c} \mid a, c \in \mathbf{F}_q\}$ of $\mathrm{Aut}(\bar{\mathbf{F}}_q(\mathcal{X}))$.

The stabilizer of $\mathcal{P}_\infty$ in $\mathrm{Aut}(\bar{\mathbf{F}}_q(\mathcal{X}))$ is the group $\bar{\mathbf{T}}\bar{\mathbf{N}}$, where $\bar{\mathbf{N}} := \{\gamma_d \mid d \in \mathbf{F}_q, d \neq 0\}$, and the normalizer $N_{\mathrm{Aut}(\bar{\mathbf{F}}_q(\mathcal{X}))}(\bar{\mathbf{N}})$ is the dihedral group generated by $\bar{\mathbf{N}}$ together with φ. Moreover, $\mathrm{Aut}(\bar{\mathbf{F}}_q(\mathcal{X}))$ contains two conjugacy classes of subgroups of Singer type, one consisting of cyclic subgroups $\bar{\mathbf{D}}^+$ of order $q+2q_0+1$ and the other of cyclic subgroups $\bar{\mathbf{D}}^-$ of order $q-2q_0+1$. The normalizer $N_{\mathrm{Aut}(\bar{\mathbf{F}}_q(\mathcal{X}))}(\bar{\mathbf{D}}^+)$ has order $4(q+2q_0+1)$ and is the semidirect product of $\bar{\mathbf{D}}^+$ by a cyclic group of order 4. All these results hold true for $\bar{\mathbf{D}}^-$.

In some cases, $\mathrm{Aut}(\bar{\mathbf{F}}_q(\mathcal{X}))$ contains subgroups isomorphic to the Suzuki group over a subfield $\mathbf{F}_{\bar{q}}$ of $\mathbf{F}_q$. This occurs if and only if $\bar{q} = 2^{2\bar{s}+1}$ with a divisor $\bar{s}$ of s such that $2\bar{s}+1$ divides $2s+1$.

Proposition 2.2. *Any subgroup of* $\mathrm{Aut}(\bar{\mathbf{F}}_q(\mathcal{X}))$ *is conjugate to either a subgroup isomorphic to* $Sz(\bar{q})$*, or to a subgroup of one of the following groups:* $\bar{\mathbf{T}}\bar{\mathbf{N}}$, $N_{\mathrm{Aut}(\bar{\mathbf{F}}_q(\mathcal{X}))}(\bar{\mathbf{N}})$, $N_{\mathrm{Aut}(\bar{\mathbf{F}}_q(\mathcal{X}))}(\bar{\mathbf{D}}^+)$, $N_{\mathrm{Aut}(\bar{\mathbf{F}}_q(\mathcal{X}))}(\bar{\mathbf{D}}^-)$.

For tame covering, that is for subgroups of odd order, an exhaustive list of quotient curves of the DLS-curve of Suzuki type has been recently obtained in [3].

Theorem 2.3. *Let $\mathcal{X}$ be a tame quotient curve of the DLS-curve. Then one of the following holds.*

I) *r is any divisor of $q-1$, $\mathcal{X}$ has genus $g = \frac{q-1}{r}q_0$ and is a non-singular model over $\mathbf{F}_q$ of the plane curve of equation*

$$Y^{(q-1)/r}\Big(1+\sum_{i=0}^{s-1} X^{2^i(2q_0+1)-(q_0+1)}(1+X)^{2^i}\Big) = (X^{q_0}+1)(Y^{2(q-1)/r}+X^{q-1}),$$

II) *r is any divisor of $q+2q_0+1$, $\mathcal{X}$ has genus $g = \frac{q_0(q-1)-1}{r}+1$ and is a non-singular model over $\mathbf{F}_{q^4}$ of the plane curve of equation*

$$Y^{(q+2q_0+1)/r}\Big(1+\sum_{i=0}^{s-1} X^{2^i q_0}(1+X)^{2^i(q_0+1)-q_0} + X^{q/2}\Big) =$$
$$X^{q+2q_0+1} + Y^{2(q+2q_0+1)/r},$$

III) *r is any divisor of $q-2q_0+1$, $\mathcal{X}$ has genus $g = \frac{q_0(q-1)+1}{r}-1$ and is a non-singular model over $\mathbf{F}_{q^4}$ of the plane curve of equation*

$$bY^{(q-2q_0+1)/r}\Big(1+\sum_{i=0}^{s-1} X^{2^i(2q_0+1)-(q_0+1)}(1+X)^{2^i}\Big) =$$
$$(X^{q-2q_0+1} + Y^{2(q-2q_0+1)/r})(X^{q_0-1}+X^{2q_0-1}),$$

where $b = \lambda^{q_0}+\lambda^{q_0-1}+\lambda^{-q_0}+\lambda^{-q_0+1}$ and $\lambda \in \mathbf{F}_{q^4}$ is an element of order $q-2q_0+1$.

A similar complete list for non-tame coverings cannot be produced because the Suzuki group contains a huge number of pairwise non-isomorphic subgroups of even order. However, the existence of non-tame quotient curves of the DLS-curve of genus g has been given in Theorem 2.4 (see [3]). For some of these curves also a plane equation has been provided, see Theorem 2.5 ([3]).

Theorem 2.4. *Let v, u, r be positive integers. For the following values of g the DLS-curve has a quotient curve $\mathcal{X}$ of genus g.*

i) $g = 2^{s-u+v}(2^{2s+1-v} - 1)$, $v \le 2s+1$, $v \le u \le v + \log_2(v+1)$,

ii) $g = \frac{1}{r}2^s(2^{2s+1-v} - 1)$, $v \le 2s+1$, $r|(q-1)$, $r|(2^{2s+1-v} - 1)$,

iii) $g = \frac{q_0(q-r-1)}{2r}$, $r|(q-1)$,

iv) $g = \frac{1}{2}\left[\frac{q_0(q-1)-1}{r} - (q_0 - 1)\right]$, $r|(q + 2q_0 + 1)$,

v) $g = \frac{1}{2}\left[\frac{q_0(q-1)+1}{r} - (q_0 + 1)\right]$, $r|(q - 2q_0 + 1)$,

vi) $g = \frac{1}{4}\left[\frac{q_0(q-1)-1}{r} - (q_0 - 1)\right]$, $r|(q + 2q_0 + 1)$,

vii) $g = \frac{1}{4}\left[\frac{q_0(q-1)+1}{r} - (q_0 + 1)\right]$, $r|(q - 2q_0 + 1)$,

viii) $g = \frac{q_0(q-1)-1+(\bar{q}^2+1)\bar{q}^2(\bar{q}-1)+\Delta}{(\bar{q}^2+1)\bar{q}^2(\bar{q}-1)}$, $\bar{q} = 2^{2\bar{s}+1}$, $\bar{s}|s$, $(2\bar{s}+1)|(2s+1)$,
$\Delta := (\bar{q}^2 + 1)[(2q_0 + 2)(\bar{q} - 1) + 2\bar{q}(\bar{q} - 1)] + \bar{q}^2(\bar{q}^2 + 1)(\bar{q} - 2) + \bar{q}^2(\bar{q} + 2\bar{q}_0 + 1)(\bar{q} - 1)(\bar{q} - 2\bar{q}_0)$,

ix) $g = 2^4(2^{9-v} - 1)$, $3 \le v \le 2s+1$, *for* $q = 512$.

Theorem 2.5. i') *For $u = v$, $v|(2s+1)$ a non-singular model over $\mathbf{F}_q$ of the plane curve of equation*

$$X^{2q_0}(X^q + X) = b \sum_{i=0}^{(2s+1/v)-1} Y^{(2^v)^i}$$

is a quotient curve of the DLS-curve of genus g as in i).

ii') *For $u = 2$, $v = 1$ a non-singular model over $\mathbf{F}_q$ of the plane curve of equation*

$$\sum_{i=0}^{2s} X^{2^i} + \sum_{i=0}^{s} X^{2^i}\Big(\sum_{j=i}^{s} X^{2^j}\Big) + \sum_{i=s+1}^{2s} X^{2^i}\Big(\sum_{j=0}^{i-s-2} X^{2^j}\Big)^{2q_0} = \sum_{i=0}^{2s} Y^{2^i}$$

is a quotient curve of the DLS-curve of genus g as in i).

iii') *A non-singular model over $\mathbf{F}_q$ of the plane curve of equation*

$$1 + \sum_{i=0}^{s-1} X^{2^i(2q_0+1)-(q_0+1)}(1+X)^{2^i} =$$

$$\sum (-1)^{i+j}\frac{(i+j-1)!k}{i!j!}Y^i(X^{rj}(X^{q_0}+1)$$

where the summation is extended over all pairs (i, j) of non-negative integers with $i + 2j = (q + 2q_0 + 1)/r$, is a quotient curve of the DLS-curve of genus g as in iii).

$$1+\sum_{i=0}^{s-1} X^{2^i q_0}(1+X)^{2^i(q_0+1)-q_0} + X^{q/2} = \sum (-1)^{i+j}\frac{(i+j-1)!}{i!j!}X^{ri}Y^j$$

is a quotient curve of the DLS-curve of genus g as in iv).

v') *Let b be as in III). A non-singular model over $\mathbf{F}_{q^4}$ of the plane curve of equation*

$$b\Big(1+\sum_{i=0}^{s-1} X^{2^i(2q_0+1)-(q_0+1)}(1+X)^{2^i}\Big) =$$

$$(X^{q_0-1}+X^{2q_0-1})\sum (-1)^{i+j}\frac{(i+j-1)!}{i!j!}X^{ri}Y^j$$

where the summation is extended over all pairs (i, j) of non-negative integers with $i + 2j = (q + 2q_0 + 1)/r$, is a quotient curve of the DLS-curve of genus g as in v).

In the following sections we are going to investigate the quotient curves of the DLS-curve arising from its automorphism groups of even order. In particular, we provide an equation for the cases not covered by the above theorems, for $q = 8$ and $q = 32$.

3. Quotient curves arising from 2-subgroups

Throughout this section the following notation will be used:

- $\mathcal{U}$ is a subgroup of $\bar{\mathbf{T}}$;
- $\mathcal{U}_2$ is the subgroup of $\mathcal{U}$ consisting of all elements of order 2 together with the identity;
- $\Phi : \bar{\mathbf{T}} \to \mathbf{F}_q$ is the map given by $\Phi(\psi_{a,c}) = a$;
- $\mathcal{X}_\mathcal{U}$ is the quotient curve of $\mathcal{X}$ arising from $\mathcal{U}$;
- $g_\mathcal{U}$ is the genus of $\mathcal{X}_\mathcal{U}$.

Proposition 3.1. [3, Proposition 7.1] *Let $\mathcal{U}$ have order 2^u. If $\mathcal{U}_2$ has order 2^v, then*

$$g_\mathcal{U} = 2^{s-u+v}(2^{2s+1-v} - 1).$$

The map Φ is a homomorphism from $\bar{\mathbf{T}}$ onto the additive subgroup of $\mathbf{F}_q$. The restriction of Φ to $\mathcal{U}$ is the homomorphism $\Phi_{|\mathcal{U}}$ with kernel $\mathrm{Ker}(\Phi_{|\mathcal{U}}) = \{\psi_{0,c} | c \in \mathbf{F}_q\}$ isomorphic to $\mathcal{U}_2$.

Lemma 3.2. *If $\mathcal{U} = \mathcal{U}_2$, then the fixed field of $\mathcal{U}$ is generated by x and $F(y)$, where $F(T) = \prod_{\psi_{0,c}\in\mathcal{U}_2}(T + c)$.*

Proof. It is straightforward to check that both x and $F(y)$ are fixed by $\mathcal{U}$. On the other hand, $\bar{\mathbf{F}}_q(x, F(y))$ cannot be a proper subfield of the fixed field of $\mathcal{U}$. In fact, the degree of the extension $\bar{\mathbf{F}}_q(x,y) \mid \bar{\mathbf{F}}_q(x, F(y))$ is less than or equal to $\#\mathcal{U}$, as $\deg F = \#\mathcal{U}$. □

Lemma 3.3. *Let $\mathcal{U}$ have order $2\#\mathcal{U}_2$, and assume that $\psi_{a,0} \in \mathcal{U}$, $a \in \mathbf{F}_q \setminus \{0\}$. Then the fixed field of $\mathcal{U}$ is generated by $x(x+a)$ and $F(y) + a^{-1}xF(a^{2q_0}x)$, where $F(T) = \prod_{\psi_{0,c} \in \mathcal{U}_2}(T+c)$.*

Proof. The group $\mathcal{U}$ is generated by $\mathcal{U}_2$ and $\psi_{a,0}$. As the set $\{c \in \mathbf{F}_q \mid \psi_{0,c} \in \mathcal{U}_2\}$ is a linear subspace of $\mathbf{F}_q \mid \mathbf{F}_2$, the polynomial $F(T)$ is such that $F(T_1 + T_2) = F(T_1) + F(T_2)$ ([11]). Also, $\psi_{a,0}^2 = \psi_{0,a^{2q_0+1}} \in \mathcal{U}_2$ implies $F(a^{2q_0+1}) = 0$. Then it is straightforward to check that $x(x+a)$ and $F(y)+a^{-1}xF(a^{2q_0}x)$ are fixed by $\mathcal{U}_2$ and $\psi_{a,0}$. On the other hand, $\bar{\mathbf{F}}_q(x(x+a), F(y) + a^{-1}xF(a^{2q_0}x))$ cannot be a proper subfield of the fixed field of $\mathcal{U}$. In fact, by the proof of Lemma 3.2 the degree of the extension $\bar{\mathbf{F}}_q(\mathcal{X}) \mid \bar{\mathbf{F}}_q(x, F(y))$ is equal to $\#\mathcal{U}_2$. As the degree of $\bar{\mathbf{F}}_q(x, F(y)) \mid \bar{\mathbf{F}}_q(x(x+a), F(y) + a^{-1}xF(a^{2q_0}x))$ is at most 2, the claim follows. □

Lemma 3.4. *Let $\mathcal{U}$ have order $4\#\mathcal{U}_2$, and assume that $\{\psi_{a,\bar{a}}, \psi_{d,\bar{d}}\} \subset \mathcal{U}$, $a, \bar{a}, d, \bar{d} \in \mathbf{F}_q$, $ad \neq 0$, $a \neq d$. Then the fixed field of $\mathcal{U}$ is generated by $x(x+a)(x+d)(x+a+d)$ and $ad(a+d)F(y) + G(x)$, where $F(T) = \prod_{\psi_{0,c} \in \mathcal{U}_2}(T+c)$, and*

$$\begin{aligned} G(T) = \quad & (T+a)^3F(a^{2q_0}T + \bar{a}) + (T+d)^3F(d^{2q_0}T + \bar{d}) + \\ & (T+a+d)^3F((a+d)^{2q_0}T + d^{2q_0}a + \bar{a} + \bar{d}). \end{aligned}$$

Proof. The group $\mathcal{U}$ is generated by $\mathcal{U}_2$, $\psi_{a,\bar{a}}$ and $\psi_{d,\bar{d}}$. As the set $\{c \in \mathbf{F}_q \mid \psi_{0,c} \in \mathcal{U}_2\}$ is a linear subspace of $\mathbf{F}_q \mid \mathbf{F}_2$, the polynomial $F(T)$ is such that $F(T_1 + T_2) = F(T_1) + F(T_2)$. Also, $\psi_{a,\bar{a}}^2 = \psi_{0,a^{2q_0+1}}$, $\psi_{d,\bar{d}}^2 = \psi_{0,d^{2q_0+1}}$ and $(\psi_{a,\bar{a}}\psi_{d,\bar{d}})^2 = \psi_{0,a^{2q_0+1}+a^{2q_0}d+ad^{2q_0}+d^{2q_0+1}}$ imply $F(a^{2q_0+1}) = F(d^{2q_0+1}) = F(a^{2q_0}d + ad^{2q_0}) = 0$. Notice that $ad(a+d)F(y) + G(x)$ can be written as

$$x^3\Big(\prod_{\psi_{0,c} \in \mathcal{U}_2}(y+c)\Big) + (x+a)^3\Big(\prod_{\psi_{0,c} \in \mathcal{U}_2}(a^{2q_0}x + y + c + \bar{a})\Big) +$$

$$(x+d)^3\Big(\prod_{\psi_{0,c} \in \mathcal{U}_2}(d^{2q_0}x + y + c + \bar{d})\Big) +$$

$$(x+a+d)^3\Big(\prod_{\psi_{0,c} \in \mathcal{U}_2}((a+d)^{2q_0}x + y + c + d^{2q_0}a + \bar{a} + \bar{d})\Big).$$

Then it is straightforward to check that $\xi := x(x+a)(x+d)(x+a+d)$ and $\eta := ad(a+d)F(y) + G(x)$ are fixed by $\mathcal{U}_2$, $\psi_{a,\bar{a}}$ and $\psi_{d,\bar{d}}$. On the other hand, $\bar{\mathbf{F}}_q(\xi, \eta)$ cannot be a proper subfield of the fixed field of $\mathcal{U}$. In fact, by the proof of Lemma 3.2 the degree of the extension $\bar{\mathbf{F}}_q(\mathcal{X}) \mid \bar{\mathbf{F}}_q(x, F(y))$

is equal to $\#\mathcal{U}_2$. By $\bar{\mathbf{F}}_q(x, F(y)) = \bar{\mathbf{F}}_q(\xi, \eta, x)$ it follows that the degree of $\bar{\mathbf{F}}_q(x, F(y)) \mid \bar{\mathbf{F}}_q(\xi, \eta)$ is at most 4, whence the claim. □

3.1. **The case** $q = 8$. Let w be a primitive element of $\mathbf{F}_8$ satisfying $w^3 = w + 1$. As a result of a computer search, a set of representatives of the conjugacy classes of 2-subgroups of $Sz(8)$ is the following:

- $\mathcal{V}_1 :=< \{\psi_{0,1}\} >$;
- $\mathcal{V}_2 :=< \{\psi_{1,0}\} >$;
- $\mathcal{V}_3 :=< \{\psi_{0,1}, \psi_{0,w}\} >$;
- $\mathcal{V}_4 := \bar{\mathbf{T}}_0$;
- $\mathcal{V}_5 :=< \{\psi_{1,0}, \psi_{0,w}\} >$;
- $\mathcal{V}_6 :=< \{\psi_{1,0}, \psi_{0,w^2}\} >$;
- $\mathcal{V}_7 :=< \{\psi_{1,0}, \psi_{0,w^4}\} >$;
- $\mathcal{V}_8 :=< \{\psi_{1,0}, \psi_{0,c} \mid c \in \mathbf{F}_8\} >$;
- $\mathcal{V}_9 :=< \{\psi_{1,0}, \psi_{w,0}, \psi_{0,c} \mid c \in \mathbf{F}_8\} >$;
- $\mathcal{V}_{10} := \bar{\mathbf{T}}$.

By Proposition 3.1, for $\mathcal{U} \in \{\mathcal{V}_4, \mathcal{V}_8, \mathcal{V}_9, \mathcal{V}_{10}\}$ the curve $\mathcal{X}_\mathcal{U}$ is rational. For $\mathcal{U} \in \{\mathcal{V}_1, \mathcal{V}_2\}$ Theorems 7.8 and 7.9 in [3] provide an equation for a plane model of $\mathcal{X}_\mathcal{U}$. Therefore only the equation of $\mathcal{X}_{\mathcal{V}_i}$ for $i \in \{3, 5, 6, 7\}$ has to be computed.

Theorem 3.5. *The curve $\mathcal{X}_{\mathcal{V}_3}$ has genus 2 and it is $\mathbf{F}_q$-birationally isomorphic to the plane curve of equation*

$$X^{12} + X^5 = Y^2 + (\omega + 1)Y.$$

Proof. The curve $\mathcal{X}_{\mathcal{V}_3}$ has genus 2 by Proposition 3.1. By Lemma 3.2 we have to prove that $x^{12} + x^5 + F(y)^2 + (\omega + 1)F(y) = 0$ in $\bar{\mathbf{F}}_q(\mathcal{X})$, where $F(y) = y(y+1)(y+\omega)(y+\omega+1) = y^4 + (\omega^2+\omega+1)y^2 + (\omega^2+\omega)y$. This follows from $x^{12} + x^5 = y^8 + y$, $\omega^2 + \omega + 1 = \omega^5$ and $\omega^2 + \omega = \omega^4$. □

Theorem 3.6. *The curve $\mathcal{X}_{\mathcal{V}_5}$ has genus 1 and it is $\mathbf{F}_q$-birationally isomorphic to the plane curve of equation*

$$X^6 + X^4 + \omega X^3 + X^2 = Y^2 + (\omega + 1)Y.$$

Proof. The curve $\mathcal{X}_{\mathcal{V}_3}$ has genus 1 by Proposition 3.1. By Lemma 3.3 we have to prove that

$$(x^2 + x)^6 + (x^2 + x)^4 + \omega(x^2 + x)^3 + (x^2 + x)^2 =$$

$$(F(y) + xF(x))^2 + (\omega + 1)(F(y) + xF(x))$$

in $\bar{\mathbf{F}}_q(\mathcal{X})$. By straightforward computation,

$$(x^2 + x)^6 + (x^2 + x)^4 + \omega(x^2 + x)^3 + (x^2 + x)^2 =$$

$$x^{12} + x^{10} + (1 + \omega)x^6 + \omega x^5 + \omega x^4 + \omega x^3 + x^2$$

and

$$(F(y)+xF(x))^2+(\omega+1)(F(y)+xF(x))=$$
$$y^8+y+x^{10}+(1+\omega)x^6+(\omega+1)x^5+\omega x^4+\omega x^3+x^2.$$

Then the claim follows from $y^8+y=x^{12}+x^5$. □

Theorem 3.7. *The curve $\mathcal{X}_{\mathcal{V}_6}$ has genus* 1 *and it is $\mathbf{F}_q$-birationally isomorphic to the plane curve of equation*

$$X^6+X^4+\omega^2X^3+X^2=Y^2+\omega^6Y.$$

Proof. The proof is similar to that of Theorem 3.6. □

Theorem 3.8. *The curve $\mathcal{X}_{\mathcal{V}_7}$ has genus* 1 *and it is $\mathbf{F}_q$-birationally isomorphic to the plane curve of equation*

$$X^6+X^4+\omega X^4+X^2=Y^2+\omega^5Y.$$

Proof. The proof is similar to that of Theorem 3.6. □

3.2. **The case $q=32$.** Let w be a primitive element of $\mathbf{F}_{32}$ satisfying $w^5=w^2+1$. As a result of a computer search, a set of representatives of the conjugacy classes of 2-subgroups of $\mathcal{S}z(32)$ is the following:

- $\mathcal{V}_1 :=< \{\psi_{0,1}\} >$;
- $\mathcal{V}_2 :=< \{\psi_{0,w^{28}},\psi_{0,w^{23}}\} >$;
- $\mathcal{V}_3 :=< \{\psi_{0,w^{28}},\psi_{0,w^{7}}\} >$;
- $\mathcal{V}_4 :=< \{\psi_{0,w^{28}},\psi_{0,w^{5}}\} >$;
- $\mathcal{V}_5 :=< \{\psi_{0,w^{28}},\psi_{0,w^{10}}\} >$;
- $\mathcal{V}_6 :=< \{\psi_{0,w^{28}},\psi_{0,w^{6}}\} >$;
- $\mathcal{V}_7 :=< \{\psi_{1,0}\} >$;
- $\mathcal{V}_8 :=< \{\psi_{0,w^{28}},\psi_{0,w^{7}},\psi_{0,w^{10}}\} >$;
- $\mathcal{V}_9 :=< \{\psi_{0,w^{28}},\psi_{0,w^{7}},\psi_{0,w^{16}}\} >$;
- $\mathcal{V}_{10} :=< \{\psi_{0,w^{28}},\psi_{0,w^{7}},\psi_{0,w^{4}}\} >$;
- $\mathcal{V}_{11} :=< \{\psi_{0,w^{28}},\psi_{0,w^{7}},\psi_{0,w^{9}}\} >$;
- $\mathcal{V}_{12} :=< \{\psi_{0,w^{28}},\psi_{0,w^{7}},\psi_{0,w^{27}}\} >$;
- $\mathcal{V}_{13} :=< \{\psi_{w,0},\psi_{0,w^{26}},\psi_{0,w^{9}}\} >$;
- $\mathcal{V}_{14} :=< \{\psi_{w,0},\psi_{0,1},\psi_{0,w^{16}}\} >$;
- $\mathcal{V}_{15} :=< \{\psi_{w,0},\psi_{0,w^{28}},\psi_{0,w^{9}}\} >$;
- $\mathcal{V}_{16} :=< \{\psi_{w,0},\psi_{0,w^{1}},\psi_{0,w^{9}}\} >$;
- $\mathcal{V}_{17} :=< \{\psi_{w,0},\psi_{0,w^{7}},\psi_{0,w^{9}}\} >$;
- $\mathcal{V}_{18} :=< \{\psi_{w,0},\psi_{0,w^{15}},\psi_{0,w^{9}}\} >$;
- $\mathcal{V}_{19} :=< \{\psi_{w,0},\psi_{0,w^{29}},\psi_{0,w^{9}}\} >$;
- $\mathcal{V}_{20} :=< \{\psi_{w,0},\psi_{0,w^{14}},\psi_{0,w^{9}}\} >$;
- $\mathcal{V}_{21} :=< \{\psi_{w,0},\psi_{0,w^{10}},\psi_{0,w^{27}}\} >$;
- $\mathcal{V}_{22} :=< \{\psi_{w,0},\psi_{0,w^{24}},\psi_{0,w^{9}}\} >$;
- $\mathcal{V}_{23} :=< \{\psi_{w,0},\psi_{0,w^{3}},\psi_{0,w^{9}}\} >$;
- $\mathcal{V}_{24} :=< \{\psi_{w,0},\psi_{0,w^{6}},\psi_{0,w^{4}}\} >$;

- $\mathcal{V}_{25} :=< \{\psi_{w,0}, \psi_{0,w^{13}}, \psi_{0,w^{9}}\} >;$
- $\mathcal{V}_{26} :=< \{\psi_{w,0}, \psi_{0,w^{22}}, \psi_{0,w^{9}}\} >;$
- $\mathcal{V}_{27} :=< \{\psi_{w,0}, \psi_{0,w^{18}}, \psi_{0,w^{9}}\} >;$
- $\mathcal{V}_{28} :=< \{\psi_{0,w^{6}}, \psi_{0,w^{7}}, \psi_{0,w^{10}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{29} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w^{7}}, \psi_{0,w^{16}}\} >;$
- $\mathcal{V}_{30} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w^{28}}, \psi_{0,w^{16}}\} >;$
- $\mathcal{V}_{31} :=< \{\psi_{w,0}, \psi_{0,w^{7}}, \psi_{0,w^{9}}, \psi_{0,w^{14}}\} >;$
- $\mathcal{V}_{32} :=< \{\psi_{w,0}, \psi_{0,w^{7}}, \psi_{0,w^{10}}, \psi_{0,w^{27}}\} >;$
- $\mathcal{V}_{33} :=< \{\psi_{w,0}, \psi_{0,w}, \psi_{0,w^{10}}, \psi_{0,w^{27}}\} >;$
- $\mathcal{V}_{34} :=< \{\psi_{w,0}, \psi_{0,w^{9}}, \psi_{0,w^{14}}, \psi_{0,w^{15}}\} >;$
- $\mathcal{V}_{35} :=< \{\psi_{w,0}, \psi_{0,w^{9}}, \psi_{0,w^{13}}, \psi_{0,w^{24}}\} >;$
- $\mathcal{V}_{36} :=< \{\psi_{w,0}, \psi_{0,w^{4}}, \psi_{0,w^{6}}, \psi_{0,w^{7}}\} >;$
- $\mathcal{V}_{37} :=< \{\psi_{w,0}, \psi_{0,w}, \psi_{0,w^{4}}, \psi_{0,w^{6}}\} >;$
- $\mathcal{V}_{38} :=< \{\psi_{w,0}, \psi_{0,w^{7}}, \psi_{0,w^{9}}, \psi_{0,w^{13}}\} >;$
- $\mathcal{V}_{39} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w}, \psi_{0,w^{16}}\} >;$
- $\mathcal{V}_{40} :=< \{\psi_{w,0}, \psi_{0,w^{9}}, \psi_{0,w^{22}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{41} :=< \{\psi_{w,0}, \psi_{0,w^{9}}, \psi_{0,w^{22}}, \psi_{0,w^{26}}\} >;$
- $\mathcal{V}_{42} :=< \{\psi_{w,0}, \psi_{0,w^{7}}, \psi_{0,w^{9}}, \psi_{0,w^{26}}\} >;$
- $\mathcal{V}_{43} :=< \{\psi_{w,0}, \psi_{0,w^{9}}, \psi_{0,w^{24}}, \psi_{0,w^{26}}\} >;$
- $\mathcal{V}_{44} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w^{3}}, \psi_{0,w^{16}}\} >;$
- $\mathcal{V}_{45} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w^{24}}, \psi_{0,w^{16}}\} >;$
- $\mathcal{V}_{46} :=< \{\psi_{w,0}, \psi_{0,w^{9}}, \psi_{0,w^{15}}, \psi_{0,w^{26}}\} >;$
- $\mathcal{V}_{47} :=< \{\psi_{w,0}, \psi_{0,w^{9}}, \psi_{0,w^{15}}, \psi_{0,w^{13}}\} >;$
- $\mathcal{V}_{48} :=< \{\psi_{w,0}, \psi_{0,w^{4}}, \psi_{0,w^{6}}, \psi_{0,w^{18}}\} >;$
- $\mathcal{V}_{49} :=< \{\psi_{w,0}, \psi_{0,w^{4}}, \psi_{0,w^{6}}, \psi_{0,w^{22}}\} >;$
- $\mathcal{V}_{50} :=< \{\psi_{w,0}, \psi_{0,w^{9}}, \psi_{0,w^{13}}, \psi_{0,w^{22}}\} >;$
- $\mathcal{V}_{51} :=< \{\psi_{w,0}, \psi_{0,w^{9}}, \psi_{0,w^{7}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{52} :=< \{\psi_{w,0}, \psi_{0,w^{4}}, \psi_{0,w^{6}}, \psi_{0,w^{26}}\} >;$
- $\mathcal{V}_{53} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w^{13}}, \psi_{0,w^{16}}\} >;$
- $\mathcal{V}_{54} :=< \{\psi_{w,0}, \psi_{0,w^{10}}, \psi_{0,w^{26}}, \psi_{0,w^{27}}\} >;$
- $\mathcal{V}_{55} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w^{6}}, \psi_{0,w^{10}}\} >;$
- $\mathcal{V}_{56} :=< \{\psi_{w,0}, \psi_{0,w^{10}}, \psi_{0,w^{18}}, \psi_{0,w^{27}}\} >;$
- $\mathcal{V}_{57} :=< \{\psi_{w,0}, \psi_{0,w^{9}}, \psi_{0,w^{14}}, \psi_{0,w^{22}}\} >;$
- $\mathcal{V}_{58} :=< \{\psi_{w,0}, \psi_{0,w^{10}}, \psi_{0,w^{22}}, \psi_{0,w^{27}}\} >;$
- $\mathcal{V}_{59} :=< \{\psi_{w,0}, \psi_{0,w^{9}}, \psi_{0,w^{14}}, \psi_{0,w^{24}}\} >;$
- $\mathcal{V}_{60} :=< \{\psi_{w,0}, \psi_{0,w^{10}}, \psi_{0,w^{27}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{61} :=< \{\psi_{w,0}, \psi_{0,w^{9}}, \psi_{0,w^{15}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{62} :=< \{\psi_{w,0}, \psi_{0,w^{4}}, \psi_{0,w^{6}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{63} :=< \{\psi_{w,0}, \psi_{0,w^{9}}, \psi_{0,w^{24}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{64} := \bar{\mathbf{T}}_0;$
- $\mathcal{V}_{65} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w^{13}}, \psi_{0,w^{16}}, \psi_{0,w^{24}}\} >;$
- $\mathcal{V}_{66} :=< \{\psi_{w,0}, \psi_{0,w^{7}}, \psi_{0,w^{10}}, \psi_{0,w^{26}}, \psi_{0,w^{27}}\} >;$
- $\mathcal{V}_{67} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w}, \psi_{0,w^{6}}, \psi_{0,w^{10}}\} >;$

- $\mathcal{V}_{68} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w^{7}}, \psi_{0,w^{13}}, \psi_{0,w^{16}}\} >;$
- $\mathcal{V}_{69} :=< \{\psi_{w,0}, \psi_{0,w^{4}}, \psi_{0,w^{6}}, \psi_{0,w^{7}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{70} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w^{6}}, \psi_{0,w^{10}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{71} :=< \{\psi_{w,0}, \psi_{0,w^{4}}, \psi_{0,w^{6}}, \psi_{0,w^{22}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{72} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w^{16}}, \psi_{0,w^{24}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{73} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w^{7}}, \psi_{0,w^{16}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{74} :=< \{\psi_{w,0}, \psi_{0,w^{10}}, \psi_{0,w^{22}}, \psi_{0,w^{27}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{75} :=< \{\psi_{w,0}, \psi_{0,w^{10}}, \psi_{0,w^{22}}, \psi_{0,w^{26}}, \psi_{0,w^{27}}\} >;$
- $\mathcal{V}_{76} :=< \{\psi_{w,0}, \psi_{0,w^{4}}, \psi_{0,w^{6}}, \psi_{0,w^{7}}, \psi_{0,w^{26}}\} >;$
- $\mathcal{V}_{77} :=< \{\psi_{w,0}, \psi_{0,1}, \psi_{0,w^{6}}, \psi_{0,w^{7}}, \psi_{0,w^{10}}\} >;$
- $\mathcal{V}_{78} :=< \{\psi_{w,0}, \psi_{0,w^{4}}, \psi_{0,w^{6}}, \psi_{0,w^{22}}, \psi_{0,w^{26}}\} >;$
- $\mathcal{V}_{79} :=< \{\psi_{w,0}, \psi_{0,w^{7}}, \psi_{0,w^{10}}, \psi_{0,w^{27}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{80} :=< \{\psi_{w,w^{13}}, \psi_{w^{17},w^{10}}, \psi_{0,w^{9}}, \psi_{0,w^{15}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{81} :=< \{\psi_{w,w^{29}}, \psi_{w^{17},w^{10}}, \psi_{0,w^{9}}, \psi_{0,w^{15}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{82} :=< \{\psi_{w,w^{29}}, \psi_{w^{24},w^{8}}, \psi_{0,w^{9}}, \psi_{0,w^{24}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{83} :=< \{\psi_{w,w^{3}}, \psi_{w^{24},w^{8}}, \psi_{0,w^{9}}, \psi_{0,w^{24}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{84} :=< \{\psi_{w,w^{24}}, \psi_{w^{2},w^{22}}, \psi_{0,w^{7}}, \psi_{0,w^{9}}, \psi_{0,w^{14}}\} >;$
- $\mathcal{V}_{85} :=< \{\psi_{w,w^{29}}, \psi_{w^{19},w^{13}}, \psi_{0,1}, \psi_{0,w}, \psi_{0,w^{16}}\} >;$
- $\mathcal{V}_{86} :=< \{\psi_{w,w^{18}}, \psi_{w^{19},w^{13}}, \psi_{0,1}, \psi_{0,w}, \psi_{0,w^{16}}\} >;$
- $\mathcal{V}_{87} :=< \{\psi_{w,w^{29}}, \psi_{w^{27},w^{19}}, \psi_{0,w^{10}}, \psi_{0,w^{26}}, \psi_{0,w^{27}}\} >;$
- $\mathcal{V}_{88} :=< \{\psi_{w,w^{14}}, \psi_{w^{27},w^{19}}, \psi_{0,w^{10}}, \psi_{0,w^{26}}, \psi_{0,w^{27}}\} >;$
- $\mathcal{V}_{89} :=< \{\psi_{w,w^{29}}, \psi_{w^{22},w^{2}}, \psi_{0,w^{7}}, \psi_{0,w^{9}}, \psi_{0,w^{14}}\} >;$
- $\mathcal{V}_{90} :=< \{\psi_{w,0}, \psi_{0,c} \mid c \in \mathbf{F}_{32}\} >;$
- $\mathcal{V}_{91} :=< \{\psi_{w,w^{29}}, \psi_{w^{24},w^{8}}, \psi_{0,w^{10}}, \psi_{0,w^{22}}, \psi_{0,w^{27}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{92} :=< \{\psi_{w,w^{29}}, \psi_{w^{17},w^{10}}, \psi_{0,1}, \psi_{0,w^{16}}, \psi_{0,w^{24}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{93} :=< \{\psi_{w,w^{29}}, \psi_{w^{17},w^{10}}, \psi_{0,w^{4}}, \psi_{0,w^{6}}, \psi_{0,w^{22}}, \psi_{0,w^{29}}\} >;$
- $\mathcal{V}_{94} :=< \{\psi_{w,w^{29}}, \psi_{w^{17},w^{10}}, \psi_{0,w^{7}}, \psi_{0,w^{10}}, \psi_{0,w^{27}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{95} :=< \{\psi_{w,w^{29}}, \psi_{w^{24},w^{8}}, \psi_{0,w^{4}}, \psi_{0,w^{6}}, \psi_{0,w^{7}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{96} :=< \{\psi_{w,w^{29}}, \psi_{w^{19},w^{13}}, \psi_{0,1}, \psi_{0,w^{7}}, \psi_{0,w^{16}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{97} :=< \{\psi_{w,w^{29}}, \psi_{w^{19},w^{13}}, \psi_{0,1}, \psi_{0,w^{13}}, \psi_{0,w^{16}}, \psi_{0,w^{24}}\} >;$
- $\mathcal{V}_{98} :=< \{\psi_{w,w^{29}}, \psi_{w^{19},w^{13}}, \psi_{0,1}, \psi_{0,w}, \psi_{0,w^{6}}, \psi_{0,w^{10}}\} >;$
- $\mathcal{V}_{99} :=< \{\psi_{w,w^{29}}, \psi_{w^{27},w^{19}}, \psi_{0,1}, \psi_{0,w^{6}}, \psi_{0,w^{10}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{100} :=< \{\psi_{w,w^{29}}, \psi_{w^{27},w^{19}}, \psi_{0,w^{7}}, \psi_{0,w^{10}}, \psi_{0,w^{26}}, \psi_{0,w^{27}}\} >;$
- $\mathcal{V}_{101} :=< \{\psi_{w,w^{29}}, \psi_{w^{27},w^{19}}, \psi_{0,w^{10}}, \psi_{0,w^{22}}, \psi_{0,w^{26}}, \psi_{0,w^{27}}\} >;$
- $\mathcal{V}_{102} :=< \{\psi_{w,w^{29}}, \psi_{w^{22},w^{2}}, \psi_{0,w^{7}}, \psi_{0,w^{10}}, \psi_{0,w^{27}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{103} :=< \{\psi_{w,w^{29}}, \psi_{w^{22},w^{2}}, \psi_{0,1}, \psi_{0,w^{7}}, \psi_{0,w^{13}}, \psi_{0,w^{16}}\} >;$
- $\mathcal{V}_{104} :=< \{\psi_{w,w^{29}}, \psi_{w^{22},w^{2}}, \psi_{0,w^{4}}, \psi_{0,w^{6}}, \psi_{0,w^{7}}, \psi_{0,w^{26}}\} >;$
- $\mathcal{V}_{105} :=< \{\psi_{w,w^{29}}, \psi_{w^{24},w^{8}}, \psi_{0,1}, \psi_{0,w^{16}}, \psi_{0,w^{24}}, \psi_{0,w^{28}}\} >;$
- $\mathcal{V}_{106} :=< \{\psi_{w,w^{29}}, \psi_{w^{24},w^{8}}, \psi_{0,c} \mid c \in \mathbf{F}_{32}\} >;$
- $\mathcal{V}_{107} :=< \{\psi_{w,w^{29}}, \psi_{w^{22},w^{2}}, \psi_{0,c} \mid c \in \mathbf{F}_{32}\} >;$
- $\mathcal{V}_{108} :=< \{\psi_{w,w^{29}}, \psi_{w^{27},w^{19}}, \psi_{0,c} \mid c \in \mathbf{F}_{32}\} >;$
- $\mathcal{V}_{109} :=< \{\psi_{w,w^{29}}, \psi_{w^{19},w^{13}}, \psi_{0,c} \mid c \in \mathbf{F}_{32}\} >;$
- $\mathcal{V}_{110} :=< \{\psi_{w,w^{29}}, \psi_{w^{17},w^{10}}, \psi_{0,c} \mid c \in \mathbf{F}_{32}\} >;$

- $\mathcal{V}_{111} :=< \{\psi_{w,w^{29}}, \psi_{w^{24},w^8}, \psi_{w^{21},w^{16}}, \psi_{0,c} \mid c \in \mathbf{F}_{32}\} >$;
- $\mathcal{V}_{112} :=< \{\psi_{w,w^{29}}, \psi_{w^{24},w^8}, \psi_{w^{18},w^2}, \psi_{0,c} \mid c \in \mathbf{F}_{32}\} >$;
- $\mathcal{V}_{113} :=< \{\psi_{w,w^{29}}, \psi_{w^{24},w^8}, \psi_{w^{27},w^{19}}, \psi_{0,c} \mid c \in \mathbf{F}_{32}\} >$;
- $\mathcal{V}_{114} :=< \{\psi_{w,w^{29}}, \psi_{w^{24},w^8}, \psi_{w^3,w^{21}}, \psi_{0,c} \mid c \in \mathbf{F}_{32}\} >$;
- $\mathcal{V}_{115} :=< \{\psi_{w,w^{29}}, \psi_{w^{24},w^8}, \psi_{w^{17},w^{10}}, \psi_{0,c} \mid c \in \mathbf{F}_{32}\} >$;
- $\mathcal{V}_{116} :=< \{\psi_{w,w^{29}}, \psi_{w^{24},w^8}, \psi_{w^{21},w^{16}}, \psi_{w^{27},w^{19}}, \psi_{0,c} \mid c \in \mathbf{F}_{32}\} >$;
- $\mathcal{V}_{117} := \bar{\mathbf{T}}$.

By Proposition 3.1, the genus $g_{\mathcal{V}_i}$ of the curve $\mathcal{X}_{\mathcal{V}_i}$ is equal to

$$g_{\mathcal{V}_i} = \begin{cases} 60 & \text{for } i = 1, \\ 28 & \text{for } 2 \le i \le 6, \\ 30 & \text{for } i = 7, \\ 12 & \text{for } 8 \le i \le 12, \\ 14 & \text{for } 13 \le i \le 27, \\ 4 & \text{for } i = 28, \\ 6 & \text{for } 29 \le i \le 63, \\ 2 & \text{for } 65 \le i \le 79, \\ 3 & \text{for } 80 \le i \le 89, \\ 1 & \text{for } 91 \le i \le 105, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that for $\mathcal{U} = \mathcal{V}_i$, $i = 64, 90$ and $106 \le i \le 117$ the curve $\mathcal{X}_{\mathcal{U}}$ is rational. For $\mathcal{U} \in \{\mathcal{V}_1, \mathcal{V}_7\}$ Theorems 7.8 and 7.9 in [3] provide an equation for a plane model of $\mathcal{X}_{\mathcal{U}}$. An equation for a plane model of $\mathcal{X}_{\mathcal{U}}$ for the remaining cases is given in the following theorem.

Theorem 3.9. *The curve $\mathcal{X}_{\mathcal{V}_i}$ is $\mathbf{F}_q$-birationally isomorphic to the plane curve of equation*

- $X^{40} + X^9 + Y^8 + w^{17}Y^4 + w^{26}Y^2 + w^{17}Y = 0$, *for* $i = 2$;
- $X^{40} + X^9 + Y^8 + w^{25}Y^4 + w^{23}Y^2 + w^{26}Y = 0$, *for* $i = 3$;
- $X^{40} + X^9 + Y^8 + w^5Y^4 + w^{25}Y^2 + w^{12}Y = 0$, *for* $i = 4$;
- $X^{40} + X^9 + Y^8 + Y^4 + w^8Y^2 + w^{13}Y = 0$, *for* $i = 5$;
- $X^{40} + X^9 + Y^8 + w^{27}Y^4 + w^{29}Y^2 + w^{15}Y = 0$, *for* $i = 6$;
- $X^{40} + X^9 + Y^4 + w^6Y^2 + w^{14}Y = 0$, *for* $i = 8$;
- $X^{40} + X^9 + Y^4 + w^{18}Y^2 + w^{16}Y = 0$, *for* $i = 9$;
- $X^{40} + X^9 + Y^4 + w^{10}Y^2 + w^2Y = 0$, *for* $i = 10$;
- $X^{40} + X^9 + Y^4 + w^{21}Y^2 + w^{26}Y = 0$, *for* $i = 11$;
- $X^{40} + X^9 + Y^4 + w^{27}Y^2 + w^3Y = 0$, *for* $i = 12$;
- $X^{12} + w^{29}X^8 + w^{14}X^6 + w^4X^5 + w^6X^4 + wX^3 + w^{10}X^2 + w^5Y^8 + w^4Y^4 + w^{27}Y^2 + w^{24}Y = 0$, *for* $i = 13$;
- $X^{12} + w^{15}X^8 + w^4X^6 + w^{21}X^5 + w^{23}X^4 + w^{30}X^3 + w^{27}X^2 + w^{22}Y^8 + w^4Y^4 + w^{30}Y^2 + w^{28}Y = 0$, *for* $i = 14$;
- $X^{12} + w^{27}X^8 + w^8X^6 + w^2X^5 + w^4X^4 + w^{24}X^3 + w^8X^2 + w^3Y^8 + w^4Y^4 + w^{22}Y^2 + w^8Y = 0$, *for* $i = 15$;

- $X^{12}+w^{21}X^8+w^{28}X^6+w^{27}X^5+w^{29}X^4+w^6X^3+w^2X^2+w^{28}Y^8+w^4Y^4+w^{25}Y^2+w^{28}Y=0$, *for* $i=16$;
- $X^{12}+w^{19}X^8+w^{16}X^6+w^{25}X^5+w^{27}X^4+w^{15}X^3+X^2+w^{26}Y^8+w^4Y^4+w^9Y^2+w^{29}Y=0$, *for* $i=17$;
- $X^{12}+w^{30}X^8+w^{20}X^6+w^5X^5+w^7X^4+w^{12}X^3+w^{11}X^2+w^6Y^8+w^4Y^4+w^4Y^2+w^8Y=0$, *for* $i=18$;
- $X^{12}+w^{17}X^8+X^6+w^{23}X^5+w^{25}X^4+w^{25}X^3+w^{29}X^2+w^{24}Y^8+w^4Y^4+w^{27}Y^2+Y=0$, *for* $i=19$;
- $X^{12}+w^{18}X^8+w^9X^6+w^{24}X^5+w^{26}X^4+w^{12}X^3+w^{30}X^2+w^{25}Y^8+w^4Y^4+w^{25}Y^2+w^{22}Y=0$, *for* $i=20$;
- $X^{12}+w^{13}X^8+w^{26}X^6+w^{19}X^5+w^{21}X^4+w^{15}X^3+w^{25}X^2+w^{20}Y^8+w^4Y^4+w^4Y^2+w^5Y=0$, *for* $i=21$;
- $X^{12}+w^3X^8+w^{13}X^6+w^9X^5+w^{11}X^4+w^{25}X^3+w^{15}X^2+w^{10}Y^8+w^4Y^4+w^5Y^2+w^6Y=0$, *for* $i=22$;
- $X^{12}+w^2X^8+w^{10}X^6+w^8X^5+w^{10}X^4+w^{21}X^3+w^{14}X^2+w^9Y^8+w^4Y^4+w^{18}Y^2+w^{29}Y=0$, *for* $i=23$;
- $X^{12}+w^5X^8+w^{11}X^6+w^{11}X^5+w^{13}X^4+w^4X^3+w^{17}X^2+w^{12}Y^8+w^4Y^4+w^{16}Y^2+w^{24}Y=0$, *for* $i=24$;
- $X^{12}+w^{28}X^8+w^6X^6+w^3X^5+w^5X^4+w^6X^3+w^9X^2+w^4Y^8+w^4Y^4+w^5Y^2+w^{25}Y=0$, *for* $i=25$;
- $X^{12}+w^{22}X^8+w^{27}X^6+w^{28}X^5+w^{30}X^4+w^{11}X^3+w^3X^2+w^{29}Y^8+w^4Y^4+w^{15}Y^2+w^6Y=0$, *for* $i=26$;
- $X^{12}+w^{26}X^8+w^{19}X^6+wX^5+w^3X^4+wX^3+w^7X^2+w^2Y^8+w^4Y^4+w^9Y^2+w^{12}Y=0$, *for* $i=27$;
- $X^{40}+X^9+Y^2+w^{20}Y=0$, *for* $i=28$;
- $X^{20}+w^{16}X^{12}+w^{10}X^{10}+w^{24}X^8+w^8X^6+wX^5+wX^4+w^{14}X^3+w^5X^2+Y^4+w^{10}Y^2+w^4Y=0$, *for* $i=29$;
- $X^{20}+w^{16}X^{12}+w^{15}X^{10}+w^{24}X^8+w^{18}X^6+X^5+wX^4+w^{24}X^3+w^5X^2+Y^4+w^{15}Y^2+w^{17}Y=0$, *for* $i=30$;
- $X^{20}+w^{16}X^{12}+w^{15}X^{10}+w^{24}X^8+w^{17}X^6+w^{27}X^5+wX^4+w^8X^3+w^5X^2+Y^4+w^{15}Y^2+w^{25}Y=0$, *for* $i=31$;
- $X^{20}+w^{16}X^{12}+w^{14}X^{10}+w^{24}X^8+w^{23}X^6+w^{15}X^5+wX^4+w^{10}X^3+w^5X^2+Y^4+w^{14}Y^2+w^8Y=0$, *for* $i=32$;
- $X^{20}+w^{16}X^{12}+X^{10}+w^{24}X^8+w^{15}X^6+w^{18}X^5+wX^4+w^{23}X^3+w^5X^2+Y^4+Y^2+w^{10}Y=0$, *for* $i=33$;
- $X^{20}+w^{16}X^{12}+w^8X^{10}+w^{24}X^8+w^{18}X^6+X^5+wX^4+w^{17}X^3+w^5X^2+Y^4+w^8Y^2+w^{17}Y=0$, *for* $i=34$;
- $X^{20}+w^{16}X^{12}+w^{27}X^{10}+w^{24}X^8+w^8X^6+wX^5+wX^4+X^3+w^5X^2+Y^4+w^{27}Y^2+w^4Y=0$, *for* $i=35$;
- $X^{20}+w^{16}X^{12}+w^{28}X^{10}+w^{24}X^8+wX^6+w^{16}X^5+wX^4+w^{13}X^3+w^5X^2+Y^4+w^{28}Y^2+w^{29}Y=0$, *for* $i=36$;
- $X^{20}+w^{16}X^{12}+w^{11}X^{10}+w^{24}X^8+w^{10}X^6+w^7X^5+wX^4+w^{16}X^3+w^5X^2+Y^4+w^{11}Y^2+w^{19}Y=0$, *for* $i=37$;

- $X^{20}+w^{16}X^{12}+w^{7}X^{10}+w^{24}X^{8}+w^{24}X^{6}+w^{17}X^{5}+wX^{4}+w^{19}X^{3}+w^{5}X^{2}+Y^{4}+w^{7}Y^{2}+Y=0$, *for* $i=38$;
- $X^{20}+w^{16}X^{12}+w^{30}X^{10}+w^{24}X^{8}+w^{12}X^{6}+w^{9}X^{5}+wX^{4}+w^{5}X^{3}+w^{5}X^{2}+Y^{4}+w^{30}Y^{2}+w^{3}Y=0$, *for* $i=39$;
- $X^{20}+w^{16}X^{12}+w^{5}X^{10}+w^{24}X^{8}+w^{15}X^{6}+w^{18}X^{5}+wX^{4}+w^{28}X^{3}+w^{5}X^{2}+Y^{4}+w^{5}Y^{2}+w^{10}Y=0$, *for* $i=40$;
- $X^{20}+w^{16}X^{12}+w^{24}X^{10}+w^{24}X^{8}+w^{30}X^{6}+w^{23}X^{5}+wX^{4}+w^{8}X^{3}+w^{5}X^{2}+Y^{4}+w^{24}Y^{2}+w^{14}Y=0$, *for* $i=41$;
- $X^{20}+w^{16}X^{12}+w^{17}X^{10}+w^{24}X^{8}+w^{10}X^{6}+w^{7}X^{5}+wX^{4}+w^{22}X^{3}+w^{5}X^{2}+Y^{4}+w^{17}Y^{2}+w^{19}Y=0$, *for* $i=42$;
- $X^{20}+w^{16}X^{12}+w^{3}X^{10}+w^{24}X^{8}+w^{19}X^{6}+w^{3}X^{5}+wX^{4}+w^{28}X^{3}+w^{5}X^{2}+Y^{4}+w^{3}Y^{2}+w^{9}Y=0$, *for* $i=43$;
- $X^{20}+w^{16}X^{12}+w^{28}X^{10}+w^{24}X^{8}+w^{6}X^{6}+w^{24}X^{5}+wX^{4}+X^{3}+w^{5}X^{2}+Y^{4}+w^{28}Y^{2}+w^{20}Y=0$, *for* $i=44$;
- $X^{20}+w^{16}X^{12}+w^{22}X^{10}+w^{24}X^{8}+w^{29}X^{6}+w^{11}X^{5}+wX^{4}+w^{21}X^{3}+w^{5}X^{2}+Y^{4}+w^{22}Y^{2}+w^{22}Y=0$, *for* $i=45$;
- $X^{20}+w^{16}X^{12}+w^{5}X^{10}+w^{24}X^{8}+w^{5}X^{6}+w^{2}X^{5}+wX^{4}+w^{23}X^{3}+w^{5}X^{2}+Y^{4}+w^{5}Y^{2}+w^{28}Y=0$, *for* $i=46$;
- $X^{20}+w^{16}X^{12}+w^{10}X^{10}+w^{24}X^{8}+w^{6}X^{6}+w^{24}X^{5}+wX^{4}+w^{13}X^{3}+w^{5}X^{2}+Y^{4}+w^{10}Y^{2}+w^{20}Y=0$, *for* $i=47$;
- $X^{20}+w^{16}X^{12}+w^{4}X^{10}+w^{24}X^{8}+w^{22}X^{6}+w^{29}X^{5}+wX^{4}+w^{15}X^{3}+w^{5}X^{2}+Y^{4}+w^{4}Y^{2}+w^{16}Y=0$, *for* $i=48$;
- $X^{20}+w^{16}X^{12}+X^{10}+w^{24}X^{8}+w^{19}X^{6}+w^{3}X^{5}+wX^{4}+w^{25}X^{3}+w^{5}X^{2}+Y^{4}+Y^{2}+w^{9}Y=0$, *for* $i=49$;
- $X^{20}+w^{16}X^{12}+w^{12}X^{10}+w^{24}X^{8}+w^{25}X^{6}+w^{14}X^{5}+wX^{4}+w^{9}X^{3}+w^{5}X^{2}+Y^{4}+w^{12}Y^{2}+w^{23}Y=0$, *for* $i=50$;
- $X^{20}+w^{16}X^{12}+w^{21}X^{10}+w^{24}X^{8}+w^{13}X^{6}+w^{5}X^{5}+wX^{4}+w^{12}X^{3}+w^{5}X^{2}+Y^{4}+w^{21}Y^{2}+w^{26}Y=0$, *for* $i=51$;
- $X^{20}+w^{16}X^{12}+w^{29}X^{10}+w^{24}X^{8}+w^{26}X^{6}+w^{8}X^{5}+wX^{4}+w^{11}X^{3}+w^{5}X^{2}+Y^{4}+w^{29}Y^{2}+w^{15}Y=0$, *for* $i=52$;
- $X^{20}+w^{16}X^{12}+w^{16}X^{10}+w^{24}X^{8}+w^{11}X^{6}+w^{22}X^{5}+wX^{4}+w^{6}X^{3}+w^{5}X^{2}+Y^{4}+w^{16}Y^{2}+w^{11}Y=0$, *for* $i=53$;
- $X^{20}+w^{16}X^{12}+w^{2}X^{10}+w^{24}X^{8}+w^{7}X^{6}+w^{13}X^{5}+wX^{4}+w^{21}X^{3}+w^{5}X^{2}+Y^{4}+w^{2}Y^{2}+w^{12}Y=0$, *for* $i=54$;
- $X^{20}+w^{16}X^{12}+w^{21}X^{10}+w^{24}X^{8}+w^{2}X^{6}+w^{6}X^{5}+wX^{4}+w^{22}X^{3}+w^{5}X^{2}+Y^{4}+w^{21}Y^{2}+w^{21}Y=0$, *for* $i=55$;
- $X^{20}+w^{16}X^{12}+w^{19}X^{10}+w^{24}X^{8}+w^{16}X^{6}+w^{28}X^{5}+wX^{4}+w^{27}X^{3}+w^{5}X^{2}+Y^{4}+w^{19}Y^{2}+w^{2}Y=0$, *for* $i=56$;
- $X^{20}+w^{16}X^{12}+w^{22}X^{10}+w^{24}X^{8}+w^{7}X^{6}+w^{13}X^{5}+wX^{4}+w^{10}X^{3}+w^{5}X^{2}+Y^{4}+w^{22}Y^{2}+w^{12}Y=0$, *for* $i=57$;
- $X^{20}+w^{16}X^{12}+w^{9}X^{10}+w^{24}X^{8}+w^{27}X^{6}+w^{19}X^{5}+wX^{4}+w^{7}X^{3}+w^{5}X^{2}+Y^{4}+w^{9}Y^{2}+w^{7}Y=0$, *for* $i=58$;

- $X^{20}+w^{16}X^{12}+w^{11}X^{10}+w^{24}X^8+w^2X^6+w^6X^5+wX^4+w^{12}X^3+w^5X^2+Y^4+w^{11}Y^2+w^{21}Y=0$, *for* $i=59$;
- $X^{20}+w^{16}X^{12}+w^{18}X^{10}+w^{24}X^8+w^4X^6+w^{26}X^5+wX^4+w^{20}X^3+w^5X^2+Y^4+w^{18}Y^2+w^5Y=0$, *for* $i=60$;
- $X^{20}+w^{16}X^{12}+w^{25}X^{10}+w^{24}X^8+w^{20}X^6+w^4X^5+wX^4+w^4X^3+w^5X^2+Y^4+w^{25}Y^2+wY=0$, *for* $i=61$;
- $X^{20}+w^{16}X^{12}+w^{24}X^{10}+w^{24}X^8+w^{17}X^6+w^{27}X^5+wX^4+w^{17}X^3+w^5X^2+Y^4+w^{24}Y^2+w^{25}Y=0$, *for* $i=62$;
- $X^{20}+w^{16}X^{12}+w^2X^{10}+w^{24}X^8+w^{23}X^6+w^{15}X^5+wX^4+w^{29}X^3+w^5X^2+Y^4+w^2Y^2+w^8Y=0$, *for* $i=63$;
- $X^{20}+w^4X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^{26}X^9+w^{24}X^8+w^{15}X^5+wX^4+w^{26}X^3+w^5X^2+Y^2+w^{24}Y=0$, *for* $i=65$;
- $X^{20}+w^4X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^{23}X^9+w^{24}X^8+w^3X^5+wX^4+wX^3+w^5X^2+Y^2+w^{21}Y=0$, *for* $i=66$;
- $X^{20}+w^4X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^3X^9+w^{24}X^8+w^{19}X^5+wX^4+w^{10}X^3+w^5X^2+Y^2+wY=0$, *for* $i=67$;
- $X^{20}+w^4X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^{25}X^9+w^{24}X^8+w^4X^5+wX^4+w^{28}X^3+w^5X^2+Y^2+w^{23}Y=0$, *for* $i=68$;
- $X^{20}+w^4X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^{20}X^9+w^{24}X^8+w^{23}X^5+wX^4+w^7X^3+w^5X^2+Y^2+w^{18}Y=0$, *for* $i=69$;
- $X^{20}+w^4X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^8X^9+w^{24}X^8+w^{17}X^5+wX^4+X^3+w^5X^2+Y^2+w^6Y=0$, *for* $i=70$;
- $X^{20}+w^4X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^{28}X^9+w^{24}X^8+w^9X^5+wX^4+w^{22}X^3+w^5X^2+Y^2+w^{26}Y=0$, *for* $i=71$;
- $X^{20}+w^4X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^{18}X^9+w^{24}X^8+w^{16}X^5+wX^4+w^{11}X^3+w^5X^2+Y^2+w^{16}Y=0$, *for* $i=72$;
- $X^{20}+w^4X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^{14}X^9+w^{24}X^8+w^2X^5+wX^4+w^{19}X^3+w^5X^2+Y^2+w^{12}Y=0$, *for* $i=73$;
- $X^{20}+w^4X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^{15}X^9+w^{24}X^8+w^8X^5+wX^4+w^{17}X^3+w^5X^2+Y^2+w^{13}Y=0$, *for* $i=74$;
- $X^{20}+w^4X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^{30}X^9+w^{24}X^8+X^5+wX^4+w^{18}X^3+w^5X^2+Y^2+w^{28}Y=0$, *for* $i=75$;
- $X^{20}+w^4X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^7X^9+w^{24}X^8+wX^5+wX^4+w^2X^3+w^5X^2+Y^2+w^5Y=0$, *for* $i=76$;
- $X^{20}+w^4X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^6X^9+w^{24}X^8+w^5X^5+wX^4+w^4X^3+w^5X^2+Y^2+w^4Y=0$, *for* $i=77$;
- $X^{20}+w^4X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^{21}X^9+w^{24}X^8+w^{11}X^5+wX^4+w^5X^3+w^5X^2+Y^2+w^{19}Y=0$, *for* $i=78$;
- $X^{20}+w^4X^{18}+w^{16}X^{12}+w^{20}X^{10}+w^{24}X^9+w^{24}X^8+w^7X^5+wX^4+w^{30}X^3+w^5X^2+Y^2+w^{22}Y=0$, *for* $i=79$;
- $X^8+w^{13}X^4+w^4X^3+w^{16}X^2+w^{11}X+w^{13}Y^4+wY^2+w^5Y+w^{21}=0$, *for* $i=80$;

- $X^8+w^{13}X^4+w^4X^3+w^{22}X^2+w^7X+w^{13}Y^4+wY^2+w^5Y+w^{30}=0$, *for* $i=81$;
- $X^8+w^{26}X^4+w^7X^3+w^{15}X^2+w^{17}X+w^{17}Y^4+w^2Y^2+w^{15}Y+w^{11}=0$, *for* $i=82$;
- $X^8+w^{26}X^4+w^7X^3+w^{25}X^2+w^{17}Y^4+w^2Y^2+w^{15}Y+w^{11}=0$, *for* $i=83$;
- $X^8+w^{18}X^4+w^2X^3+w^{25}X^2+w^{13}X+w^{10}Y^4+w^{30}Y^2+w^{27}Y+w^{27}=0$, *for* $i=84$;
- $X^8+w^{10}X^4+w^{19}X^3+w^8X^2+w^{25}X+w^{15}Y^4+w^{27}Y^2+w^{22}Y+w^{10}=0$, *for* $i=85$;
- $X^8+w^{10}X^4+w^{19}X^3+w^{11}X^2+w^{18}X+w^{15}Y^4+w^{27}Y^2+w^{22}Y+w^{11}=0$, *for* $i=86$;
- $X^8+w^{17}X^4+w^{11}X^3+w^{20}X^2+w^{21}X+w^{26}Y^4+w^{18}Y^2+w^{23}Y+w^{19}=0$, *for* $i=87$;
- $X^8+w^{17}X^4+w^{11}X^3+w^{29}X^2+w^{15}X+w^{26}Y^4+w^{18}Y^2+w^{23}Y+w^{19}=0$, *for* $i=88$;
- $X^8+w^{18}X^4+w^2X^3+w^{28}X^2+w^{16}X+w^{10}Y^4+w^{30}Y^2+w^{27}Y=0$, *for* $i=89$;
- $X^{10}+w^{17}X^8+w^3X^6+w^{13}X^5+w^{25}X^4+w^{20}X^3+w^5X^2+w^{26}X+w^{17}Y^2+w^6Y=0$, *for* $i=91$;
- $X^{10}+w^{30}X^8+w^{29}X^6+w^{16}X^5+w^7X^4+w^{26}X^3+w^{25}X^2+w^{25}X+w^6Y^2+w^{19}Y+w^5=0$, *for* $i=92$;
- $X^{10}+w^{30}X^8+w^{29}X^6+w^{26}X^5+w^7X^4+w^{10}X^3+w^{18}X^2+X+w^6Y^2+w^{29}Y+w^{23}=0$, *for* $i=93$;
- $X^{10}+w^{30}X^8+w^{29}X^6+w^{22}X^5+w^7X^4+w^4X^3+w^{16}X^2+w^5X+w^6Y^2+w^{25}Y+w^{20}=0$, *for* $i=94$;
- $X^{10}+w^{17}X^8+w^3X^6+w^{18}X^5+w^{25}X^4+w^{12}X^3+w^{17}X^2+w^2X+w^{17}Y^2+w^{11}Y+w^{24}=0$, *for* $i=95$;
- $X^{10}+w^{21}X^8+w^{11}X^6+w^{12}X^5+w^{21}X^4+w^{24}X^3+w^{18}X^2+wX+w^{18}Y^2+w^{21}Y=0$, *for* $i=96$;
- $X^{10}+w^{21}X^8+w^{11}X^6+w^{24}X^5+w^{21}X^4+w^{11}X^3+X^2+w^3X+w^{18}Y^2+w^2Y+w^{21}=0$, *for* $i=97$;
- $X^{10}+w^{21}X^8+w^{11}X^6+wX^5+w^{21}X^4+w^{23}X^3+w^4X^2+w^{26}X+w^{18}Y^2+w^{10}Y+w^6=0$, *for* $i=98$;
- $X^{10}+w^{25}X^8+w^{19}X^6+w^6X^5+w^{22}X^4+w^2X^3+w^2X^2+w^{22}X+w^{10}Y^2+w^{11}Y+w^{20}=0$, *for* $i=99$;
- $X^{10}+w^{25}X^8+w^{19}X^6+w^{21}X^5+w^{22}X^4+w^9X^3+w^{25}X^2+w^{11}X+w^{10}Y^2+w^{26}Y+w^8=0$, *for* $i=100$;
- $X^{10}+w^{25}X^8+w^{19}X^6+w^{28}X^5+w^{22}X^4+w^4X^3+w^{30}X^2+w^{29}X+w^{10}Y^2+w^2Y+w^{10}=0$, *for* $i=101$;
- $X^{10}+w^{11}X^8+w^{22}X^6+w^{22}X^5+w^{20}X^4+w^{21}X^3+w^{18}X^2+w^9X+w^{26}Y^2+w^4Y+w^{29}=0$, *for* $i=102$;

- $X^{10}+w^{11}X^8+w^{22}X^6+w^{23}X^5+w^{20}X^4+w^7X^3+w^3X^2+w^{20}X+w^{26}Y^2+w^5Y+w^9=0$, *for* $i=103$;
- $X^{10}+w^{11}X^8+w^{22}X^6+w^5X^5+w^{20}X^4+w^{11}X^3+w^{28}X^2+w^{22}X+w^{26}Y^2+w^{18}Y+w^{11}=0$, *for* $i=104$;
- $X^{10}+w^{17}X^8+w^3X^6+w^{16}X^5+w^{25}X^4+w^9X^3+w^{16}X^2+w^{27}X+w^{17}Y^2+w^9Y+w^{20}=0$, *for* $i=105$.

Proof. The proof is a straightforward computation based on Lemmas 3.2, 3.3, and 3.4. □

4. Quotient curves arising from subgroups of even order of $\bar{\mathbf{T}}\bar{\mathbf{N}}$

Thanks to a computer research, for both $q=8$ and $q=32$ there are only two conjugacy classes of subgroups of even order of $\bar{\mathbf{T}}\bar{\mathbf{N}}$ which are not 2-subgroups, and they have order $q(q-1)$ and $q^2(q-1)$. By [3, Theorem 8.1] the quotient curves of the DLS-curve associated to those subgroups are rational.

5. Quotient curves arising from subgroups of even order of $N_{\mathrm{Aut}(\bar{\mathbb{F}}_q(\mathcal{X}))}\bar{\mathbf{N}}$

Theorem 9.2 in [3] gives an equation for the quotient curve of the DLS-curve associated to subgroups of even order of $N_{\mathrm{Aut}(\bar{\mathbb{F}}_q(\mathcal{X}))}\bar{\mathbf{N}}$.

6. Quotient curves arising from subgroups of even order of $N_{\mathrm{Aut}(\bar{\mathbb{F}}_q(\mathcal{X}))}\bar{\mathbf{D}}^+$

Any subgroup of $N_{\mathrm{Aut}(\bar{\mathbb{F}}_q(\mathcal{X}))}\bar{\mathbf{D}}^+$ has order $2^i r$, for some $i=0,1,2$ and for a certain divisor r of $q+2q_0+1$. Theorem 10.1 in [3] gives an equation of the quotient curve associated to a subgroup of $N_{\mathrm{Aut}(\bar{\mathbb{F}}_q(\mathcal{X}))}\bar{\mathbf{D}}^+$ for the case $i=1$. For both $q=8$ and $q=32$ the integer $q+2q_0+1$ is a prime number, therefore by [3, Proposition 11.1] the case $i=2$ gives rise to rational curves.

7. Quotient curves arising from subgroups of even order of $N_{\mathrm{Aut}(\bar{\mathbb{F}}_q(\mathcal{X}))}\bar{\mathbf{D}}^-$

Any subgroup of $N_{\mathrm{Aut}(\bar{\mathbb{F}}_q(\mathcal{X}))}\bar{\mathbf{D}}^-$ has order $2^i r$, for some $i=0,1,2$ and for some divisor r of $q-2q_0+1$. Theorem 10.2 in [3] gives an equation of the quotient curve associated to a subgroup of $N_{\mathrm{Aut}(\bar{\mathbb{F}}_q(\mathcal{X}))}\bar{\mathbf{D}}^+$ for the case $i=1$. For $q=8$ the integer $q-2q_0+1$ is a prime number, therefore by [3, Proposition 11.1] the case $i=2$ gives rise to rational curves. For $q=32$ we have $q-2q_0+1=25$. Therefore for $i=2$ we have the cases $r=25$ giving rise to a rational curve, and $r=5$ giving rise to a curve of genus 5.

8. Classification of elliptic and hyperelliptic curves of genus 2 covered by the DLS-curve of Suzuki type for $q = 32$

In this section we will provide canonical equations for both elliptic and hyperelliptic curves of genus 2 which are quotient curves of the Suzuki curve defined over the finite field with 32 elements.

A computer based investigation, together with Theorem 8.1 below, has proved that the hyperelliptic curves of genus 2 are pairwise non-isomorphic.

Theorem 8.1. [10] *Assume K to be of even characteristic. Let Γ and Δ be two hyperelliptic curves of genus g given in their canonical form, that is*

$$\Gamma = \mathbf{v}(Y^2 + h(X)Y + g(X)), \quad \deg g(X) = 2g+1, \quad \deg h(X) \le g;$$

$$\Delta = \mathbf{v}(Y^2 + h_1(X)Y + g_1(X)), \quad \deg g_1(X) = 2g+1, \quad \deg h_1(X) \le g.$$

Then Γ and Δ are birationally equivalent if and only if each of the following two conditions are satisfied.

i) *$n = m$, and there is rational function $w(X) = (aX+b)/(cX+d)$ with $ad - bc \neq 0$ which maps the set $\{\alpha_0, \dots, \alpha_n, \infty\}$ onto the set $\{\beta_0, \dots, \beta_n, \infty\}$.*

ii) *There is a rational function $v(X) \in K(X)$ such that*

$$v(X)^2 + v(X) = \frac{g(X)}{h(X)^2} + \frac{g_1(w(X))}{h_1(w(X))^2}.$$

8.1. Canonical equations for elliptic curves $\mathcal{X}_{\nu_i}$, $i = 91, \dots, 105$.

- $w^{29}X^3 + w^6X + Y^2 + Y + w^{30} = 0$, for $i = 91$;
- $w^2X^3 + w^{11}X + Y^2 + Y + w^{13} = 0$, for $i = 92$;
- $w^{13}X^3 + w^{16}X + Y^2 + Y + w^{10} = 0$, for $i = 93$;
- $w^{21}X^3 + w^{20}X + Y^2 + Y + w^{24} = 0$, for $i = 94$;
- $w^{19}X^3 + w^{30}X + Y^2 + Y + w^{28} = 0$, for $i = 95$;
- $w^{13}X^3 + wX + Y^2 + Y + w^{27} = 0$, for $i = 96$;
- $w^{23}X^3 + w^{29}X + Y^2 + Y + 1 = 0$, for $i = 97$;
- $w^7X^3 + w^5X + Y^2 + Y + w^{20} = 0$, for $i = 98$;
- $w^{24}X^3 + w^3X + Y^2 + Y + w^9 = 0$, for $i = 99$;
- $w^{25}X^3 + w^{19}X + Y^2 + Y + w^3 = 0$, for $i = 100$;
- $w^{11}X^3 + w^{14}X + Y^2 + Y + w^{21} = 0$, for $i = 101$;
- $X^3 + w^{13}X + Y^2 + Y + 1 = 0$, for $i = 102$;
- $w^{29}X^3 + w^6X + Y^2 + Y + w^{14} = 0$, for $i = 103$;
- $w^3X^3 + wX + Y^2 + Y + w^{11} = 0$, for $i = 104$;
- $w^{23}X^3 + w^9X + Y^2 + Y + w^{12} = 0$, for $i = 105$.

8.2. Canonical equations for hyperelliptic curves $\mathcal{X}_{\nu_i}$, $i = 65, \dots, 79$.

- $w^{30}X^5 + w^{11}X^4 + w^8X^3 + w^5X^2 + Y^2 + w^{24}Y = 0$, for $i = 65$;
- $w^{30}X^5 + w^{19}X^4 + w^{12}X^3 + w^5X^2 + Y^2 + w^{21}Y = 0$, for $i = 66$;

- $w^{30}X^5 + w^{24}X^4 + w^{29}X^3 + w^5X^2 + Y^2 + wY = 0$, for $i = 67$;
- $w^{30}X^5 + w^{30}X^4 + w^{25}X^3 + w^5X^2 + Y^2 + w^{23}Y = 0$, for $i = 68$;
- $w^{30}X^5 + w^{4}X^4 + w^{22}X^3 + w^5X^2 + Y^2 + w^{18}Y = 0$, for $i = 69$;
- $w^{30}X^5 + X^4 + w^{14}X^3 + w^5X^2 + Y^2 + w^{6}Y = 0$, for $i = 70$;
- $w^{30}X^5 + w^{28}X^4 + w^{16}X^3 + w^5X^2 + Y^2 + w^{26}Y = 0$, for $i = 71$;
- $w^{30}X^5 + w^{7}X^4 + w^{10}X^3 + w^5X^2 + Y^2 + w^{16}Y = 0$, for $i = 72$;
- $w^{30}X^5 + w^{13}X^4 + w^{17}X^3 + w^5X^2 + Y^2 + w^{12}Y = 0$, for $i = 73$;
- $w^{30}X^5 + w^{16}X^4 + w^{24}X^3 + w^5X^2 + Y^2 + w^{13}Y = 0$, for $i = 74$;
- $w^{30}X^5 + w^{21}X^4 + w^{13}X^3 + w^5X^2 + Y^2 + w^{28}Y = 0$, for $i = 75$;
- $w^{30}X^5 + w^{10}X^4 + w^{23}X^3 + w^5X^2 + Y^2 + w^{5}Y = 0$, for $i = 76$;
- $w^{30}X^5 + w^{25}X^4 + X^3 + w^5X^2 + Y^2 + w^{4}Y = 0$, for $i = 77$;
- $w^{30}X^5 + w^{18}X^4 + w^{28}X^3 + w^5X^2 + Y^2 + w^{19}Y = 0$, for $i = 78$;
- $w^{30}X^5 + w^{6}X^4 + w^{21}X^3 + w^5X^2 + Y^2 + w^{22}Y = 0$, for $i = 79$.

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