Super-edge-connected and Optimally Super-edge-connected Bi-Cayley graphs

Fengxia Liu and Jixiang Meng*
College of Mathematics and Systems Science,
Xinjiang University, Urumqi, Xinjiang 830046, P.R.China
Email:xjulfx@163.com

Abstract

Let G be a finite group, S(possibly, contains the identity element) be a subset of G. The Bi-Cayley graph $BC(G,S)$ is a bipartite graph with vertex set $G \times \{0,1\}$ and edge set $\{\{(g,0),(gs,1)\},\ g \in G,\ s \in S\}$. A graph X is said to be super-edge-connected if every minimum edge cut of X is a set of edges incident with some vertex. The restricted edge connectivity $\lambda'(X)$ of X is the minimum number of edges whose removal disconnects X into nontrivial components. A k-regular graph X is said to be optimally super-edge-connected if X is super-edge-connected and its restricted edge connectivity attains the maximum $2k-2$. In this paper, we show that all connected Bi-Cayley graphs, except even cycles, are optimally super-edge-connected.

Keywords: Super-edge-connected; Optimally super-edge-connected; Bi-Cayley graphs; Vertex-transitive; Orbit

1 Introduction

Fault-tolerance is one of the main factors which should be taken into account in the design of an interconnection network. Indeed, it is generally expected that the system be able to work even if several of its links fail. Thus, it is often required that the graph associated to the interconnection network be sufficiently connected. In most cases, a good design requires that the graph has maximum connectivity. Observe that the maximum edge connectivity of k-regular graphs is k, and that if X satisfies $\lambda(X) = k$, then every set of edges incident with some vertex is a minimum

*Correspondence author, the research of this author is supported by NSFC, FUKT and SPUX.

edge cut. It is natural to introduce the following definition (see [3]).

Definition 1.1. A k-regular graph X is said to be *super-edge-connected* (or simply *super-*λ) if every minimum edge cut is the set of edges incident with some vertex.

As a natural generalization of classical connectivity, Harary [9] proposed the concept of conditional edge connectivity of graphs. Let P be a graph property. The P-edge connectivity $\lambda(X,P)$ of X is the minimum number of edges whose removal disconnects X into components with property P. In particular, Esfahanian and Hakimi [6] considered a special kind of conditional edge connectivity which we call restricted edge connectivity.

Definition 1.2. Let $X = (V, E)$ be a connected graph. A set C of edges of X is called a *restricted edge cut* if C is an edge cut and every component of $X \backslash C$ has at least two vertices. The minimum cardinality of all restricted edge cuts, denoted by $\lambda'(X)$, is called the *restricted edge connectivity.*

Proposition 1.3.[6] *Let $X = (V, E)$ be a connected graph with at least four vertices and it is not a Star graph $K_{1,m}$. Then X has restricted edge cuts and so $\lambda'(X)$ is well defined. Furthermore, $\lambda(X) \le \lambda'(X) \le \xi(X)$, where $\xi(X) = min\{d(x) + d(y) - 2 : e = xy \in E\}$.*

Clearly, a graph X is super-λ if and only if $\lambda'(X) > \lambda(X)$. If $X = (V, E)$ is a k-regular connected graph with at least four vertices, then X has restricted edge cuts and $\lambda(X) \le \lambda'(X) \le 2k - 2$.

Definition 1.4. Let X be a k-regular connected graph with at least four vertices. Then X is said to be *optimally super-edge-connected* (or simply *optimally super-*λ) if X is super-λ and $\lambda'(X) = 2k - 2$.

Definition 1.5. A bipartite graph $X = (A, B)$ is called *semiregular* if the vertices in the same partition class have the same degree.

A graph X is said to be *vertex-transitive* if for every two vertices u and v of X, there is an automorphism of X that maps u to v. Let $x \in V(X)$, we call the set $\{x^g : g \in Aut(X)\}$ an *orbit* of $Aut(X)$. Clearly, the automorphism group $Aut(X)$ acts transitively on each orbit of $Aut(X)$. A graph X is vertex-transitive if and only if $Aut(X)$ has a unique orbit.

Let G be a finite group and S be a subset of $G\backslash\{1\}$ with $S = S^{-1}$. The *Cayley graph* $C(G, S)$ is the graph with vertex set G and edge set $\{gh : g^{-1}h \in S\}$. Xu [14] proposed the definition of Bi-Cayley graphs. Let G be a finite group, S(possibly, contains the identity element) be a subset of G. The *Bi-Cayley graph* $BC(G, S)$ is a bipartite graph with vertex set $G \times \{0, 1\}$ and edge set $\{\{(g, 0), (gs, 1)\},\ g \in G,\ s \in S\}$. Obviously, the graph $BC(G, S)$ is $|S|$-regular. In this paper, we always assume that

$|S| = k$.

For any element a in G, the left multiplication

$$l_a : (g, i) \to (ag, i), \ g \in G, \ i = 0, 1,$$

is clearly an automorphism of $BC(G, S)$. All these left mutiplications constitute a group L_G which acts transitively on $G \times \{0\}$ and $G \times \{1\}$ respectively. Thus, if $Aut[BC(G, S)]$ has a unique orbit, then $BC(G, S)$ is a vertex-transitive graph, and if $BC(G, S)$ is not vertex-transitive, then $Aut[BC(G, S)]$ has exact two orbits.

Tindell [13] characterized super-λ transitive graphs and Cayley graphs. Meng [12] characterized optimally super-λ transitive graphs and Cayley graphs. Here, we prove that all connected Bi-Cayley graphs, except even cycles, are optimally super-λ.

For details on connectivity of bipartite graphs and digraphs, see [1,2,8] for references. For connectivity of transitive graphs, see [11-13] for references.

2 Super-λ Bi-Cayley graphs

In this section, we study super-λ Bi-Cayley graphs.

Recall that $Aut[BC(G, S)]$ has at most two orbits. If $Aut[BC(G, S)]$ has a unique orbit, then $BC(G, S)$ is a vertex-transitive graph. Tindell [13] characterized super-λ transitive graphs. In fact, he proved:

Theorem 2.1. *Let X be a k-regular-connected vertex-transitive graph which is neither a complete graph nor a cycle. Then X is not super-λ if and only if it contains k-cliques.*

For vertex-transitive Bi-Cayley graph, we have the following result.

Corollary 2.2. *Let $X = BC(G, S)$ be a connected vertex-transitive Bi-Cayley graph which is not a cycle. Then X is super-λ.*

Proof. If X is a complete graph, then $X \cong K_2$. Thus X is super-λ. If X is not a complete graph which is not super-λ, then by Theorem 2.1, it contains k-cliques. Since X is not a cycle, we have $k \geq 3$. Thus X contains triangles, contradicting that $BC(G, S)$ is a bipartite graph. The result follows. □

Corollary 2.2 assures that vertex-transitive Bi-Cayley graphs which are not cycles are super-λ. However, not all Bi-Cayley graphs are vertex-transitive. Lu [10] constructed several infinite families of Bi-Cayley graphs which are edge-transitive but not vertex-transitive (such graphs are called semisymmetric graphs). In the following, we show that non vertex-transitive Bi-Cayley graphs are also super-λ. We first introduce some notation.

Let $X = (V, E)$ be a connected graph and $F \subset V$ be a non-empty set. We use $\omega(F)$ to denote the set of edges with exactly one end vertex in F. The vertex set F is called an *edge fragment* if $|\omega(F)| = \lambda(X)$. It is easy to see that F is an edge fragment if and only if $V - F$ is an edge fragment. An edge fragment F is called a *strict edge fragment* if $\omega(F)$ is a restricted edge cut. A strict edge fragment of X with minimum cardinality is called a *super edge atom* of X. Clearly, the cardinality of a super edge atom of X is at least 2. Moreover, X has a super edge atom if and only if X is not super-λ.

Let $X = BC(G, S)$ be a connected Bi-Cayley graph which is not vertex-transitive, and A be a super edge atom of X. Then the intersection of A and each partition class of X is nonempty. Otherwise, if A is contained in only one partition class, then $|\omega(A)| = k|A| = \lambda \leq \delta = k$, which gives $|A| \leq 1$, contradicting that A is a super edge atom. So, in this paper, we use $A = A_0 \cup A_1$ to denote the super edge atom of a connected Bi-Cayley graph which is not vertex-transitive, and we always assume that $A_0 = A \cap [G \times \{0\}]$, $A_1 = A \cap [G \times \{1\}]$, $|A_0| \geq 1$ and $|A_1| \geq 1$. Tindell [13] proved the following proposition.

Proposition 2.3. *If $X = (V, E)$ is a connected graph which is not a cycle, is not super-λ and has $\delta(X) > 2$, then distinct super edge atoms of X are disjoint.*

If X is a connected graph which is not super-λ, then clearly any automorphic image of a super edge atom of X is again a super edge atom of X. Thus, if X is a connected Bi-Cayley graph which is not vertex-transitive, the vertex set A is a super edge atom of X, and ϕ is an automorphism of X, then $\phi(A)$ is a super edge atom, and by Proposition 2.3 we have either $\phi(A) = A$ or $\phi(A) \cap A = \emptyset$.

For $i = 0, 1$, we define the operation of $G \times \{i\}$ as follows.

$$(g_1, i) \cdot (g_2, i) = (g_1 \cdot g_2, i) \quad g_1, g_2 \in G, \quad i = 0, 1.$$

Then $G \times \{i\}$ is a group isomorphic to G.

Proposition 2.4. *Let $X = BC(G, S)$ be a connected Bi-Cayley graph which is not vertex-transitive. Let $A = A_0 \cup A_1$ be a super edge atom of X and $Y = X[A]$. Then*
(i) The automorphism group $Aut(Y)$ acts transitively on both A_0 and A_1.
(ii) If X is edge-transitive, then A is an independent subset of X.
(iii) If A_i contains the identity of $G \times \{i\}$, then A_i is a subgroup of $G \times \{i\}$, for $i = 0, 1$.

Proof. (i) Given $(u,0),(v,0) \in A_0$, there is an automorphism φ of X with $\varphi((u,0)) = (v,0)$, and so $(v,0) \in \varphi(A) \cap A \neq \emptyset$. Since $\varphi(A)$ is a super edge atom, by Proposition 2.3 we have $\varphi(A) = A$. The automorphism φ of X induces an automorphism $\varphi|_Y$ of $X[A] = Y$ satisfying $\varphi|_Y((u,0)) = (v,0)$. Thus $Aut(Y)$ acts transitively on A_0. Similarly, the automorphism group $Aut(Y)$ acts transitively on A_1.
(ii) Suppose there is an edge $e = xy$ with $x, y \in A$. Since X is edge-transitive, there must be an automorphism φ of X such that $\varphi(e) \in \omega_X(A)$ because of Proposition 2.3. But then we have $\varphi(x)$ or $\varphi(y)$ belongs to $\varphi(A) \cap A \neq \emptyset$; and $\varphi(x)$ or $\varphi(y)$ belongs to $\varphi(A) \cap (V(X) \backslash A) \neq \emptyset$, which is a contradiction since $\varphi(A)$ is a super edge atom.
(iii) If A_0 contains the identity $(1,0)$ of $G \times \{0\}$, since for any $(g,0) \in A_0$, $l_{g^{-1}} \in Aut(X)$, we have $(1,0) = l_{g^{-1}}((g,0)) \in l_{g^{-1}}(A_0) \subset l_{g^{-1}}(A) = g^{-1}A$ and $(1,0) \in A_0 \subset A$, so $g^{-1}A \cap A \neq \emptyset$. Since $g^{-1}A$ is also a super edge atom of X, we have $g^{-1}A = A$. It follows that $g^{-1}A_0 = A_0$. Thus $(g,0),(h,0) \in A_0$ implies that $(g^{-1},0)\cdot(h,0) = (g^{-1}h,0) = g^{-1}(h,0) \in g^{-1}A_0 = A_0$, that is $(g^{-1},0)\cdot(h,0) \in A_0$. Thus A_0 is a subgroup of $G \times \{0\}$. □

Proposition 2.5. *If $X = BC(G,S)$ is a k-regular connected Bi-Cayley graph which is neither vertex-transitive nor super-λ, then*
(i) $\lambda(X) = |S| = k$
(ii) The vertex set $A = A_0 \cup A_1$ is a super edge atom of X if and only if $Y = X[A] \cong K_{k,(k-1)}$.

Proof. Suppose $A = A_0 \cup A_1$ is a super edge atom of X, then $|A_0| \geq 1, |A_1| \geq 1$. Since $Aut(X[A])$ acts transitively on A_i, the vertices in A_i have the same degree, say k_i $(i = 0,1)$, in $X[A]$. Clearly, $k \geq k_i$, $|A_0| \geq k_1, |A_1| \geq k_0$, and $k_0|A_0| = k_1|A_1|$. Without loss of generality, assume that $k_0 \geq k_1$. Then

$$|A_0|(k-k_0) + |A_1|(k-k_1) = |\omega(A)| = \lambda(X) \leq k = k - k_0 + k_0,$$

this gives

$$(|A_0|-1)(k-k_0) + |A_1|(k-k_1) \leq k_0 \leq |A_1|,$$

that is

$$(|A_0|-1)(k-k_0) + |A_1|(k-k_1-1) \leq 0.$$

If $k = k_1$, then $k_0 = k_1 = k$ because $k \geq k_0 \geq k_1$. Since X is connected, $X = X[A]$, contradicting that A is a super edge atom. So $k - k_1 - 1 \geq 0$. We thus have

$$(|A_0|-1)(k-k_0) + |A_1|(k-k_1-1) = 0.$$

Case 1: $|A_0| - 1 = 0$ and $k - k_1 - 1 = 0$, that is $|A_0| = 1, k = k_1 + 1$. Since $k_1 \leq |A_0|$, we have $k \leq 2$. If $k = 1$, then $X \cong K_2$. If $k = 2$, then

X is a cycle. Both K_2 and cycle are vertex-transitive, contradicting our assumption.
Case 2: $k - k_0 = 0$ and $k - k_1 - 1 = 0$, that is $k = k_0$ and $k_1 = k - 1$. Then

$$k \geq \lambda(X) = |A_0|(k - k_0) + |A_1|(k - k_1) = |A_1|.$$

Since $|A_1| \geq k_0 = k$, we have $|A_1| = k$, and so $\lambda(X) = k$. Therefore (i) is established. To prove the necessity of (ii) notice that $|A_0| = k - 1$ because $k_0|A_0| = k_1|A_1|$. Conversely, as X is not super-λ, we have $k \geq 2$, this gives $|A| = k + k - 1 = 2k - 1 \geq 3$. By item (i) we know $|\omega(A)| = k$. If $X - \omega(A)$ has isolated vertices, then $X \cong K_{k,k}$, which is vertex-transitive. Thus $\omega(A)$ is a strict edge fragment. By the necessity of (ii) we see that the super edge atom of X has cardinality $2k - 1$, thus A is a super edge atom. □

Clearly, if $X = BC(G, S)$ is super-λ, then $\lambda(X) = k$. Since the edge connectivity of a connected vertex-transitive graph attains its regular degree, we have

Corollary 2.6. *If $X = BC(G, S)$ is a connected Bi-Cayley graph, then $\lambda(X) = |S|$.*

Corollary 2.7. *Let $X = BC(G, S)$ be a connected Bi-Cayley graph which is neither vertex-transitive nor a complete bipartite graph and $k = |S|$. Then X is not super-λ if and only if X contains $K_{k,(k-1)}$ as a subgraph.*

Proposition 2.8. *Let $X = BC(G, S)$ be a connected Bi-Cayley graph which is not vertex-transitive. Then X is super-λ.*

Proof. Suppose that X is not super-λ, then X has a super edge atom. Without loss of generality, suppose $A = A_0 \cup A_1$ is a super edge atom of X such that A_0 contains the identity $(1, 0)$ of $G \times \{0\}$, where $A_0 = A \cap [G \times \{0\}]$ and $A_1 = A \cap [G \times \{1\}]$. By Proposition 2.5, $Y = X[A] \cong K_{k,(k-1)}$, and by Proposition 2.4, we know that A_0 is a subgroup of $G \times \{0\}$. Let $S = \{s_1, \cdots, s_k\}$.
Case 1: $|A_0| = k - 1$, $|A_1| = k$. Since A_0 contains the identity $(1, 0)$ of $G \times \{0\}$, we have $A_1 = S \times \{1\}$. Further, since Y is a complete bipartite graph, for any $s_i \in S$, we have $A_0 s_i \subset A_1$, and thus

$$|A_0 s_1 \cup A_0 s_2 \cup \cdots \cup A_0 s_k| = |A_1| = k.$$

Clearly $|A_0 s_i| = |A_0| = k - 1$. Thus, for any $s_i, s_j \in S$, $A_0 s_i \cap A_0 s_j \neq \emptyset$, and so $A_0 s_i = A_0 s_j$. This gives

$$|A_0 s_1 \cup A_0 s_2 \cup \cdots \cup A_0 s_k| = |A_0 s_i| = |A_0| = k - 1,$$

a contradiction.
Case 2: $|A_0| = k$, $|A_1| = k - 1$. Clearly, the neighbor set $N(A_0)$ of A_0 is

contained in $\omega(A) \cup A_1$. Thus

$$|A_0s_1 \cup A_0s_2 \cup \cdots \cup A_0s_k| \leq 2k - 1.$$

Since $|A_0s_i| = |A_0| = k$, for any $s_i, s_j \in S$, we have $A_0s_i \cap A_0s_j \neq \emptyset$, and so $A_0s_i = A_0s_j$. This gives

$$|A_0s_1 \cup A_0s_2 \cup \cdots \cup A_0s_k| = |A_0s_i| = |A_0| = k,$$

which implies that X contains $K_{k,k}$ as a subgraph. Since X is connected, we have $X \cong K_{k,k}$. But then X is vertex transitive, a contradiction. □

Now, combining Corollary 2.2 and Proposition 2.8, we have the following

Theorem 2.9. *Except cycles, all connected Bi-Cayley graphs are super-λ.*

3 Optimally super-λ Bi-Cayley graphs

In this section, we study optimally super-λ Bi-Cayley graphs, and we always assume that $|V(X)| \geq 4$. Meng [12] characterized optimally super-λ transitive graphs. In fact, he proved:

Theorem 3.1. *Let X be a k-regular-connected vertex-transitive graph which is neither a complete graph nor a cycle. Then X is not optimally super-λ if and only if it contains a $(k-1)$-regular subgraph Y satisfying*

$$k \leq |V(Y)| \leq 2k - 3.$$

For vertex-transitive Bi-Cayley graphs, we have the following result.

Corollary 3.2. *Let $X = BC(G, S)$ be a connected vertex-transitive Bi-Cayley graph which is not a cycle. Then X is optimally super-λ.*

Proof. If X is a complete graph, then $X \cong K_2$ and so $|V(X)| = 2$, contradicting our assumption that $|V(X)| \geq 4$. In the following we assume that X is not a complete graph. Suppose that X is not optimally super-λ, then by Theorem 3.1, it contains a $(k-1)$-regular subgraph Y satisfying $k \leq |V(Y)| \leq 2k-3$. Since X is a Bi-Cayley graph, the graph X is bipartite and so each $(k-1)$-regular subgraph Y of X satisfies $|V(Y)| \geq 2k-2$, a contradiction. Thus X is optimally super-λ. □

Corollary 3.2 assures that vertex-transitive Bi-Cayley graphs are optimally super-λ. In the following we show that non vertex-transitive Bi-Cayley graphs are optimally super-λ as well.

Let $X = (V, E)$ be a connected graph, and $F \subset V$ be a non-empty set. The vertex set F is called a *λ'-fragment*, if $\omega(F)$ is a restricted edge cut and $|\omega(F)| = \lambda'(X)$. It is easy to see that F is a λ'-fragment if and only if $V - F$ is a λ'-fragment. A λ'-fragment of X with minimum cardinality is called *λ'-atom* of X. The cardinality of a λ'-atom of X is denoted by $\omega'(X)$. Clearly, $\omega'(X) \geq 2$. By the definition of optimally super-λ and Proposition 2.8, it is easy to see that if $X = BC(G, S)$ is a k-regular connected Bi-Cayley graph which is neither vertex-transitive nor optimally super-λ, then $k < \lambda'(X) \leq 2k-3$. Let A be a λ'-atom of a Bi-cayley graph X. Then, as in the last section, we may write $A = A_0 \cup A_1$, and we always assume that $A_0 = A \cap [G \times \{0\}]$, $A_1 = A \cap [G \times \{1\}]$, $|A_0| \geq 1$ and $|A_1| \geq 1$. In [12] the following result on λ'-atoms was proved.

Proposition 3.3. *Let $X = (V, E)$ be a k-regular k-edge-connected graph with $\omega'(X) \geq 3$. Then any two distinct λ'-atoms of X are disjoint.*

By Corollary 2.6, if X is a connected Bi-Cayley graph, then $\lambda(X) = k$.

Next, we have the following proposition which is similar to Proposition 2.4.

Proposition 3.4. *Let $X = BC(G, S)$ be a connected Bi-Cayley graph which is not vertex-transitive and $\omega'(X) \geq 3$. Let $A = A_0 \cup A_1$ be a λ'-atom of X and $Y = X[A]$. Then*
(i) The automorphism group $Aut(Y)$ acts transitively on both A_0 and A_1.
(ii) If X is edge-transitive, then A is an independent subset of X.
(iii) If A_i contains the identity of $G \times \{i\}$, then A_i is a subgroup of $G \times \{i\}$, for $i = 0, 1$.

Before proceeding, we first establish a lemma.

Lemma 3.5. *Let $X = BC(G, S)$ be a connected Bi-Cayley graph which is not vertex-transitive. If X is not optimally super-λ, then the subgraph Y induced by a λ'-atom is semiregular bipartite graph with degree k, $k-1$ and*

$$2k \leq |V(Y)| \leq 4k - 7.$$

Proof. Let $A = A_0 \cup A_1$ be a λ'-atom of X, $|A_0| \geq 1$, $|A_1| \geq 1$. Since $Aut(X[A])$ acts transitively on both A_0 and A_1, the vertices of A_i have the same degree, say k_i $(0 \leq i \leq 1)$, in $X[A]$. Clearly, $k \geq k_i$, $|A_0| \geq k_1$, $|A_1| \geq k_0$ and $k_0|A_0| = k_1|A_1|$. Since X is not optimally super-λ, we have $k < \lambda'(X) \leq 2k - 3$. Clearly

$$\lambda'(X) = |\omega(A)| = |A_0|(k - k_0) + |A_1|(k - k_1).$$

If $|A_0| = |A_1|$, since $k_0|A_0| = k_1|A_1|$, we have $k_0 = k_1$ and so $\lambda'(X)$ is even. This gives

$$\lambda'(X) = 2|A_0|(k - k_0) \leq 2k - 4,$$

$$|A_0|(k-k_0) \leq k-2 = k-k_0+k_0-2,$$

$$(|A_0|-1)(k-k_0) \leq k_0-2 \leq |A_1|-2 = |A_0|-2 = |A_0|-1-1,$$

$$(|A_0|-1)(k-k_0-1) \leq -1.$$

Since $|A_0| \geq 1$, we derive that $k-k_0-1 \leq -1$. Then $k_0 \geq k$ and so $k_0 = k_1 = k$. Since X is connected, $X = X[A]$, contradicting that A is a λ'-atom. So, $|A_0| \neq |A_1|$. Without loss of generality, we may assume that $|A_0| > |A_1|$. Since $k_0|A_0| = k_1|A_1|$, we have $k_0 < k_1$.

$$\lambda'(X) = |A_0|(k-k_0)+|A_1|(k-k_1) \leq 2k-3 = (k-k_0)+(k-k_1)+k_0+k_1-3,$$

$$(|A_0|-1)(k-k_0)+(|A_1|-1)(k-k_1) \leq k_0+k_1-3 \leq |A_0|+|A_1|-3$$

$$= (|A_0|-1)+(|A_1|-1)-1,$$

$$(|A_0|-1)(k-k_0-1)+(|A_1|-1)(k-k_1-1) \leq -1.$$

Since $|A_0| > |A_1| \geq 1$ and $k \geq k_1 > k_0$, we have $|A_0|-1 > 0$, $|A_1|-1 \geq 0$, $k-k_0-1 \geq 0$ and $k-k_1-1 \geq -1$. If $k-k_1-1 > -1$, then $k-k_1-1 \geq 0$, and so

$$(|A_0|-1)(k-k_0-1)+(|A_1|-1)(k-k_1-1) \geq 0,$$

a contradiction. So, $k-k_1-1 = -1$, that is $k = k_1$. This gives

$$\lambda'(X) = |A_0|(k-k_0)+|A_1|(k-k_1) = |A_0|(k-k_0).$$

If $k-k_0 \geq 2$, since $|A_0| \geq k_1 = k$, we have $\lambda'(X) = |A_0|(k-k_0) \geq 2k$, contradicting $\lambda'(X) \leq 2k-3$. So, $k-k_0 \leq 1$. Since $k = k_1 > k_0$, we have $k_0 = k-1$ and so $k < \lambda'(X) = |A_0| \leq 2k-3, k+1 \leq |A_0| \leq 2k-3$. Thus $k-1 = k_0 \leq |A_1| < |A_0| \leq 2k-3, k-1 \leq |A_1| \leq 2k-4$. It follows that $2k \leq |V(Y)| = |A_0|+|A_1| \leq 4k-7$. The result follows. □

Proposition 3.6. *Let $X = BC(G,S)$ be a connected Bi-Cayley graph which is not vertex-transitive. Then X is not optimally super-λ if and only if X contains a semiregular bipartite subgraph Y with degree k and $k-1$ satisfying*

$$2k \leq |V(Y)| \leq 4k-7.$$

Proof. The necessity follows directly from Lemma 3.5. Now we prove the sufficiency. Let $A = V(Y)$, and the sets A_0 and A_1 be the intersection of A and the two partition classes of X. Then $A = A_0 \cup A_1$. Without loss of generality, we assume that the degree of vertices of A_0 in Y is $k-1$ and the degree of vertices of A_1 in Y is k. Then $|\omega(A)| = |A_0|$. Since $(k-1)|A_0| = k|A_1|$ and $|A_0|+|A_1| = |V(Y)| \geq 2k$, we have $|A_0| > k$. We claim that $k < |A_0| \leq 2k-3$. Otherwise, $|A_0| \geq 2k-2$. Then $k|A_1| = (k-1)|A_0| \geq 2k^2-4k+2$ and so $|A_1| > 2k-4$. It follows that $|V(Y)| = |A_0| + |A_1| > 2k - 2 + 2k - 4 = 4k - 6$, contradicting the assumption of the proposition. Thus, $k < |\omega(A)| \leq 2k - 3$. Since Y is semiregular bipartite subgraph with degree k and $k-1$, every component of Y has at least $2k-1$ vertices. Thus, if Y has at least two components, then Y has at least $4k - 2$ vertices, contradicting $|V(Y)| \leq 4k - 7$. It follows that Y is connected. We claim that there exists at least one non-trivial component in $X \backslash \omega(A)$ other than Y. Otherwise, all components other than Y in $X \backslash \omega(A)$ are trivial. If there is only one isolated vertex in $X \backslash \omega(A)$, then $|\omega(A)| = k$. If there are at least two isolated vertices in $X \backslash \omega(A)$, then $|\omega(A)| \geq 2k$. Both contradicts the inequality $k < |\omega(A)| \leq 2k - 3$. Let Z be a non-trivial component in $X \backslash \omega(A)$ other than Y and $B = V(Z)$, then $\omega(B) \subset \omega(A)$, and $\omega(B)$ is a restricted edge cut. It follows that $\lambda'(X) \leq |\omega(B)| \leq |\omega(A)| \leq 2k - 3$. The result follows. □

Corollary 3.7. *Let $X = BC(G, S)$ be a connected Bi-Cayley graph which is not vertex-transitive. Then X is optimally super-λ.*

Proof. By contradiction. If X is not optimally super-λ, by Lemma 3.5, it contains a semiregular bipartite subgraph Y with degree k and $k - 1$ which is induced by a λ'-atom $A = A_0 \cup A_1$, and $(k - 1)|A_0| = k|A_1|$. Clearly, $|A_0| = ka$ and $|A_1| = (k - 1)a$ for some positive integer a. These are impossible since $k + 1 \leq |A_0| \leq 2k - 3$ and $k - 1 \leq |A_1| \leq 2k - 4$. □

Combining Corollary 3.2 and Corollary 3.7, we have the following result.

Theorem 3.8. *Except cycles, all connected Bi-Cayley graphs are optimally super-λ.*

Acknowledgements

We are grateful to an anonymous referee for his valuable suggestions which improved the presentation.

References

[1] C. Balbuena and A. Carmona, On the connectivity and superconnectivity of bipartite digraphs and graphs, Ars Combin. 61 (2001) 3-21.

[2] C. Balbuena, A. Carmona, J. Fàbrega and M.A. Fiol, Superconnectivity of bipartite digraphs and graphs, Discrete Math. 197/198 (1999) 61-75.

[3] D. Bauer, F. Boesch, C. Suffel, R. Tindell, Connectivity extremal problems and the design of reliable probabilistic networks, The Theory and Application of Graphs, Wiley, New York, 1981, pp. 89-98.

[4] F.T. Boesch, Synthesis of reliable networks-A survey, IEEE Trans. Reliab. 35 (1986) 240-246.

[5] F.T. Boesch and R. Tindell, Circulants and their connectivities, J. Graph Theory 8 (1984) 487-499.

[6] A.H. Esfahanian and S.L. Hakimi, On computing a conditional edge-connectivity of a graph, *Inf. Process. Lett.* 27 (1988) 195-199.

[7] M.A. Fiol, J. Fàbrega, and M. Escudero, Short paths and connectivity in graphs and digraphs, Ars Combin. 29B (1990) 17-31.

[8] M.A. Fiol, On super-edge-connected digraphs and bipartite digraphs, J. Graph Theory 16 (1992) 545-555.

[9] F. Harary, Conditional connectivity, Networks 13 (1983) 346-357.

[10] Z. Lu, C. Wang, M. Xu, Semisymmetric cubic graphs constructed from Bi-Cayley graphs of A_n, Ars Combin, to appear.

[11] J. Meng, Connectivity of vertex and edge transitive graphs, Discrete Applied Mathematics. 127 (2003) 601-613.

[12] J. Meng, Optimally super-edge-connected transitive graphs, Discrete Math. 260 (2003) 239-248.

[13] R. Tindell, Connectivity of Cayley graphs, in: D.Z. Du, D.F. Hsu(Eds.), Combinatorial Network Theory, Kluwer, Dordrecht, 1996, pp. 41-64.

[14] M. Xu, Introduction of Finite Groups, Volume II, Science Press. Beijing, 1999.