Edge-antimagicness for a class of disconnected graphs

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Abstract

Suppose G is a finite graph with vertex-set $V(G)$ and edge-set $E(G)$. An (a,d)*-edge-antimagic total labeling* on G is a one-to-one map f from $V(G) \cup E(G)$ onto the integers $1, 2, \ldots, |V(G)| + |E(G)|$ with the property that the edge-weights $w(uv) = f(u) + f(v) + f(uv)$, $uv \in E(G)$, form an arithmetic progression starting from a and having common difference d. Such a labeling is called *super* if the smallest labels appear on the vertices. In this paper, we investigate the existence of super (a,d)-edge-antimagic total labelings of disjoint union of multiple copies of complete bipartite graph.

1 Introduction and Definitions

All graphs are finite, simple and undirected. The graph G has vertex-set $V(G)$ and edge-set $E(G)$. Unless otherwise noted, $|V(G)| = p$ and $|E(G)| = q$. General references for graph-theoretic notions are [15] or [16].

A labeling of a graph is any map that carries some set of graph elements to numbers (usually to the positive or non-negative integers). If the domain is the vertex-set or the edge-set, the labelings are called, respectively, *vertex labelings* or *edge labelings*. Moreover, if the domain is $V(G) \cup E(G)$ then the labeling is called *total labeling*.

We define *edge-weight* of an edge uv under a vertex labeling to be the sum of the vertex labels corresponding to the vertices u an v. Under a total labeling, we also add the label of the edge uv.

By an (a,d)*-edge-antimagic vertex labeling* of graph G we mean a one-to-one map $f : V(G) \to \{1, 2, \ldots, p\}$ such that the set of edge-weights of all edges in G,

$\{f(u)+f(v) : uv \in E(G)\}$, is $\{a, a+d, a+2d, \ldots, a+(q-1)d\}$, for two integers $a > 0$ and $d \geq 0$. (a,d)-edge-antimagic vertex labeling is equivalent to a strongly (a,d)-indexable labeling defined by Hegde in his Ph.D. thesis (see Acharya and Hegde [1]).

An (a,d)-*edge-antimagic total labeling* on a (p,q) graph G is a one-to-one map f from $V(G) \cup E(G)$ onto the integers $1, 2, \ldots, p+q$ with the property that the edge-weights $w(uv) = f(u) + f(uv) + f(v)$, $uv \in E(G)$, form an arithmetic progression $\{a, a+d, a+2d, \ldots, a+(q-1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers. Such a labeling is called *super* if the vertex labels are the integers $\{1, 2, \ldots, p\}$.

A graph G is called (a,d)-edge-antimagic total or super (a,d)-edge-antimagic total if there exists an (a,d)-edge-antimagic total or super (a,d)-edge-antimagic total labeling of G.

The $(a,0)$-edge-antimagic total labelings are usualy called *edge-magic* in the literature (see [2], [8], [12] and [14]).

Definitions of (a,d)-edge-antimagic total labeling and super (a,d)-edge-antimagic total labeling were introduced by Simanjuntak *et al.* [13]. These labelings are natural extensions of the notion of edge-magic labeling (see [11], where edge-magic labeling is called magic valuation) and super edge-magic labeling introduced by Enomoto *et al.* in [7].

Many other researchers investigated different forms of antimagic graphs. For example, see Bodendiek and Walther [4] and [5], and Hartsfield and Ringel [10].

In this paper, we investigate the existence of super (a,d)-edge-antimagic total labelings of disjoint union of multiple copies of complete bipartite graph.

2 Upper bound for difference

In [3] it is proved that the complete bipartite graph $K_{n,n}$ has super (a,d)-edge-antimagic total labeling if and only if $d = 1$ and $n \geq 2$. Let $mK_{n,n}$ be a disjoint union of m copies of $K_{n,n}$ with $V(mK_{n,n}) = \{x_i^k : 1 \leq i \leq n \text{ and } 1 \leq k \leq m\} \cup \{y_j^k : 1 \leq j \leq n \text{ and } 1 \leq k \leq m\}$ and $E(mK_{n,n}) = \bigcup_{k=1}^{m} \{x_i^k y_j^k : 1 \leq i \leq n \text{ and } 1 \leq j \leq n\}$.

Thus $|V(mK_{n,n})| = 2mn$ and $|E(mK_{n,n})| = mn^2$.

The next theorem provides an upper bound for the parameter d for a super (a,d)-edge-antimagic total labeling of $mK_{n,n}$.

Theorem 1 *If $mK_{n,n}$, $n \geq 1$ and $m \geq 2$, is super (a,d)-edge-antimagic total, then $d \leq 5$.*

Proof Let $mK_{n,n}$, $n \geq 1$ and $m \geq 2$, be super (a,d)-edge-antimagic total with a super (a,d)-edge-antimagic total labeling $f : V(mK_{n,n}) \cup E(mK_{n,n}) \to \{1, 2, \ldots, 2mn+mn^2\}$. Then $\{w(uv) = f(u)+f(uv)+f(v) : uv \in E(mK_{n,n})\} = \{a, a+d, a+2d, \ldots, a+(mn^2-1)d\}$ is the set of edge-weights and

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m} w(x_i^k y_j^k) = \sum_{t=0}^{mn^2-1} (a+td) = mn^2\left(a + \frac{(mn^2-1)d}{2}\right) \quad (1)$$

is the sum of all the edge-weights.
In the computation of the edge-weights of $mK_{n,n}$ under the map f the label of each vertex is used n times and the label of each edge is used once. Thus

$$n\sum_{i=1}^{n}\sum_{k=1}^{m} f(x_i^k) + n\sum_{j=1}^{n}\sum_{k=1}^{m} f(y_j^k) + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m} f(x_i^k y_j^k) =$$

$$= mn^2\left(1 + 2mn + \frac{mn^2+4mn+1}{2}\right). \quad (2)$$

Combining (1) and (2) gives

$$2a + (mn^2-1)d = mn^2 + 8mn + 3. \quad (3)$$

The minimum possible edge-weight under the labeling f is at least $2mn+4$. So, $a \geq 2mn+4$, and from (3) it follows that

$$d \leq 1 + \frac{4mn-4}{mn^2-1}. \quad (4)$$

If $n \geq 1$ and $m \geq 2$, then $\frac{4(mn-1)}{mn^2-1} \not> 4$ and thus $d \leq 5$. □

Theorem 2 *The graph $mK_{n,n}$, $m \geq 2$, is super $(3m+3,3)$-edge-antimagic total and super $(2m+4,5)$-edge-antimagic total if and only if $n = 1$.*

Proof Suppose that $mK_{n,n}$, $m \geq 2$ is super (a,d)-edge-antimagic total for $d \in \{3,5\}$. If $d=3$ or $d=5$ then from (3) it follows that

$$a = 4mn - mn^2 + 3 \quad (5)$$

or

$$a = 4mn - 2mn^2 + 4, \quad (6)$$

respectively. Since $a \geq 2mn+4$ then (5) and (6) give inequalities

$$mn(2-n) \geq 1 \quad (7)$$

and

$$mn(1-n) \geq 0, \quad (8)$$

respectively. The inequalities (7) and (8) are true only for $n=1$ and $m \geq 2$. Using $n=1$ and $d=3$ or $d=5$ the equations (5) and (6) imply that $a = 3m+3$ or $a = 2m+4$, respectively.

Now, define the vertex labeling $f_1 : V(mK_{1,1}) \to \{1, 2, \ldots, 2m\}$ in the following way:

$$f_1(x_1^k) = 2k \quad \text{for} \quad 1 \le k \le m$$
$$f_1(y_1^k) = 2k - 1 \quad \text{for} \quad 1 \le k \le m.$$

The set of edge-weights is $\{f_1(x_1^k) + f_1(y_1^k) : 1 \le k \le m\} = \{4k - 1 : 1 \le k \le m\}$. Let us now construct the edge labelings g_1 and g_2 of $mK_{1,1}$ with values in the set $\{2m + 1, 2m + 2, \ldots, 3m\}$ as follows:

$$g_1(x_1^k y_1^k) = 3m + 1 - k \quad \text{for} \quad 1 \le k \le m$$

$$g_2(x_1^k y_1^k) = 2m + k \quad \text{for} \quad 1 \le k \le m.$$

It can be seen that combining the vertex labeling f_1 and the edge labeling g_1 or g_2 gives a super $(3m+3, 3)$-edge-antimagic total labeling or a super $(2m+4, 5)$-edge-antimagic total labeling, respectively. □

Theorem 3 *The graph $mK_{n,n}$ has a super $(a, 4)$-edge-antimagic total labeling if and only if $n = 1$, m is odd, $m \ge 3$, and $a = \frac{5m+7}{2}$.*

Proof Assume that $mK_{n,n}$ has a super $(a, 4)$-edge-antimagic total labeling. If $d = 4$ then from equation (3) we have

$$a = 4mn + \frac{7 - 3mn^2}{2}. \tag{9}$$

So, the minimum edge-weight a is an integer if and only if m and n are odd. Since $a \ge 2mn + 4$ then from (9) it follows that $mn(4 - 3n) \ge 1$ and it holds only for $n = 1$.
If $n = 1$, then (9) becames $a = \frac{5m+7}{2}$. The wanted super $(\frac{5m+7}{2}, 4)$-edge-antimagic total labeling f_2 can be defined as follows:

$$f_2(x_1^k) = \begin{cases} \frac{3m+3}{2} - 2k, & \text{if } 1 \le k \le \frac{m+1}{2} \\ m + 1 - k, & \text{if } \frac{m+3}{2} \le k \le m \end{cases}$$

$$f_2(y_1^k) = \begin{cases} 2m + 1 - k, & \text{if } 1 \le k \le \frac{m+1}{2} \\ \frac{5m+3}{2} - 2k, & \text{if } \frac{m+3}{2} \le k \le m \end{cases}$$

$$f_2(x_1^k y_1^k) = g_1(x_1^k y_1^k) \quad \text{for} \quad 1 \le k \le m.$$

□

Theorem 4 *There is a super $(4mn + 2, 1)$-edge-antimagic total labeling for $mK_{n,n}$ for every $n \ge 1$ and every $m \ge 2$.*

Proof If $d = 1$ then from (3) it follows that $a = 4mn + 2$. Construct the one-to-one map $f_3 : V(mK_{n,n}) \cup E(mK_{n,n}) \rightarrow \{1, 2, \ldots, 2mn + mn^2\}$ as follows:

$$f_3(x_i^k) = (i-1)m + k \quad \text{for} \quad 1 \le i \le n \quad \text{and} \quad 1 \le k \le m$$

$$f_3(y_j^k) = (n + j - 1)m + k \quad \text{for} \quad 1 \le j \le n \quad \text{and} \quad 1 \le k \le m.$$

If $1 \le k \le m$ then

$$f_3(x_i^k y_j^k) = (i - j + 3)mn - m\left(j - 1 + \sum_{t=0}^{i-j} t\right) - k + 1$$

for $1 \le j \le n$ and $j \le i \le n$, and

$$f_3(x_i^k y_j^k) = \left(\frac{n+5}{2} + j - i\right)mn - m\left(i - 1 + \sum_{t=1}^{j-i} t\right) - k + 1$$

for $1 \le i \le n-1$ and $i + 1 \le j \le n$.
Thus, for $1 \le j \le n$ and $j \le i \le n$ the edge-weight of $x_i^k y_j^k$, where $1 \le k \le m$, is given by

$$w(x_i^k y_j^k) = f_3(x_i^k) + f_3(y_j^k) + f_3(x_i^k y_j^k) = mn(4 + i - j) + m\left(i - 1 - \sum_{t=0}^{i-j} t\right) + k + 1$$

and for $1 \le i \le n-1$ and $i+1 \le j \le n$ the edge-weight of $x_i^k y_j^k$, where $1 \le k \le m$, is given by

$$\overline{w}(x_i^k y_j^k) =$$

$$f_3(x_i^k) + f_3(y_j^k) + f_3(x_i^k y_j^k) = \tfrac{mn}{2}(n + 7 + 2j - 2i) + m\left(j - 1 - \sum_{t=1}^{j-i} t\right) + k + 1.$$

The edge-weights of edges of $mK_{n,n}$ we can exhibit by a system of square matrices $H^k = (h_{i,j}^k)$, for $k = 1, 2, \ldots, m$, with dimensions $(n \times n)$ each, where

$$h_{i,j}^k = \begin{cases} w(x_i^k y_j^k), & \text{if } 1 \le j \le n \text{ and } j \le i \le n \\ \overline{w}(x_i^k y_j^k), & \text{if } 1 \le i \le n-1 \text{ and } i+1 \le j \le n. \end{cases}$$

We are setting $A = 4mn + k + 1$, $B = A + mn$, $C = \frac{mn^2 + 9mn}{2} + k + 1$, $D = \frac{mn^2 + 11mn}{2} + k + 1$, $E = A + 2mn$ and $F = mn^2 + 4mn + k + 1$.

$$H^k = \begin{pmatrix} A & C & D-m & \ldots & F-3m & F-m \\ B & A+m & C+m & \ldots & F-5m & F-2m \\ E-m & B+m & A+2m & \ldots & F-8m & F-4m \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ C-3m & C-5m & C-8m & \ldots & B-2m & D-2m \\ C-m & C-2m & C-4m & \ldots & E-2m & B-m \end{pmatrix}.$$

It is a routine procedure to verify that the system of square matrices H^k, $k = 1, 2, \ldots, m$, is formed from consecutive integers $4mn + 2, 4mn + 3, 4mn +$

$4, \ldots, mn^2 + 4mn, mn^2 + 4mn + 1$. We deduce that f_3 is a super $(4mn + 2, 1)$-edge-antimagic total labeling of $mK_{n,n}$. □

Lemma 1 *If $mK_{n,n}$ is super (a, d)-edge-antimagic total for $d \in \{0, 2\}$, then $n = 1$ or $n = 3$ and m is odd, $m \geq 3$.*

Proof *Case 1* $d = 0$.
Figueroa-Centeno *et al.* [8] showed that a (p, q) graph G is super $(a, 0)$-edge-antimagic total if and only if there exists $(a - p - q, 1)$-edge-antimagic vertex labeling.
Suppose that $f : V(mK_{n,n}) \to \{1, 2, \ldots, p\}$ is an $(a - p - q, 1)$-edge-antimagic vertex labeling and let $\{a-p-q, a-p-q+1, \ldots, a-p-q+q-1\}$ be the set of all edge-weights in $mK_{n,n}$, where $p = 2mn$ and $q = mn^2$. The sum of all vertex labels used to calculate the edge-weights is equal to the sum of the edge-weights. So,

$$\sum_{uv \in E(mK_{n,n})} w(uv) = n \sum_{u \in V(mK_{n,n})} f(u)$$

which is obviously equivalent to the equation

$$amn^2 - \frac{mn^2}{2}\left(4mn + mn^2 + 1\right) = mn^2(2mn + 1)$$

and

$$a - p - q = a - 2mn - mn^2 = \frac{mn(4 - n) + 3}{2}.$$

It can be seen that the minimum possible edge-weight $a - p - q$ is a positive integer if and only if m is odd and $n = 1$ or $n = 3$.

Case 2 $d = 2$.
Let $mK_{n,n}$ be super $(a, 2)$-edge-antimagic total. For $d = 2$ from (3) we obtain

$$a = 4mn + \frac{5 - mn^2}{2} \tag{10}$$

and it is an integer if and only if m and n are odd.
Using $a \geq 2mn + 4$ the equation (10) implies that $mn(4 - n) \geq 3$. In light the previous condition for m and n we can see that the last inequality is true only for $n \in \{1, 3\}$ and m odd, $m \geq 3$.
This completes the proof. □

Super $(a, 0)$-edge-antimagic total labeling of $mK_{1,1}$ can be found in [6] and [9].

Theorem 5 *If m is odd, $m \geq 3$, then $mK_{1,1}$ has a super $(\frac{7m+5}{2}, 2)$-edge-antimagic total labeling.*

Proof If $n = 1$ then from (10) it follows that $a = \frac{7m+5}{2}$. The required super $(\frac{7m+5}{2}, 2)$-edge-antimagic total labeling we define in the following way:

$$f_4(x_1^k) = m + 1 - k \quad \text{for} \quad 1 \le k \le m$$

$$f_4(y_1^k) = \begin{cases} m + \frac{k+1}{2}, & \text{if } k \text{ is odd} \\ \frac{3m+1+k}{2}, & \text{if } k \text{ is even} \end{cases}$$

$$f_4(x_1^k y_1^k) = \begin{cases} \frac{5m+2-k}{2}, & \text{if } k \text{ is odd} \\ 3m + 1 - \frac{k}{2}, & \text{if } k \text{ is even.} \end{cases}$$

□

For $mK_{3,3}$, $m \ge 3$ odd, we have not found any super $(a, 0)$-edge-antimagic total labeling. Therefore we propose the following open problem.

Open Problem 1 *For $mK_{3,3}$, $m \ge 3$ odd, determine if there is a super $(a, 0)$-edge-antimagic total labeling.*

The following three tables for $k = 1, 2, 3$ descibe a super $(25, 2)$-edge-antimagic total labeling for $3K_{3,3}$. Each table contains the vertex labels x_1^k, x_2^k and x_3^k in the first row, the vertex labels y_1^k, y_2^k and y_3^k in the first column and the edge labels $x_i^k y_j^k$ for $i, j \in \{1, 2, 3\}$ in the internal array of order 3.

	4	5	6
1	28	19	20
2	29	30	31
3	34	21	22

	12	13	14
8	23	24	25
9	32	33	36
10	35	26	27

	11	17	18
7	45	37	40
15	41	43	44
16	42	38	39

We have not yet found a convenient construction that will produce super $(a, 2)$-edge-antimagic total labeling for $mK_{3,3}$, $a = \frac{15m+5}{2}$, for all odd m. However, the existence of super $(25, 2)$-edge-antimagic total labeling for $3K_{3,3}$ leads us to suggest the following

Conjecture 1 *There is a super $(\frac{15m+5}{2}, 2)$-edge-antimagic total labeling for $mK_{3,3}$ for all m odd.*

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