Regular matroids without disjoint circuits

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June 29, 2006

Abstract

A cosimple regular matroid M does not have disjoint circuits if and only if $M \in \{M(K_{3,3}), M^*(K_n)\ (n \geq 3)\}$. This extends a former result of Erdös and Pósa on graphs without disjoint circuits.

Key words: regular matroid, disjoint circuits.

1 Introduction

We shall assume familiarity with graph theory and matroid theory. For terms that are not defined in this note, see Bondy and Murty [1] for graphs, and Oxley [3] or Welsh [6] for matroids. We allow graphs to have multiple edges but we forbid loops. To be consistent with the matroid terminology, a *circuit* in a graph is a nontrivial 2-regular connected subgraph, and a *cycle* is a disjoint union of circuits.

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If G is a graph and if V_1, V_2 are two disjoint vertex subsets of G, then $[V_1, V_2]$ denote the set of edges in G with one end in V_1 and the other end in V_2. For a vertex $v \in V(G)$, let

$$E_G(v) = \{e \in E(G) : e \text{ is incident with } v\}.$$

Let M and N denote two matroids. If $\{e, f\}$ is a circuit of M^* and if $M/f = N$, then M is a *serial extension* of N. In this case, we say that f is serial to e. Note that being serial is an equivalence relation on $E(M)$ for a matroid M. The corresponding equivalence classes are the *serial classes* of M. Dually, two elements e, f are *parallel* in M if they are serial in M^*; being parallel is an equivalence relation on $E(M)$ and the equivalence classes are the *parallel classes* of M. An equivalence class is *nontrivial* if it has more than one elements.

In 1960, Erdös and Pósa consider the problem of determining all connected graphs that do not have edge-disjoint circuits. We view the complete graph K_3 as a plane graph and let K_3^* denote the geometric dual of the plane graph K_3.

Theorem 1.1 *(Erdös and Pósa [2]) Let G be a graph with $\delta(G) \geq 3$. The following are equivalent.*
(i) G does not have edge-disjoint circuits.
(ii) $G \in \{K_{3,3}, K_3^*, K_4\}$.

Since a graph G does not have disjoint circuits if and only if any subdivision of G does not have disjoint circuits, the following corollary follows immediately.

Corollary 1.2 *(Erdös and Pósa [2]) Let G be a simple graph of order $n \geq 3$.*
(i) If $|E(G)| \geq n + 4$, then G has 2 edge-disjoint circuits.
(ii) The graph G with $|E(G)| = n + 3$ does not have edge-disjoint circuits if and only if G can be obtained from a subdivision G_0 of $K_{3,3}$ by adding a forest and exactly one edge, joining each tree of the forest to G_0.

Theorem 1.1 can be viewed as a result on cosimple graphic matroids. Thus we consider generalizing Theorem 1.1. to matroids. Our main results of this note are the following.

Theorem 1.3 *Let M be a connected cosimple regular matroid. The following are equivalent.*

(i) M does not have disjoint circuits.
(ii) $M \in \{M(K_{3,3})\} \cup \{M^*(K_n), n \geq 3\}$.

Corollary 1.4 *Let M be a regular matroid. Then M has no disjoint circuits if and only if one of the following holds:*
(i) $M = U_{m,m}$, *for some integer* $m > 0$, *or*
(ii) M is a serial extension of a member in $\{M(K_{3,3}), U_{0,1}\} \cup \{M^*(K_n), n \geq 3\}$, *or*
(iii) $M = M_1 \bigoplus M_2$ *is the direct sum of two matroids* M_1 *and* M_2, *where* M_1 *is a serial extension of a member in* $\{M(K_{3,3}), U_{0,1}\} \cup \{M^*(K_n), n \geq 3\}$ *and where* $M_2 \cong U_{m,m}$, *for some* $m = |E(M)| - |E(M_1)| \geq 1$.

2 Proof of the Main Result

We follow Seymour [5] to introduce the notion of binary matroid sums. Given two sets X and Y, the symmetric difference of X and Y, is

$$X \Delta Y = (X \cup Y) - (X \cap Y).$$

Let M_1 and M_2 be two binary matroids where $E(M_1)$ and $E(M_2)$ may intersect. Define $M_1 \Delta M_2$ to be the binary matroid on $E = E(M_1) \Delta E(M_2)$ whose cycles are the nonempty, minimal subsets of E of the form $X_1 \Delta X_2$, where for each $i = 1, 2$, X_i is a disjoint union of circuits of M_i. The binary matroid sums are defined as follows.
(i) If $E(M_1) \cap E(M_2) = \emptyset$, then $M_1 \Delta M_2$ is the *1-sum* of M_1 and M_2 (also referred as a direct sum).
(ii) If $E(M_1) \cap E(M_2) = \{e_0\}$, such that, for each $i \in \{1, 2\}$, the element e_0 is neither a loop nor a coloop of M_i, then $M_1 \Delta M_2$ is the *2-sum* of M_1 and M_2.
(iii) If $E(M_1) \cap E(M_2) = C$, where C is a 3-circuit of both M_1 and M_2, such that C includes no cocircuit of either M_1 or M_2, and such that for $i \in \{1, 2\}$, $|E(M_i)| \geq 7$, then $M_1 \Delta M_2$ is the *3-sum* of M_1 and M_2.

For $k = 1, 2, 3$, we also use $M_1 \bigoplus_k M_2$ to denote the k-sum of two matroids M_1 and M_2. If each of M_1 and M_2 is isomorphic to a proper minor of $M_1 \bigoplus_k M_2$, then we say that M is a *proper k-sum* of M_1 and M_2. For the case k=1, we also use $M_1 \bigoplus M_2$ for $M_1 \bigoplus_1 M_2$ to denote the direct sum of M_1 and M_2.

Let A denote the matrix below

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix},$$

and let R_{10} denote the binary matroid $M_2[A]$.

Seymour's regular matroid decomposition theorem can be applied to cosimple matroids in the following form.

Theorem 2.1 *(Seymour [4]) Let M be a cosimple connected regular matroid. Then one of the following holds.*
(i) M is cosimple and graphic.
(ii) M is cosimple and cographic.
(iii) M is isomorphic to R_{10}.
(iv) For $i \in \{2,3\}$, $M = M_1 \bigoplus_k M_2$ is the proper 2-sum or 3-sum of two cosimple regular matroids M_1 and M_2, where both M_1 and M_2 are isomorphic to proper minors of M.

The following lemma is straightforward.

Lemma 2.2 *Let G be a graph. If $M(G)$ is cosimple, then $\delta(G) \geq 3$.*

Proof: Note that any edge incident with a degree 1 vertex in G must be a loop of $M^*(G)$, and that the edges incident with a degree 2 vertex in G must be in a parallel class of $M^*(G)$. Since $M(G)$ is cosimple, $M^*(G)$ does not have loops or nontrivial parallel classes. Hence we must have $\delta(G) \geq 3$. □

Proof of Theorem 1.3 We first show that Theorem 1.3(i) implies Theorem 1.3(ii), and so we assume the M is a connected cosimple regular matroid with no disjoint circuits. By Theorem 2.1, one of the conclusions in Theorem 2.1 must hold.

If M is graphic, then we may assume that for some connected graph G, $M = M(G)$. By Lemma 2.2, $\delta(G) \geq 3$. Since G has no disjoint circuits, by Theorem 1.1, $G \in \{K_{3,3}, K_3^*, K_4\}$, and so Theorem 1.3(ii) holds.

If M is cographic, then we may assume that for some graph G, $M = M^*(G)$, where G is a connected graph with $n = r(M) + 1$ vertices. Since M is cosimple, G is a simple graph, and so G is a spanning subgraph of K_n, the complete graph on n vertices. Let $V(G) = \{v_1, v_2, \cdots v_n\}$. If $G \neq K_n$, then we may assume that $v_1 v_2 \notin E(G)$. In this case, $E_G(v_1) \cap E_G(v_2) = \emptyset$, contrary to Theorem 1.3(i). Therefore, we must have $G = K_n$, and so $M \in \{M^*(K_n), n \geq 3\}$.

If M is isomorphic to R_{10}, then it is well known that R_{10} is a disjoint union of a 4-circuit and a 6-circuit, contrary to Theorem 1.3(i). Thus $M \cong R_{10}$ is impossible.

Now suppose that 2.1(iv) holds. We argue by induction on $|E(M)|$. Since any matroid with at most 3 elements must be graphic, we assume that $|E(M)| = n \geq 4$, and Theorem 1.3(ii) holds for any matroid M satisfying Theorem 1.3(i) with $|E(M)| < n$.

Since Theorem 2.1(iv) holds, for some $i \in \{2,3\}$, $M = M_1 \bigoplus_i M_2$ is the proper i-sum of two cosimple regular matroids M_1 and M_2, where both M_1 and M_2 are proper minors of M.

If one of M_1 or M_2 has two disjoint circuits, then by the definition of binary matroid sums, M would also have disjoint circuits, contrary to Theorem 1.3(i). Therefore, for each i, M_i does not have disjoint circuits. Since M_i is a proper minor of M, by induction, $M_1, M_2 \in \{M(K_{3,3})\} \cup \{M^*(K_n), n \geq 3\}$.

If $i = 2$, then we may assume that $e_0 \in E(M_1) \cap E(M_2)$. By the definition of 2-sum and by the fact that $M_1, M_2 \in \{M(K_{3,3})\} \cup \{M^*(K_n), n \geq 3\}$, $\exists C_1 \in \mathcal{C}(M_1)$ and $C_2 \in \mathcal{C}(M_2)$ such that $e_0 \notin C_i$. It follows that $C_1 \cap C_2 = \emptyset$ and so Theorem 1.3(i) is violated. Thus this is impossible.

Now assume that $i = 3$, and $Z = E(M_1) \cap E(M_2)$ is a 3 element circuit of both M_1 and M_2. Recall that $M_1, M_2 \in \{M(K_{3,3})\} \cup \{M^*(K_n), n \geq 3\}$. By the definition of a 3-sum, for any $i \in \{1,2\}$, $|E(M_i)| \geq 7$ and so $M_i \notin \{M^*(K_3), M^*(K_4)\}$. Since there is no 3-circuits in either $M(K_{3,3})$ or a $M^*(K_n)$ with $n > 4$, it is impossible that both $|Z| = 3$ and $Z \in \mathcal{C}(M_1) \cap \mathcal{C}(M_2)$. This contradiction shows that this case is also impossible.

Thus if Theorem 1.3(i) holds, then we must have $M \in \{M(K_{3,3})\} \cup \{M^*(K_n), n \geq 3\}$.

Conversely, suppose $M \in \{M(K_{3,3})\} \cup \{M^*(K_n), n \geq 3\}$. Since $K_{3,3}$ is a bipartite simple graph, any circuit of $K_{3,3}$ has length at least 4. Suppose that $K_{3,3}$ has two disjoint circuits C_1 and C_2, then since $K_{3,3}$ is 3-regular, we must have $V(C_1) \cap V(C_2) = \emptyset$, and so $6 = |V(K_{3,3})| \geq |V(C_1)| + |V(C_2)| \geq 8$, a contradiction. Hence $M(K_{3,3})$ cannot have disjoint circuits. Suppose that $M = M^*(K_n), n \geq 3$ and write $V(K_n) = \{v_1, v_2, \cdots, v_n\}$. Suppose that C_1 and C_2 are two circuits of $M^*(K_n)$. Then C_1 is an edge cut of K_n and so $C_1 = [V_1, V_2]$, for some proper vertex subset $V_1 \subseteq V(G)$ and $V_2 = V(G) - V_1$. Similarly, $C_2 = [W_1, W_2]$, where $\emptyset \neq W_1 \subseteq V(G)$ and $W_2 = V(G) - W_1 \neq \emptyset$. We may assume that $v_1 \in V_1 \cap W_1$. If $V_2 \cap W_2 \neq \emptyset$, say $v_2 \in V_2 \cap W_2$, then $v_1v_2 \in C_1 \cap C_2$. If $V_2 \cap W_2 = \emptyset$, then we have $W_2 \subseteq V_1, V_2 \subseteq W_1$. Since $\emptyset \neq [V_2, W_2] \subseteq [V_2, V_1] = C_1$ and $\emptyset \neq [V_2, W_2] \subseteq [W_1, W_2] = C_2$, then $C_1 \cap C_2 \neq \emptyset$. This proves that $M^*(K_n)$ does not have disjoint circuits. □

Proof of Corollary 1.4 It suffices to show, by induction on $|E(M)|$, that if M has no disjoint circuits, then one of (i), (ii) and (iii) holds. Let M be a regular matroid that does not have disjoint circuits.

We first assume that M is connected. If M has a loop or a coloop, then since M is connected, we must have $M \in \{U_{0,1}, U_{1,1}\}$, and so Corollary 1.4 (i) or (ii) must hold. Thus we assume that M is loopless and coloopless.

If M is connected and cosimple, then by Theorem 1.3, M is a member of $\{M(K_{3,3})\} \cup \{M^*(K_n), n \geq 3\}$ and so Corollary 1.4(ii) holds. Otherwise, M has nontrivial serial classes. Let $\{e_1, e_2\}$ be a pair of serial elements in M. Since the intersection of any circuit and any cocircuit in a matroid M cannot have exactly one element, any circuit in M containing e_1 must also contain e_2. This implies that M has no disjoint circuits if and only if M/e_2 has no disjoint circuits. By induction, M/e_2 is a serial extension of a member in $\{M(K_{3,3}), U_{0,1}\} \cup \{M^*(K_n), n \geq 3\}$. Since M is a serial extension of M/e_2, M is also a serial extension of a member in $\{M(K_{3,3}), U_{0,1}\} \cup \{M^*(K_n), n \geq 3\}$.

Now suppose that M is not connected. Then $M = M_1 \bigoplus M_2 \bigoplus \cdots \bigoplus M_k$, where $M_1, M_2, \cdots, M_k$ are connected components of M. If $\forall i$, M_i contains no circuits, then Corollary 1.4(i) holds. Otherwise, since M has no disjoint circuits, exactly one connected component, say M_1, has at least one circuit. It follows that $M_2 \bigoplus \cdots \bigoplus M_k \cong U_{n,n}$ and so Corollary 1.4 (iii) must hold. □

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