Orientations of graphs and minimum degrees of graphs

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Abstract. Katerinis established the following result in [1]. Let G be a simple graph with $\delta(G) \geq \left\lfloor \frac{|V(G)|}{2} \right\rfloor + k$, where k is a non-negative integer. Let $f : V(G) \to \mathbb{Z}^+$ be a function having the following properties:

(1) $\frac{1}{2}(d_G(v) - (k+1)) \leq f(v) \leq \frac{1}{2}(d_G(v) + (k+1))$ for every $v \in V(G)$,

(2) $\sum_{v \in V(G)} f(v) = |E(G)|$.

Then G has an orientation D such that $d_D^+(v) = f(v)$, for every $v \in V(G)$. In this paper, we focus on the sharpness of the above two inequalities.

1 Introduction

Let G be a simple graph. For $v \in V(G)$, $N_G(v)$ is the set of neighbours of v in G and $d_G(v)$, the degree of v in G, is the cardinality of $N_G(v)$. For a graph G, $\delta(G)$ is the minimum degree of G. For $v \in V(G)$ and $S \subset V(G)$, $d_{G[S \cup \{v\}]}(v)$ is the degree of v in $G[S \cup \{v\}]$, the subgraph induced by $S \cup \{v\}$ in G. If S and T are disjoint sets of vertices of G, we write $e_G(S,T)$ for the number of edges of G having one end in S and the other end in T. For $S \subseteq V(G)$, $e_G(S,S)$ denotes the number of edges of G having both ends in S, i.e., the number of edges of the induced subgraph $G[S]$.

The out-degree of a vertex v in a digraph D is denoted by $d_D^+(v)$. An *orientation* of a graph G is a digraph obtained from G by assigning to each edge in G a direction. We use $\mathbb{Z}^+$ for the set of positive integers.

The following theorem was proved in [1].

Theorem 1.1. (Katerinis [1]). *Let* G *be a simple graph with* $\delta(G) \geq \left\lfloor \frac{|V(G)|}{2} \right\rfloor + k$, *where* k *is a non-negative integer. Let* $f : V(G) \to \mathbb{Z}^+$ *be a function having the following properties:*

(1) $\frac{1}{2}(d_G(v)-(k+1)) \leq f(v) \leq \frac{1}{2}(d_G(v)+(k+1))$ *for every* $v \in V(G)$,

(2) $\sum_{v \in V(G)} f(v) = |E(G)|$.

Then G *has an orientation* D *such that* $d_D^+(v) = f(v)$, *for every* $v \in V(G)$. ■

The condition $\delta(G) \geq \left\lfloor \frac{|V(G)|}{2} \right\rfloor + k$ of Theorem 1.1 is best possible as it was shown in [1], since there exist a graph G_1 and a function f_1 with

$$\delta(G_1) = \left\lfloor \frac{|V(G_1)|}{2} \right\rfloor - 1 + k,$$
$$\frac{1}{2}(d_{G_1}(v) - (k+1)) \leq f_1(v) \leq \frac{1}{2}(d_{G_1}(v) + (k+1)),$$
$$\sum_{v \in V(G_1)} f_1(v) = |E(G_1)|$$

and there is no orientation D_1 of G_1 such that $d_{D_1}^+(v) = f_1(v)$, for every $v \in V(G_1)$. Similarly, the condition

$$\frac{1}{2}(d_G(v) - (k+1)) \leq f(v) \leq \frac{1}{2}(d_G(v) + (k+1))$$

of Theorem 1.1 is also best possible as it was shown in [1], since there exist a graph G_2 and a function f_2 with

$$\delta(G_2) \geq \left\lfloor \frac{|V(G_2)|}{2} \right\rfloor + k,$$
$$\left\lfloor \frac{1}{2}(d_{G_2}(v) - (k+1)) \right\rfloor \leq f_2(v) \leq \left\lceil \frac{1}{2}(d_{G_2}(v) + (k+1)) \right\rceil,$$
$$\sum_{v \in V(G_2)} f_2(v) = |E(G_2)|$$

and there is no orientation D_2 of G_2 such that $d_{D_2}^+(v) = f_2(v)$, for every $v \in V(G_2)$. In this paper, we consider these two extremal cases with the following additional condition:

$$e_G(W, V(G) - W) \geq \left\lfloor \frac{|V(G)|}{2} \right\rfloor (k+i), \qquad i \in \{1, 2\},$$

for every $W \subseteq V(G)$ with $|W| = \left\lfloor \frac{|V(G)|}{2} \right\rfloor$.

2 Orientations of graphs with preassigned out-degrees

We first state the results.

Theorem 2.1. *Let* G *be a simple graph such that* *(i)* $\delta(G) = \left\lfloor \frac{|V(G)|}{2} \right\rfloor - 1 + k$, *where* k *is a non-negative integer,*

(ii) $e_G(W, V(G) - W) \geq \left\lfloor \frac{|V(G)|}{2} \right\rfloor (k+1)$, *for every* $W \subset V(G)$ *with* $|W| = \left\lfloor \frac{|V(G)|}{2} \right\rfloor$;

further, there exists a function $f : V(G) \to \mathbb{Z}^+$ *with the following properties:*

(1) $\frac{1}{2}(d_G(v) - (k+1)) \leq f(v) \leq \frac{1}{2}(d_G(v) + (k+1))$ *for every* $v \in V(G)$,

(2) $\sum_{v \in V(G)} f(v) = |E(G)|$.

Then G *has an orientation* D *such that* $d_D^+(v) = f(v)$, *for every* $v \in V(G)$. ■

Theorem 2.2. *Let* G *be a simple graph such that*

(i) $\delta(G) \geq \left\lfloor \frac{|V(G)|}{2} \right\rfloor + k$, *where* k *is a non-negative integer,*

(ii) $e_G(W, V(G) - W) \geq \left\lfloor \frac{|V(G)|}{2} \right\rfloor (k+2)$, *for every* $W \subset V(G)$ *with* $|W| = \left\lfloor \frac{|V(G)|}{2} \right\rfloor$;

further, there exists a function $f : V(G) \to \mathbb{Z}^+$ *with the following properties:*

(1) $\frac{1}{2}(d_G(v) - (k+2)) \leq f(v) \leq \frac{1}{2}(d_G(v) + (k+2))$ *for every* $v \in V(G)$,

(2) $\sum_{v \in V(G)} f(v) = |E(G)|$.

Then G *has an orientation* D *such that* $d_D^+(v) = f(v)$, *for every* $v \in V(G)$. ■

The arguments in the proofs of Theorems 2.1 and 2.2 are similar to those used for the proof of Theorem 1.1. For the proofs of Theorems 2.1 and 2.2 we use the following Lemma, which can be found in [1].

Lemma 2.1. *Let* G *be a graph and let function* $f : V(G) \to \mathbb{Z}^+$. *Then* G *has an orientation* D *such that* $d_D^+(v) = f(v)$ *for every* $v \in V(G)$ *if and only if* $\sum_{v \in V(G)} f(v) = |E(G)|$ *and* $\sum_{v \in X} f(v) \leq e_G(X, X) + e_G(X, V(G) - X)$ *for every* $X \subseteq V(G)$. ■

Proof of Theorem 2.1. Suppose that G does not have an orientaion D as stated in the theorem. Then from Lemma 2.1, there exists a $T \subseteq V(G)$ such that

$\sum_{v \in T} f(v) > e_G(T,T) + e_G(T,S)$, (1)

where $S = V(G) - T$.

Claim 1. $(k+1)\,|T| > \sum_{v \in T} d_{G[S \cup \{v\}]}(v)$.

By condition (1) of the hypothesis of the theorem,

$\sum_{v \in T} f(v) \le \sum_{v \in T} \frac{1}{2}(d_G(v) + (k+1))$. (2)

But

$\sum_{v \in T} \frac{1}{2}(d_G(v) + (k+1)) = \frac{1}{2}\sum_{v \in T} d_G(v) + \frac{1}{2}(k+1)\,|T|$ (3)

and

$$e_G(T,T) + e_G(T,S) = \frac{1}{2}\sum_{v \in T} d_{G[T]}(v) + \sum_{v \in T} d_{G[S \cup \{v\}]}(v)$$

$$= \frac{1}{2}\sum_{v \in T} d_G(v) + \frac{1}{2}\sum_{v \in T} d_{G[S \cup \{v\}]}(v). \qquad (4)$$

Combining equations (3), (2), (1) and (4) in order, we get

$\frac{1}{2}(k+1)\,|T| > \frac{1}{2}\sum_{v \in T} d_{G[S \cup \{v\}]}(v)$.

Claim 2. $(k+1)\,|S| > \sum_{v \in T} d_{G[S \cup \{v\}]}(v)$.

By condition (1) of the theorem,

$\sum_{v \in S} \frac{1}{2}(d_G(v) - (k+1)) \le \sum_{v \in S} f(v)$.

This implies, $\frac{1}{2}\sum_{v \in S} d_G(v) - \frac{1}{2}(k+1)\,|S| \le \sum_{v \in S} f(v)$.

Thus, $\frac{1}{2}\sum_{v \in S} d_G(v) - \sum_{v \in S} f(v) \le \frac{1}{2}(k+1)\,|S|$. (5)

By condition (2) of the hypothesis of the theorem,
$\sum_{v \in V(G)} f(v) = |E(G)|$.

As S and T is a partition of $V(G)$,
$\sum_{v \in T} f(v) + \sum_{v \in S} f(v) = \frac{1}{2}\sum_{v \in V(G)} d_G(v)$.

This implies,
$\sum_{v \in T} f(v) + \sum_{v \in S} f(v) = \frac{1}{2}\sum_{v \in T} d_G(v) + \frac{1}{2}\sum_{v \in S} d_G(v)$.

Thus, $\sum_{v \in T} f(v) - \frac{1}{2}\sum_{v \in T} d_G(v) = \frac{1}{2}\sum_{v \in S} d_G(v) - \sum_{v \in S} f(v)$. (6)

Hence, by (5) and (6),

$\sum_{v \in T} f(v) - \frac{1}{2}\sum_{v \in T} d_G(v) \le \frac{1}{2}(k+1)\,|S|$. (7)

But, by (1) and (4),

$$\sum_{v \in T} f(v) > e_G(T,T) + e_G(T,S)$$
$$= \frac{1}{2}\sum_{v \in T} d_G(v) + \frac{1}{2}\sum_{v \in T} d_{G[S \cup \{v\}]}(v).$$

This implies that

$$\sum_{v \in T} f(v) - \frac{1}{2}\sum_{v \in T} d_G(v) > \frac{1}{2}\sum_{v \in T} d_{G[S \cup \{v\}]}(v). \tag{8}$$

Hence, by (7) and (8),

$$\frac{1}{2}(k+1)\,|S| > \frac{1}{2}\sum_{v \in T} d_{G[S \cup \{v\}]}(v).$$

To complete the proof we consider the following four cases, and in each case we get a contradiction, thereby establishing the result.

Case 1. $|T| \le \left\lfloor \frac{|V(G)|}{2} \right\rfloor - 1.$

By hypothesis $\delta(G) = \left\lfloor \frac{|V(G)|}{2} \right\rfloor - 1 + k$ and hence $d_{G[S \cup \{v\}]}(v) \ge k+1$ for every $v \in T$. Therefore $\sum_{v \in T} d_{G[S \cup \{v\}]}(v) \ge (k+1)\,|T|$, a contradiction to Claim 1.

Case 2. $|T| \ge \left\lceil \frac{|V(G)|}{2} \right\rceil + 1.$

Suppose that $|T| = \left\lceil \frac{|V(G)|}{2} \right\rceil + \ell$, where ℓ is a positive integer. This implies that $|S| = \left\lfloor \frac{|V(G)|}{2} \right\rfloor - \ell$. So $e_G(v,T) \ge k+\ell$ for every $v \in S$ because $\delta(G) = \left\lfloor \frac{|V(G)|}{2} \right\rfloor - 1 + k$. This implies that $e_G(S,T) \ge (k+\ell)\,|S|$. Hence $(k+1)\,|S| \le (k+\ell)\,|S| \le e_G(S,T) = \sum_{v \in T} d_{G[S \cup \{v\}]}(v)$, a contradiction to Claim 2.

Case 3. $|T| = \left\lfloor \frac{|V(G)|}{2} \right\rfloor.$

By condition (ii) of the hypothesis of the theorem, $e_G(T,S) \ge \left\lfloor \frac{|V(G)|}{2} \right\rfloor (k+1)$. Hence $\sum_{v \in T} d_{G[S \cup \{v\}]}(v) = e_G(T,S) \ge \left\lfloor \frac{|V(G)|}{2} \right\rfloor (k+1) = |T|\,(k+1)$, again a contradiction to Claim 1.

Case 4. $|V(G)|$ is odd and $|T| = \frac{|V(G)|+1}{2}.$

In this case, $|S| = \left\lfloor \frac{|V(G)|}{2} \right\rfloor$. By condition (ii) of the theorem, $e_G(S,T) \ge \left\lfloor \frac{|V(G)|}{2} \right\rfloor (k+1)$. Hence $\sum_{v \in T} d_{G[S \cup \{v\}]}(v) = e_G(S,T) \ge \left\lfloor \frac{|V(G)|}{2} \right\rfloor (k+1) = |S|\,(k+1)$, again a contradiction to Claim 2. ■

Proof of Theorem 2.2. Suppose that G does not have such an orientation D as stated in the theorem. Then from Lemma 2.1, there exists $T \subseteq V(G)$ such that

$$\sum_{v \in T} f(v) > e_G(T,T) + e_G(T,S), \tag{9}$$

where $S = V(G) - T$.

Claim 1. $(k+2)\,|T| > \sum_{v \in T} d_{G[S \cup \{v\}]}(v).$

$\frac{1}{2}\sum_{v \in T} d_G(v) + \frac{1}{2}\sum_{v \in T} d_{G[S\cup\{v\}]}(v)$

$= e_G(T,T) + e_G(T,S)$ by using (4)

$< \sum_{v \in T} f(v)$ by using (9)

$\leq \sum_{v \in T} \frac{1}{2}(d_G(v) + (k+2))$ by condition (1) of the theorem

$= \frac{1}{2}\sum_{v \in T} d_G(v) + \frac{1}{2}(k+2)\,|T|.$

Hence

$$\frac{1}{2}(k+2)\,|T| > \frac{1}{2}\sum_{v \in T} d_{G[S\cup\{v\}]}(v).$$

Claim 2. $(k+2)\,|S| > \sum_{v \in T} d_{G[S\cup\{v\}]}(v)$.

By condition (1) of the hypothesis of the theorem,

$$\sum_{v \in S} \frac{1}{2}(d_G(v) - (k+2)) \leq \sum_{v \in S} f(v).$$

This implies, $\frac{1}{2}\sum_{v \in S} d_G(v) - \frac{1}{2}(k+2)\,|S| \leq \sum_{v \in S} f(v)$.

Thus, $\frac{1}{2}\sum_{v \in S} d_G(v) - \sum_{v \in S} f(v) \leq \frac{1}{2}(k+2)\,|S|$. (10)

By condition (2) of the hypothesis of the theorem,

$$\sum_{v \in V(G)} f(v) = |E(G)|.$$

This implies as in (6),

$\sum_{v \in T} f(v) - \frac{1}{2}\sum_{v \in T} d_G(v) = \frac{1}{2}\sum_{v \in S} d_G(v) - \sum_{v \in S} f(v)$. (11)

Hence, by (11) and (10),

$\sum_{v \in T} f(v) - \frac{1}{2}\sum_{v \in T} d_G(v) \leq \frac{1}{2}(k+2)\,|S|$. (12)

But by using (9) and (4),

$$\sum_{v \in T} f(v) > e_G(T,T) + e_G(T,S) = \frac{1}{2}\sum_{v \in T} d_G(v) + \frac{1}{2}\sum_{v \in T} d_{G[S\cup\{v\}]}(v)$$

implies that

$\sum_{v \in T} f(v) - \frac{1}{2}\sum_{v \in T} d_G(v) > \frac{1}{2}\sum_{v \in T} d_{G[S\cup\{v\}]}(v)$. (13)

Hence, by (12) and (13),

$$\frac{1}{2}(k+2)\,|S| > \frac{1}{2}\sum_{v \in T} d_{G[S\cup\{v\}]}(v).$$

To complete the proof we consider the following four cases, and in each case we get a contradiction, thereby establishing the result.

Case 1. $|T| \leq \left\lfloor \frac{|V(G)|}{2} \right\rfloor - 1$.

By hypothesis $\delta(G) \geq \left\lfloor \frac{|V(G)|}{2} \right\rfloor + k$ and hence $d_{G[S\cup\{v\}]}(v) \geq k+2$ for every $v \in T$. Therefore $\sum_{v\in T} d_{G[S\cup\{v\}]}(v) \geq (k+2)\,|T|$, a contradiction to Claim 1.

Case 2. $|T| \geq \left\lceil \frac{|V(G)|}{2} \right\rceil + 1$.

Suppose that $|T| = \left\lceil \frac{|V(G)|}{2} \right\rceil + 1 + \ell$, where ℓ is a non-negative integer. This implies that $|S| = \left\lfloor \frac{|V(G)|}{2} \right\rfloor - 1 - \ell$. So $e_G(v,T) \geq k + 2 + \ell$ for every $v \in S$ because $\delta(G) \geq \left\lfloor \frac{|V(G)|}{2} \right\rfloor + k$. This implies that $e_G(S,T) \geq (k+2+\ell)\,|S|$. Hence $(k+2)\,|S| \leq (k+2+\ell)\,|S| \leq e_G(S,T) = \sum_{v\in T} d_{G[S\cup\{v\}]}(v)$, a contradiction to Claim 2.

Case 3. $|T| = \left\lfloor \frac{|V(G)|}{2} \right\rfloor$.

By condition *(ii)* of the hypothesis of the theorem, $e_G(T,S) \geq \left\lfloor \frac{|V(G)|}{2} \right\rfloor (k+2)$. Hence $\sum_{v\in T} d_{G[S\cup\{v\}]}(v) = e_G(T,S) \geq \left\lfloor \frac{|V(G)|}{2} \right\rfloor (k+2) = |T|\,(k+2)$, a contradiction to Claim 1.

Case 4. $|V(G)|$ is odd and $|T| = \frac{|V(G)|+1}{2}$.

In this case, $|S| = \left\lfloor \frac{|V(G)|}{2} \right\rfloor$. By condition *(ii)* of the theorem, $e_G(S,T) \geq \left\lfloor \frac{|V(G)|}{2} \right\rfloor (k+2)$. Hence $\sum_{v\in T} d_{G[S\cup\{v\}]}(v) = e_G(S,T) \geq \left\lfloor \frac{|V(G)|}{2} \right\rfloor (k+2) = |S|\,(k+2)$, a contradiction to Claim 2. ■

Remark 2.1 *The condition* $e_G(W, V(G) - W) \geq \left\lfloor \frac{|V(G)|}{2} \right\rfloor (k+i)$, $i \in \{1,2\}$, *for every* $W \subseteq V(G)$ *with* $|W| = \left\lfloor \frac{|V(G)|}{2} \right\rfloor$ *is true for several families of graphs, including those* G *which contains a complete tripartite graph* $K_{r,r,r}$ *as a spanning subgraph. It can be verified easily.* ■

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References

[1] P. Katerinis, Minimum degree and the orientation of a graph, Ars Combinatoria 75 (2005) 65-73.