Clique domination in graphs*

Guangjun Xu, Erfang Shan† Min Zhao
Department of Mathematics, Shanghai University, Shanghai 200444, China

ABSTRACT. A set S of vertices in a graph G is a clique dominating set of G if S contains at least one vertex of every clique C of G. The clique domination number $\gamma_q(G)$ and the upper clique domination number $\Gamma_q(G)$ are, respectively, the minimum and maximum cardinalities of a minimal clique dominating set of G. In this paper, we prove that the problem of computing $\gamma_q(G)$ is NP-complete even for split graphs and the problem of computing $\Gamma_q(G)$ is NP-complete even for chordal graphs. In addition, for a block graph BG we show that the clique domination number is bounded above by the vertex independence number ($\gamma_q(BG) \leq \beta_0(BG)$) and give a linear algorithm for computing $\gamma_q(BG)$.

Key words: Dominating set; Clique dominating set; Block graphs; Algorithm

AMS subject classification: 05C69; 05C85

1 Introduction

All graphs considered here are simple, i.e., finite, undirected, and loopless. For standard graph theory terminology not given here we refer to [2]. Let $G = (V, E)$ be a graph with vertex set V and edge set E, the *open neighborhood* $N(v)$ of the vertex v consists of vertices adjacent to v, i.e., $N(v) = \{u \in V \mid (u,v) \in E\}$, and the *closed neighborhood* of v is $N[v] = \{v\} \cup N(v)$. For a subset $S \subseteq V$, we define $N[S] = \cup_{x \in S} N[x]$. The

*This research was partly supported by the National Natural Science Foundation of China under grant 10571117 and and the Development Foundation of Shanghai Education Committee under grant 05AZ04.

† *Corresponding author. Email address: efshan@staff.shu.edu.cn (E.F. Shan)*

subgraph induced by S is denoted by $G[S]$. The *distance* $d_G(u, v)$ of two vertices u and v is the minimum length of a path between u and v. The *degree* of a vertex v of G is denoted by $d_G(v) = |N_G(v)|$, and a vertex with degree one is called a *leaf.* The minimum and maximum degrees of vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For any connected graph G, a vertex $v \in V(G)$ is called a *cut-vertex* of G if $G - v$ is no longer connected. A maximal connected induced subgraph without a cut-vertex is called a *block* of G. A graph G is a *block graph* if every block in G is complete.

A set S is *a dominating set* if every vertex of $V - S$ has at least one neighbor in S. The *domination number* $\gamma(G)$ is defined to be the minimum cardinality of a dominating set S in G, and the *upper domination number* $\Gamma(G)$ be the maximum cardinality of a minimal dominating set S in G. We call a minimal dominating set of G with minimum cardinality a γ-set, and we will use similar notation for other parameters. A set S is *independent* if no two vertices in S are adjacent. A dominating set S of G is a *total/connected/independent dominating set* of G if $G[S]$ has no isolated vertices, $G[S]$ is connected and S is an independent set, respectively. The minimum cardinality of a *total/connected/independent dominating set* is denoted by $\gamma_t(G)$, $\gamma_c(G)$ and $i(G)$, respectively. The *independence number* $\beta_0(G)$ is the maximum cardinality of an independent set of vertices of G. The *matching number* is the maximum cardinality among the independent sets of edges of G and is denoted by $\alpha_1(G)$. In recent years, many domination-related parameters have been defined (see [4, 6, 7]), for a comprehensive review on this subject we refer to [8, 9].

This paper introduces a new invariant of the domination concept. A *clique* in a graph is a subgraph of G which is complete and is not a proper subgraph of another complete subgraph of G. A set $S \subseteq V$ in G is called a *clique dominating set* if for every clique C of G, $S \cap V(C) \neq \emptyset$. The *clique domination number* $\gamma_q(G)$ is defined to be the minimum cardinality of a clique dominating set S in G, while the *upper clique domination number* $\Gamma_q(G)$ is defined to be the maximum cardinality of a minimal clique dominating set S in G. The applications for this parameter may be seen as follows. Consider a communication network in which a clique is usually represents a cluster of sites which has best possible ability for rapid information exchanging among the members of the cluster. The clique domination claims not only dominating the network but also including at least one core role for every cluster, therefore it is convenient to control all clusters as well as keep the ability of dominating the whole network. Another application may be seen in social networks theory. Kelleher and Cozzens [10] have studied the application of dominating sets in social networks, where a vertex represents an actor and an edge represents a relationship between two actors. Note that a clique in a social networks can be viewed as a maximal group of members which have the same property. Suppose a dominating set is some kind of organization in the social networks, then the clique dominating sets claim each clique in the social networks owns at least one position in this organization.

It should be noted that another clique-related invariant of dominating set, namely, dominating clique, have been previously defined and studied [5], which requires that a dominating set induces a complete subgraph and thus is different from the clique dominating set introduced in current paper.

2 Clique domination number

2.1 Properties of clique domination

The existence of clique dominating sets is easy to see, since V itself is such a set for every graph $G = (V, E)$. So we have the following observation.

Observation 1 *The clique dominating set is well-defined for each graph G.*

In the following we consider only connected graphs. The number of cliques in a graph G is denoted by $n_q(G)$. Clearly every clique dominating set is a dominating set and a subset comprising exactly one vertex of each clique is a clique dominating set. So we have

Observation 2 $\gamma(G) \leq \gamma_q(G) \leq n_q(G)$ *and* $\gamma_q(G) = 1$ *if and only if* $\gamma(G) = 1$.

Next we give some exact values of clique domination number in specific classes of graphs, the proofs are straightforward and are omitted. Here K_n is a complete graph, C_n, P_n and W_n are the cycle, path and wheel on n vertices respectively.

Observation 3

(1) $\gamma_q(K_n) = \gamma(W_n) = 1$,
(2) $\gamma_q(C_n) = \lceil \frac{n}{2} \rceil$, *where* $n \geq 4$,
(3) $\gamma_q(P_n) = \lceil \frac{n-1}{2} \rceil$, *where* $n \geq 2$.

Next we show some relationships between clique domination number and other domination-related parameters. We can see that $i(G)$, $\gamma_t(G)$, $\gamma_c(G)$, $\alpha_1(G)$, $\beta_0(G)$, $\Gamma(G)$ are incomparable with $\gamma_q(G)$ (Here we denoted by $\{i(G), \gamma_t(G), \gamma_c(G), \alpha_1(G), \beta_0(G), \Gamma(G)\} \diamond \gamma_q(G)$). For example, it is not difficult to show that the Petersen graph PG has $i(PG) = 3 < 6 = \gamma_q(PG)$, while a double star G, obtained from adding an edge between two central vertices of T_1 and T_2, where $T_1 \cong T_2 \cong K_{1,p}$, $p \geq 2$, has $i(G) > 2 = \gamma_q(G)$. Also, $\gamma_t(G)$, $\gamma_c(G)$, $\Gamma(G)$ are incomparable with $\gamma_q(G)$. For example, the Petersen Graph PG has $\gamma_t(PG) = \gamma_c(PG) = \Gamma(PG) = 5 < 6 = \gamma_q(PG)$; while a path P_5 has $\gamma_c(P_5) = \gamma_t(P_5) = \Gamma(P_5) = 3 > 2 = \gamma_q(P_5)$. Both $\alpha_1(G)$ and $\beta_0(G)$ are incomparable with $\gamma_q(G)$ may be seen as follows. For a 5-cycle C_5, we have $\alpha_1(C_5) = \beta_0(C_5) = 2 < 3 = \gamma_q(C_5)$; while the complete graph K_{2n} with $n \geq 2$ has $\alpha_1(K_{2n}) = n \geq 2 > 1 = \gamma_q(K_{2n})$ and the path P_{2n+1} with $n \geq 1$ has $\beta_0(P_{2n+1}) = n + 1 > n = \gamma_q(P_{2n+1})$. Thus,

Observation 4 *For any graph,* $\{i(G), \gamma_t(G), \gamma_c(G), \alpha_1(G), \beta_0(G), \Gamma(G)\} \diamond \gamma_q(G)$.

A set $U \subseteq V$ is called a *vertex cover* of G if every edge of G is incident with a vertex in U. The minimum cardinality of a vertex cover is denoted by $\alpha_0(G)$. The maximum number of vertices of a clique in G is denoted by $w(G)$. Then we have the following observation immediately,

Observation 5 *For any graph, we have* $\gamma_q(G) \leq \alpha_0(G)$*; and if* $w(G) \leq 2$ *then any vertex cover is precisely a clique dominating set of* G*, i.e.,* $\alpha_0(G) = \gamma_q(G)$.

Proof. For any graph G, since every vertex cover is also a clique dominating set of G, we have $\gamma_q(G) \leq \alpha_0(G)$; On the other hand, if G is a graph with $w(G) \leq 2$, i.e., each clique in G is a K_2, then every clique dominating set is a vertex cover, it follows that $\alpha_0(G) \leq \gamma_q(G)$, so $\alpha_0(G) = \gamma_q(G)$. ■

Many domination-related parameters remain NP-complete when restricted to bipartite graphs, while for clique domination problem things are different. From the above observation, for any bipartite graph G we have $\alpha_0(G) = \gamma_q(G) = \alpha_1(G)$.

2.2 Complexity results

In this subsection we will show that the clique dominating set problem is NP-complete when restricted to split graphs (a subclass of chordal graphs).

Consider the following decision problem

CLIQUE DOMINATING SET (CDS)

INSTANCE: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.

QUESTION: Does G have a clique dominating set of cardinality at most k?

To establish the NP-completeness of the above clique dominating set problem, we describe a polynomial transformation from the following well-known NP-complete problem

EXACT COVER BY 3-SETS (X3C)

INSTANCE: A finite set X with $|X| = 3q$ and a collection $\mathcal{C}$ of 3-element subsets of X. Each element $x \in X$ appears in at least two subsets.

QUESTION: Does $\mathcal{C}$ contain an exact cover (a subcollection $\mathcal{C}' \subseteq \mathcal{C}$ such that every element of X occurs in exactly one member of $\mathcal{C}'$) for X? (Note that if $\mathcal{C}'$ exists, then its cardinality is precisely q.)

Theorem 6 **CDS** *is NP-complete, even for split graphs.*

Proof. It is clearly that **CDS** is in NP, since we can verify that a 'yes' instance is a clique dominating set of cardinality at most k in polynomial time. Next we show that the clique dominating set problem is NP-complete. Let $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $\mathcal{C}= \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of **X3C**, we will construct a split graph G such that this instance of **X3C** will have an exact three cover if and only if G has a clique dominating set of cardinality at most k. The split graph G is constructed as follows. Corresponding to each subset C_i and variable x_j are vertices c_i and v_j, now adding an edge between c_i and c_j for any two distinct $i \neq j$; then joining c_i and v_j if and only if the variable $x_j \in C_i$. Set $k = q$.

Let $C = \cup_{i=1}^{m}\{c_i\}$, $V = \cup_{i=1}^{3q}\{v_i\}$, then the resulting graph is $G = (C \cup V, E_1 \cup E_2)$ where $E_1 = \{(c_i, c_j) \mid c_i, c_j \in C, i \neq j\}$ and $E_2 = \{(v_i, c_j) \mid x_i \in C_j\}$.

It is easy to see that G is a split graph and the constructing of G is completed in polynomial time. Suppose $\mathcal{C}' \subseteq \mathcal{C}$ is an exact 3-cover for $X = \{x_1, x_2, \ldots, x_{3q}\}$, then it is not difficult to verify that $S = \{c_j \mid C_j \in \mathcal{C}'\}$ is a clique dominating set of cardinality k. Conversely, assume S is a minimum clique dominating set of cardinality at most k and for which $|S \cap V|$ is minimized. Then we must have $S \cap V = \emptyset$, for otherwise we can replace each vertex $v_i \in S \cap V$ by its one neighbor in C. Let $C' = \{C_i \mid c_i \in S\}$. Since S is a clique dominating set of G, every vertex $v_i \in V$ must be dominated by some vertex $c_j \in S$ and thus C' is a cover for X. Since C' is a cover of X such that $|C'| = |S| \leq k$, it follows immediately that $C_i \cap C_j = \emptyset$ for distinct C_i and C_j in C'. So C' is an exact 3-cover for X. ■

UPPER CLIQUE DOMINATING SET (UCDS)

INSTANCE: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.

QUESTION: Does G have a minimal clique dominating set of cardinality at least k?

Theorem 7 **UCDS** *is NP-complete, even for chordal graphs.*

Proof. We use the similar notation as in the proof for Theorem 6. Let G' be the graph constructed in the proof for Theorem 6. Let G be the graph constructed from G' as follows. For each $i \in \{1, 2, \ldots, 3q\}$, let u_i and w_i be two new vertices and join each of them to the vertex v_i. For each $j \in \{1, 2, \ldots, m\}$, let a_j and b_j be two new vertices and join each of them to the vertex c_j. Set $k = 2m + 5q$. Clearly G is a chordal graph of order $3m + 9q$.

Let $A = \cup_{i=1}^{m}\{a_i\}$, $B = \cup_{i=1}^{m}\{b_i\}$, $C = \cup_{i=1}^{m}\{c_i\}$, $U = \cup_{i=1}^{3q}\{u_i\}$, $V = \cup_{i=1}^{3q}\{v_i\}$, $W = \cup_{i=1}^{3q}\{w_i\}$.

Now we show that the above instance of **X3C** contains an exact three cover if and only if G has an upper clique dominating set of cardinality at least k. Suppose $C' \subseteq C$ is an exact 3-cover for $X = \{x_1, x_2, \ldots, x_{3q}\}$, then $S = U \cup W \cup \{c_j \mid C_j \in C'\} \cup \{a_j \mid C_j \notin C'\} \cup \{b_j \mid C_j \notin C'\}$ is a minimal clique dominating set of cardinality at least k. Conversely, assume S is a minimal clique dominating set of cardinality at least k and for which $|S \cap V|$ is minimized. We claim that $S \cap V = \emptyset$. Otherwise suppose there exists a vertex $v_i \in S \cap V$, then by the minimality of S we have that $u_i, w_i \notin S$. If $N(v_i) \cap S \neq \emptyset$, then $S' = (S \backslash \{v_i\}) \cup \{u_i, w_i\}$ is a minimal clique dominating set with $|S' \cap V| < |S \cap V|$, which is a contradiction. Hence assume $N(v_i) \cap C = \emptyset$, and let c_j be any vertex of $N(v_i) \cap C$. Since S is a clique dominating set, we must have that $a_j, b_j \in S$, then $S' = (S \backslash \{v_i, a_j, b_j\}) \cup \{c_j, u_i, w_i\}$ is also a minimal clique dominating set with $|S' \cap V| < |S \cap V|$. Both cases lead to contradictions, therefore $S \cap V = \emptyset$. Consequently, we must have $U \cup W \subseteq S$ and $|S \cap (A \cup B \cup C)| \geq k - |U \cup W| = 2m - q$. Since each vertex $v_i \in V$ corresponds exactly to one clique in the subgraph $G[C \cup V]$ and $S \cap V = \emptyset$, it follows that $|S \cap C| \geq q$. We now claim that $|S \cap C| = q$. Otherwise assume $|S \cap C| \geq q + 1$, then by the minimality of S, we have that

$|S \cap (A \cup B)| = |\{a_j, b_j \mid c_j \notin S\}| = 2(m - |S \cap C|) \leq 2(m-q-1)$, then $|S| = 2(m-|S \cap C|) + |S \cap C| + |S \cap (U \cup W)| = 2m+6q-|S \cap C| \leq 2m+5q-1 < k$, which is a contradiction. Thus, $|S \cap C| = q$, and $C' = \{C_i \mid c_i \in S\}$ is an exact 3-cover of cardinality q for X. ∎

3 Clique domination in block graphs

In this section, we investigate clique domination in block graphs, which is a superclass of trees. It is easy to observe that for a tree, $\gamma_q(T) = \alpha_0(T) = \alpha_1(T)$, while for block graphs things are different, because block graphs usually contain large cliques.

In what follows, we always use a tree-like decomposition structure, named *refined cut-tree*, of a block graph. Let G be a block graph with h blocks $BK_1,...,BK_h$ and p cut-vertices $v_1,...,v_p$. The *cut-tree* of G, denoted by $T^B(V^B, E^B)$, is defined as $V^B = \{BK_1, ..., BK_h, v_1, ..., v_p\}$ and $E^B = \{(BK_i, v_j) \mid v_j \in BK_i, 1 \leq i \leq h, 1 \leq j \leq p\}$. It is shown in [1] that the cut-tree of a block graphs can be constructed in linear time by the depth-first-search method. For any block BK_i of G, define $B_i = \{v \in BK_i \mid v \text{ is not a cut-vertex}\}$, where $1 \leq i \leq h$. We define the *refined cut-tree* $T^B(V^B, E^B)$ as $V^B = \{B_1, ..., B_h, v_1, ..., v_p\}$ and $E^B = \{(B_i, v_j) \mid v_j \in BK_i, 1 \leq i \leq h, 1 \leq j \leq p\}$, and each B_i is called a *block-vertex*. It should be noted that a block-vertex in the refined cut-tree of a block graph may be empty. A block graph G with five blocks, $BK_1 = G[\{a, b, d\}]$, $BK_2 = G[\{c, e\}]$, $BK_3 = G[\{d, e\}]$, $BK_4 = G[\{d, g, h\}]$ and $BK_5 = G[\{e, f, i, j\}]$ is shown in Fig. 1, the corresponding cut-tree and refined cut-tree of G are shown in Fig. 2, where $B_1 = \{a, b\}$, $B_2 = \{c\}$, $B_3 = \emptyset$, $B_4 = \{g, h\}$ and $B_5 = \{f, i, j\}$.

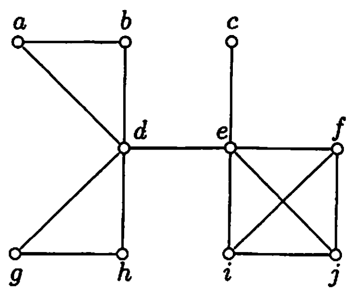

Fig. 1. A block graph G with five blocks.

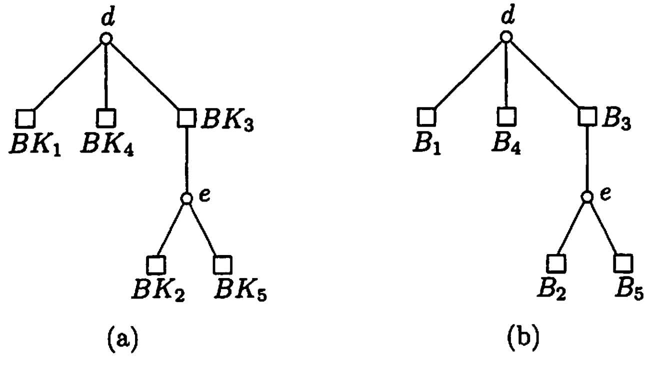

Fig. 2. (a) The cut-tree of G in Fig. 1 (b) The refined cut-tree of G in Fig. 1.

Let $T^B(V^B, E^B)$ be the refined cut-tree of a block graph G. We just treat $T^B(V^B, E^B)$ as an ordinary tree regardless of the fact that every block-vertex is actually subset of vertices of the original block graph. For clarity, we denote a block vertex B_i by v_i^B in $T^B(V^B, E^B)$, here the superscript B of v_i^B indicates that this vertex is a block vertex. Furthermore, v_i^B is corresponding to B_i one by one. Since we will view $T^B(V^B, E^B)$ as an ordinary tree, we need some other concepts with tree.

Let T be a tree rooted at r and v is a vertex of T, the *level number* of v, denoted by $l(v)$, is the length of the unique r-v path in T. If a vertex v of T is adjacent to u and $l(u) > l(v)$, then u is called a *child* of v and v is the *parent* of u. A vertex w is a *descendant* of v (and v is an *ancestor* of w) if the level numbers of the vertices on the v-w path are monotonically increasing. Let $D(v)$ denote the set of descendants of v, and define $D[v] = D(v) \cup \{v\}$. The *maximal subtree of T rooted at v* is the subtree of T induced by $D[v]$ and is denoted by T_v.

3.1 $\gamma_q(BG) \leq \beta_0(BG)$

For general graphs, the clique domination number $\gamma_q(G)$ is incomparable with the vertex independence number $\beta_0(G)$ (see Observation 4). However, the following result shows that for block graph BG, $\gamma_q(BG)$ is bounded above by $\beta_0(BG)$.

Theorem 8 *For any block graph BG, $\gamma_q(BG) \leq \beta_0(BG)$, and this bound is sharp.*

Proof. We proceed by induction on the number of cliques n_q of BG. Clearly the result holds for $n_q = 1, 2$ establishing the base case. Now let $n_q \geq 3$, and assume that $\gamma_q(BG') \leq \beta_0(BG')$ holds for every block graph BG' with $n'_q < n_q$. Let BG be a block graph with n_q cliques, and $T^B(V^B, E^B)$ be the refined cut-tree of BG. If T^B is a star, then $\gamma_q(BG) = 1 < n_q = \beta_0(BG)$ and hence the result is valid. So assume that T^B is not a star. Root T^B at any cut-vertex r and let v^B be any endvertex of T^B for which $d_{T^B}(v^B, r)$ is a maximum. Clearly $d_{T^B}(v^B, r) \geq 3$. Let u be the parent of v^B and w^B be the parent of u. If u has some other children, say $v_1^B, \ldots, v_i^B$, $i \geq 1$, which are different from v^B. Let $T' = T^B \backslash \{v^B, v_1^B, \ldots, v_i^B\}$ and BG' be the corresponding block graph of T', then $\gamma_q(BG') \leq \beta_0(BG')$ and we have $\gamma_q(BG) \leq \gamma_q(BG') + 1 \leq \beta_0(BG') + i \leq \beta_0(BG)$. Next assume v^B is the only child of u. Let $u_1, \ldots, u_i$ be some other children of w^Bdifferent from u, where $i \geq 0$. Without loss of generality, assume each u_i has exactly one child. Let $T' = T^B \backslash D[w^B]$ and BG' be the corresponding block graph of T', then $\gamma_q(BG') \leq \beta_0(BG')$ and we have $\gamma_q(BG) \leq \gamma_q(BG') + i + 1 \leq \beta_0(BG') + i + 1 \leq \beta_0(BG)$. This bound is sharp, since $\gamma_q(P_4) = \beta_0(P_4)$. ∎

There exists an infinite class of block graphs in which the differences $\beta_0 - \gamma_q$ can be made arbitrary large. For example, let G be any block graph of order n, then attach $k \geq 2$ pendent vertices to each vertex v of G, denote the resulting graph by G', clearly we have $\beta_0(G') - \gamma_q(G') = n(k-1)$.

3.2 A simple linear algorithm for $\gamma_q(BG)$

In this subsection we present a linear algorithm for solving the clique domination problem on block graphs. Some other papers gave efficient algorithms for solving domination-related problems on block graphs [3, 11]. For block graphs, we first show the following lemma.

Lemma 9 *Let G be a block graph, then there exists a $\gamma_q(G)$-set in which every vertex is a cut-vertex of G.*

Proof. Let G be a block graph, S a $\gamma_q(G)$-set. Suppose there exists some vertex $v \in S$ such that v is not a cut-vertex of G. If every cut-vertex that is adjacent to v is contained in S, then $S \setminus \{v\}$ is a smaller clique dominating set than S, which is a contradiction. If there exists a cut-vertex u such that u is adjacent to v and $u \notin S$, then $(S \setminus \{v\}) \cup \{u\}$ is a $\gamma_q(G)$-set of G, so the result follows. ∎

In the following, we give a linear algorithm for finding a minimum clique dominating set in a block graph, our algorithm accept the refined cut-tree $T^B(V^B, E^B)$ of the original block graph G as the input of our problem and is presented as a color-marking process. We first give a brief overview on this algorithm. Let G be a connected block graph, and $T^B(V^B, E^B)$ the refined cut-tree of G. Suppose $T^B(V^B, E^B)$ is a rooted tree with vertices $v_1,\ldots,v_n$ such that $l(v_i) \leq l(v_j)$ for $i > j$, and the root is a cut-vertex v_n (in G). Let H_i be the set of all vertices with level number i, and H_k be the set of all vertices with largest level number k. By the definition of $T^B(V^B, E^B)$, it is clear that $H_0 = \{v_n\}$ and H_i contains only cut-vertices of G for even i and block-vertices for odd i. Moreover, k is an odd number. By Lemma 9, there exists a $\gamma_p(G)$-set contains only cut-vertices for a block graph G. Our algorithm consider directly the cut-vertices in the rooted tree $T^B(V^B, E^B)$. It starts from the largest even level of $T^B(V^B, E^B)$ and works upward to the root of the tree. Initially, all vertices of T^B are marked with white (which means all needed to be dominated), and eventually, every white vertex will be marked with black or gray. In the end of the algorithm, all black cut-vertices form a minimum clique dominating set of G.

Algorithm CDS-BG. Find a minimum CDS of a connected block graph G.

Input: A block graph G of order $n \geq 3$.
Output: A minimum CDS of G.

Construct a refined cut-tree $T^B(V^B, E^B)$ with vertices $v_1,\ldots,v_n$ of G so that $l(v_i) \leq l(v_j)$ for $i > j$, and the root is a cut-vertex v_n (in G). For every vertex v_j lies in the odd levels $H_1, H_3, \ldots, H_k$, relabel v_j as v_j^B (the superscript B of v_j^B indicates that it is a block-vertex and v_i^B is in correspondence with B_i one by one).

Initialization: Mark all vertices of V^B with white, $S := \emptyset$.
for $i := k-1$ **down to** 0 by step-length 2 **do**
  for every $v_j \in H_i$ **do**
  if $N(v_j) \cap H_{i+1}$ contains at least one white vertex **then**
    {mark v_j with black;
    $S := S \cup \{v_j\}$;
    mark the parent of v_j with gray;
    mark all white vertices in $N(v_j) \cap H_{i+1}$ with gray. }
  else mark v_j with gray
  end for
end for
output S.

Theorem 10 *Algorithm CDS-BG computes in linear time a minimum CDS of a given connected block graph G.*

Proof. Let G be a connected block graph, and $T^B(V^B, E^B)$ be the refined cut-tree of G. And $T^B(V^B, E^B)$ is a rooted tree with vertices $v_1,\ldots,v_n$ such that $l(v_i) \leq l(v_j)$ for $i > j$, and the root is a cut-vertex v_n(in G). Let S be the set computed by Algorithm CDS-BG. From the Algorithm, S is a CDS of G. We now show that $\gamma_q(G) = |S|$. Let $S = \{v_{i_1}, v_{i_2}, \ldots, v_{i_m}\}$, where $i_1 < i_2 < \cdots < i_m$. Suppose that $\gamma_q(G) < |S|$. Among all $\gamma_q(G)$-sets, let S^* be chosen so that the first integer j $(1 \leq j \leq m)$ with $v_{i_j} \notin S^*$ is as large as possible. Let $T^B_{v_{i_j}}$ be the subtree induced by $D[v_{i_j}]$ where v_{i_j} is the root, and $G^T_{v_{i_j}}$ be the corresponding sub-block graph of $T^B_{v_{i_j}}$. If $(T^B_{v_{i_j}} \cap S^*) \setminus S \neq \emptyset$, then replaced any vertex of $(T^B_{v_{i_j}} \cap S^*) \setminus S$ by v_{i_j} to form a new $\gamma_q(G)$-set which contains all vertices in $\{v_{i_1}, v_{i_2}, \ldots, v_{i_j}\}$, which contradicts our choice of S^*. Thus, we have $(T^B_{v_{i_j}} \cap S^*) \setminus S = \emptyset$. Furthermore, let w^B_f be the father of v_{i_j} in $T^B(V^B, E^B)$, then S^* contains no vertex of B_f. Otherwise, we can also get a new $\gamma_q(G)$-set by replacing that vertex with v_{i_j}, which contains all vertices in $\{v_{i_1}, v_{i_2}, \ldots, v_{i_j}\}$. By the algorithm, v_{i_j} was added into S because $N(v_{i_j}) \cap H_{i+1}$ contains at least one white vertex (a clique which is not dominated). Since $(T^B_{v_{i_j}} \cap S^*) \setminus S = \emptyset$ and S^* contains no vertex of B_f, it is clear that such cliques in $G^T_{v_{i_j}}$ cannot be observed by S^*, which contradicts the assumption that S^* is a $\gamma_q(G)$-set.

The running time of algorithm CDS-BG can be estimated as follows. The running time is linear to the size of refined cut-tree of G, while the time for constructing a refined cut-tree of G is linear [1]. Therefore, it follows that the total time needed to perform algorithm CDS-BG is linear. ■

References

[1] A.V. Aho, J.E. Hopcropt, J.D. Ullman, The Design and Analysis of Computer Algorithms, Addison-Wesley, Reading, MA, 1974.

[2] J.A. Bondy, U.S.R. Murty, *Graph Theory with Applications*, Elsevier North Holland, 1976.

[3] G.J. Chang, Total domination in block graphs, *Oper. Res. Lett.* 8 (1989), 53-57.

[4] X. Chen, L. Sun, A. McRae, Tree domination in graphs, *Ars Combin.* 73 (2004), 193-203.

[5] M.B. Cozzens, L.L. Kelleher, Dominating cliques in graphs, *Discrete Math.* 86 (1990) 101-116.

[6] J.E. Dunbar, J.W. Grossman, J. H. Hattingh, S.T. Hedetniemi, A.A. McRae, On weakly connected domination in graphs, *Discrete Math.* 167/168 (1997) 261-269.

[7] F. Harary, T.W. Haynes, Double domination in graphs, *Ars Combin.* 55 (2000) 201-213.

[8] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[9] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.

[10] L.L. Kelleher, M.B. Cozzens, Dominating sets in social network graphs, *Math. Social Sci.* 16 (1988) 267-279.

[11] W.C.K. Yen, The bottleneck independent domination on the classes of bipartite graphs and block graphs, *Information Sciences* 157 (2003), 199-215.