THE PERIODS OF K-NACCI SEQUENCES IN CENTRO-POLYHEDRAL GROUPS AND RELATED GROUPS

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Abstract

The *centro-polyhedral group* $\langle l,m,n\rangle$, for $l,m,n\in\mathbb{Z}$, is defined by the presentation

$$\langle x,y,z : x^l = y^m = z^n = xyz\rangle.$$

In this paper, we obtain the periods of *k*-nacci sequences in centro-polyhedral groups and related groups.

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1. Introduction

The Fibonacci sequences and related higher-order (tribonacci, *k*-nacci) sequences are generally viewed as sequences of integers. In [6] the Fibonacci length of a 2-generator group is defined, thus extending the idea of forming a sequence of group elements based on a Fibonacci-like recurrence relation first introduce by Wall in [17], where he considered the Fibonacci length of the cyclic group C_n. Lü and Wang contributed to the study of the Wall number for the *k*-step Fibonacci sequence [15]. The concept of Fibonacci length for more than two generators has also been considered, see for example [3]. Also, the theory has been expanded to nilpotent groups, see for example [1,2,10]. Knox proved that the period of *k*-nacci (*k*-step Fibonacci) sequences in dihedral groups is equal to $2k+2$ [14]. In [4] the Fibonacci lengths of certain centro-polyhedral groups are calculated. Other works on Fibonacci length are discussed in, for example, [5,9,11,12,13].

This paper discusses the periods of k-nacci sequences in centro-polyhedral groups and related groups.

Definition 1.1: A *k-nacci sequence* in a finite group is a sequence of group elements $x_1, x_2, x_3, \cdots, x_n, \cdots$ for which, given an initial (seed) set $x_1, x_2, x_3, \cdots, x_j$, each element is defined by

$$x_n = \begin{cases} x_1 x_2 \cdots x_{n-1} & \text{for } j < n \leq k \\ x_{n-k} x_{n-k+1} \cdots x_{n-1} & \text{for } n > k \end{cases}.$$

We also require that the initial elements of the sequence, $x_1, x_2, x_3, \cdots, x_j$, generate the group, thus forcing the k-nacci sequence to reflect the structure of the group. The k-nacci sequence of a group generated by $x_1, x_2, x_3, \cdots, x_j$ is denoted by $F_k(G; x_1, x_2, \cdots, x_j)$ and its period is denoted by $P_k(G; x_1, x_2, \cdots, x_j)$.

For more information see [14].

A *2*-step Fibonacci sequence in the integers modulo m can be written as $F_2(\mathbb{Z}_m; 0, 1)$. A *2*-step Fibonacci sequence of group elements is called a *Fibonacci sequence of a finite group.* A finite group G is ***k-nacci sequenceable*** if there exists a k-nacci sequence of G such that every element of the group appears in the sequence [14]. A sequence of group elements is *periodic* if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is called the *period of the sequence.* For example, the sequence $a, b, c, d, e, b, c, d, e, b, c, d, e, \cdots$ is periodic after the initial element a and has period 4. A sequence of group elements is *simply periodic* with period k if the first k elements in the sequence form a repeating subsequence. For example, the sequence $a, b, c, d, e, f, a, b, c, d, e, f, a, b, c, d, e, f, \cdots$ is simply periodic with period 6. It is important to note that the period of a k-nacci sequence depends on the chosen generating n- tuple for a group.

Definition 1.2: For a finitely generated group $G = \langle A \rangle$ where $A = \{a_1, a_2, \ldots, a_n\}$ the sequence $x_1 = a_1, \cdots, x_{n-1} = a_{n-1}$, $x_{i+n} = \prod_{j=1}^{n} x_{i+j-1}$, $i \geq 0$, is called the *Fibonacci orbit* of G with respect to the generating set A, denoted $F_A(G)$.

Definition 1.3: If $F_A(G)$ is periodic then the length of the period of the sequence is called the *Fibonacci length* of G with respect to the generating set A, written $LEN_A(G)$.

Notice that the orbit of a k-generated group is a k-nacci sequence. The orbits of certain *centro-polyhedral groups,* for any $n > 2$, have been studied in [4].

Definition 1.4: Let $f_n^{(k)}$ denote the nth member of the k-step Fibonacci sequence defined as

$$f_n^{(k)} = \sum_{j=1}^{k} f_{n-j}^{(k)} \text{ for } n > k \tag{1}$$

with boundary conditions $f_i^{(k)} = 0$ for $1 \le i < k$ and $f_k^{(k)} = 1$. Reducing this sequence by a modulus m, we can get a repeating sequence, which we denote by

$$f(k,m) = \left(f_1^{(k,m)}, f_2^{((k,m)}, \cdots, f_n^{(k,m)} \cdots\right),$$

where $f_i^{(k,m)} = f_i^{(k)} \pmod{m}$. We then have that $\left(f_1^{(k,m)}, f_2^{(k,m)}, \cdots, f_k^{(k,m)}\right) = (0, 0, \cdots 0, 1)$ and it has the same recurrence relation as in (1) [15].

Theorem 1.1: $f(k,m)$ is a periodic sequence [15].

Let $h_k(m)$ denote the smallest period of $f(k,m)$, called the period of $f(k,m)$ or the Wall number of the k-step Fibonacci sequence modulo m. For more information see [15].

Definition 1.5: Let $h_{k(a_1,a_2,\cdots,a_k)}(m)$ denote the smallest period of the integer-valued recurrence relation $u_n = u_{n-1} + u_{n-2} + \cdots + u_{n-k}$, $u_1 = a_1, u_2 = a_2, \cdots, u_k = a_k$ when each entry is reduced modulo m.

For example we choose $u_1 = 2, u_2 = 3$ to calculate $h_{2(2,3)}(m)$, that is we choose the boundary conditions $f_1^{(2,m)} = 2$, $f_2^{(2,m)} = 3$ or we choose $u_1 = 0, u_2 = 0, u_3 = 0, u_4 = 2, u_5 = 3$ to calculate $h_{5(0,0,0,2,3)}(m)$, that is we choose the boundary conditions $f_1^{(5,m)}=0$, $f_2^{(5,m)}=0$, $f_3^{(5,m)}=0$, $f_4^{(5,m)}=2$, $f_5^{(5,m)}=3$.

Lemma 1.1: For $a_1,a_2,\cdots,a_k,x_1,x_2,\cdots,x_k \in \mathbb{Z}$ with $m>0$, $a_1,a_2,\cdots a_k$ not all congruent to zero modulo m and $x_1,x_2,\cdots,x_k$ not all congruent to zero modulo m,

$$h_{k(a_1,a_2,\cdots a_k)}(m)=h_{k(x_1,x_2,\cdots,x_k)}(m).$$

Proof: The following is due to Lü and Wang, see [15]. Let $U_n=[u_n,u_{n+1},\cdots u_{n+k-1}]$ and

$$G=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}_{k\times k}.$$

Then it follows that $U_n=U_1\left(G^T\right)^n$ where "T" denotes the transpose of a matrix. Since the integers modulo m form a finite set of equivalence classes, there exist integers n and r such that $\left(G^T\right)^{n+r}$ is congruent, elementwise, to $\left(G^T\right)^r$ modulo m. Since $\det\left(G^T\right)=1$ is a unit modulo m, $\left(G^T\right)^n$ is the $k\times k$ identity matrix. So $U_n\equiv U_1 \bmod m$, in the natural way.

Corollary 1.1: Let $a_1,a_2,\cdots,a_k,x_1,x_2,\cdots,x_k,\alpha,\beta\in\mathbb{Z}$ with $\alpha,\beta>0$, $a_1,a_2,\cdots a_k$ not all congruent to zero modulo α and $x_1,x_2,\cdots,x_k$ not all congruent to zero modulo α. Then we have

$$h_{k(a_1,a_2,\cdots a_k)}(\alpha)\Big|h_{k(x_1,x_2,\cdots,x_k)}(\alpha\beta).$$

Proof: By Lemma 1.1 we have that $h_{k(a_1,a_2,\cdots a_k)}(\alpha)=h_{k(x_1,x_2,\cdots,x_k)}(\alpha)$ and from the fact that if $m=\prod_{i=1}^{t}p_i^{e_i}$ $(t\geq 1)$ where the p_i's are distinct primes and the e_i's are positive integers, then $h_k(m)$ equals the least common multiple of the $h_k(p_i^{e_i})$'s, see [15], we find that $h_{k(a_1,a_2,\cdots a_k)}(\alpha)\Big|h_{k(x_1,x_2,\cdots,x_k)}(\alpha\beta)$.

Definition 1.6: The *polyhedral group* (l,m,n), for $l,m,n>1$, is defined by the presentation

$$\langle x,y,z : x^l=y^m=z^n=xyz=1\rangle.$$

The *polyhedral group* (l,m,n) is finite if, and only if, the number $\mu=lmn\left(\frac{1}{l}+\frac{1}{m}+\frac{1}{n}-1\right)=mn+nl+lm-lmn$ is positive. Its order is $2lmn/\mu$.

For more information on these groups see [7] and [8, pp.67-68].

2. Main Results and Proofs

Definition 2.1: The *centro-polyhedral group* $\langle l,m,n\rangle$, for $l,m,n\in\mathbb{Z}$, is defined by the presentation

$$\langle x,y,z : x^l=y^m=z^n=xyz\rangle.$$

For more information on these groups see [4,7].

Theorem 2.1: The periods of the k-nacci sequences in the groups $\langle -2,n,2\rangle,\langle 2,n,-2\rangle,\langle n,-2,2\rangle$ and $\langle n,2,-2\rangle$, for $n>2$, are $h_k(4(n-1))$.

Proof: These groups have orders $4n(n-1)$. Let us consider the group given by the presentation $\langle -2,n,2\rangle$. We first note in the group defined this presentation both x^{-2} and z^2 are central, $|x|=|z|=4(n-1)$, $|y|=2n(n-1)$ and $x^{-3}=yz$.

If $k = 2$, consider the recurrence relations defined by the following:

$$u_m = u_{m-2} + u_{m-1},\ u_3 = 0, u_4 = 3;$$

$$v_m = v_{m-2} + v_{m-1},\ v_3 = 1, v_4 = 0.$$

Then a routine induction shows that the number of x^{-1}'s and z's in m th entry of the *k*-nacci sequence is given by u_m and v_m respectively.

Here the start of the *2*-nacci sequence is

$$x_1 = x, x_2 = y, x_3 = z, x_4 = yz = x^{-3}, x_5 = x^{-2}zx^{-1},$$

$$x_6 = x^{-4}x^{-1}zx^{-1}, x_7 = x^{-8}x^{-1}z^2 \cdots.$$

For $m > 5$ we can see that the *2*-nacci sequence will separate into some natural layers and each layer will be of the form

$$x_m = \begin{cases} x^{-(u_m-1)}x^{-1}z^{v_m}, & m \equiv 1 \bmod 6, \\ x^{-(u_m-1)}x^{-1}zz^{v_m-1}, & m \equiv 2 \bmod 6, \\ x^{-u_m}zz^{v_m-1}, & m \equiv 3 \bmod 6, \\ x^{-(u_m-1)}x^{-1}z^{v_m}, & m \equiv 4 \bmod 6, \\ x^{-(u_m-1)}zx^{-1}z^{v_m-1}, & m \equiv 5 \bmod 6, \\ x^{-(u_m-2)}x^{-1}zx^{-1}z^{v_m-1}, & m \equiv 0 \bmod 6. \end{cases}$$

Now the proof is finished when we note that the *2*-nacci sequence will repeat when $x_{h_2+3} = z$ and $x_{h_2+4} = x^{-3}$, where h_2 represents the period of the *2*-nacci sequence. Since the *2*-nacci sequence can be said to form layers lenth six then the period is $6.\mu$, $(\mu \in \mathbb{N})$ that is $P+3 \equiv 3 \bmod 6$ and $P+4 \equiv 4 \bmod 6$. Where we denote $P_2(\langle -2, n, 2\rangle; x, y, z)$ by P. Examining this statement in more detail gives

$$x_{P+3} = x^{-u_{P+3}}zz^{(v_{P+3}-1)},$$

$$x_{P+4} = x^{-(u_{P+4}-1)}x^{-1}z^{v_{P+4}}.$$

Using $P+3 \equiv 3 \bmod 6$ and $P+4 \equiv 4 \bmod 6$ we obtain $u_{p+3} = u_3 = 0, v_{p+3} = v_3 = 1, u_{p+4} = u_4 = 3$ and $v_{p+4} = v_4 = 0$. In the case the first of the above equalities gives

$$x_{P+3} = x^{-u_{P+3}} z^{v_{P+3}} = z .$$

The second equality gives

$$x_{P+4} = x^{-u_{P+4}} z^{v_{P+4}} = x^{-3} .$$

The smallest non-trivial integer satisfying the above conditions occurs when the period is $h_2(4(n-1))$.

If $k = 3$, see [4] for a proof. If $k \geq 4$, consider the recurrence relations defined by the following:

$$u_m = u_{m-k} + u_{m-(k-1)} + u_{m-(k-2)} + \cdots + u_{m-1},$$

$$u_3 = 0, u_4 = 0, \cdots, u_{k+1} = 0, u_{k+2} = 3;$$

$$v_m = v_{m-k} + v_{m-(k-1)} + v_{m-(k-2)} + \cdots + v_{m-1},$$

$$v_3 = 1, v_4 = 2, v_5 = 2^2, \cdots, v_{k+1} = 2^{k-2}, v_{k+2} = 2 + 2^2 + \cdots + 2^{k-2} .$$

Then a routine induction shows that the number of x^{-1}'s and z's in m th entry of the k-nacci sequence is given by u_m and v_m respectively.

Here the start of the k-nacci sequence is

$$x_1 = x, x_2 = y, x_3 = z, x_4 = z^2, x_5 = z^{2^2}, \cdots, x_k = z^{2^{k-3}},$$

$$x_{k+1} = z^{2^{k-2}}, x_{k+2} = x^{-2}x^{-1}z^{\left(2+2^2+\cdots+2^{k-2}\right)}, x_{k+3} = x^{-2}zx^{-1}z^{\left(2^2+2^3+\cdots+2^{k-1}\right)},$$

$$x_{k+4} = x^{-4}x^{-1}zx^{-1}z^{\left(2^3+2^4+\cdots+2^{k}\right)}, x_{k+5} = x^{-12}z^{\left(2^4+2^5+\cdots+2^{k+1}\right)}, \cdots .$$

For $m > k+3$ we can see that the k-nacci sequence will separate into some natural layers and each layer will be of the form

$$x_m = \begin{cases} x^{-(u_m-1)}x^{-1}z^{v_m}, & m \equiv 1 \bmod 2k+2, \\ x^{-(u_m-1)}x^{-1}zz^{v_m-1}, & m \equiv 2 \bmod 2k+2, \\ x^{-u_m}zz^{v_m-1}, & m \equiv 3 \bmod 2k+2, \\ x^{-u_m}z^{v_m}, & m \equiv 4 \bmod 2k+2, \\ x^{-u_m}z^{v_m}, & m \equiv 5 \bmod 2k+2, \\ \vdots, & \vdots, \\ x^{-u_m}z^{v_m}, & m \equiv k \bmod 2k+2, \\ x^{-u_m}z^{v_m}, & m \equiv k+1 \bmod 2k+2, \\ x^{-(u_m-1)}x^{-1}z^{v_m}, & m \equiv k+2 \bmod 2k+2, \\ x^{-(u_m-1)}zx^{-1}z^{v_m-1}, & m \equiv k+3 \bmod 2k+2, \\ x^{-(u_m-2)}x^{-1}zx^{-1}z^{v_m-1}, & m \equiv k+4 \bmod 2k+2, \\ x^{-u_m}z^{v_m}, & m \equiv k+5 \bmod 2k+2, \\ x^{-u_m}z^{v_m}, & m \equiv k+6 \bmod 2k+2, \\ \vdots, & \vdots, \\ x^{-u_m}z^{v_m}, & m \equiv 2k+1 \bmod 2k+2, \\ x^{-u_m}z^{v_m}, & m \equiv 0 \bmod 2k+2. \end{cases}$$

Now the proof is finished when we note that the k-nacci sequence will repeat when $x_{h_k+3} = z, x_{h_k+4} = z^2, x_{h_k+5} = z^{2^2}, \cdots, x_{h_k+k} = z^{2^{k-3}}, x_{h_k+k+1} = z^{2^{k-2}}$ and $x_{h_k+k+2} = x^{-3}z^{\left(2+2^2+\cdots+2^{k-2}\right)}$ where h_k represents the period of the k-nacci sequence. Examining this statement in more detail gives

$$x_{P+3} = x^{-u_{P+3}} z z^{v_{P+3}-1},$$
$$x_{P+4} = x^{-u_{P+4}} z^{v_{P+4}},$$
$$x_{P+5} = x^{-u_{P+5}} z^{v_{P+5}},$$
$$\vdots, \qquad \vdots,$$
$$x_{P+k} = x^{-u_{P+k}} z^{v_{P+k}},$$
$$x_{P+k+1} = x^{-u_{P+k+1}} z^{v_{P+k+1}},$$
$$x_{P+k+2} = x^{-(u_{P+k+2}-1)} x^{-1} z^{v_{P+k+2}}.$$

where we denote $P_k(\langle -2,n,2\rangle; x,y,z)$ by P. Using Lemma 1.1 the above equalities give

$$x_{P+3} = x^{-u_{P+3}} z^{v_{P+3}} = z,$$
$$x_{P+4} = x^{-u_{P+4}} z^{v_{P+4}} = z^2,$$
$$x_{P+5} = x^{-u_{P+5}} z^{v_{P+5}} = z^{2^2},$$
$$\vdots, \qquad \vdots,$$
$$x_{P+k} = x^{-u_{P+k}} z^{v_{P+k}} = z^{2^{k-3}},$$
$$x_{P+k+1} = x^{-u_{P+k+1}} z^{v_{P+k+1}} = z^{2^{k-2}},$$
$$x_{P+k+2} = x^{-u_{P+k+2}} z^{v_{P+k+2}} = x^{-3} z^2.$$

The smallest non-trivial integer satisfying the above conditions occurs when the period is $h_k(4(n-1))$.

The proofs for the groups $\langle 2,n,-2\rangle$, $\langle n,-2,2\rangle$ and $\langle n,2,-2\rangle$ are similar and are omitted.

Theorem 2.2: The periods of the *k*-nacci sequences in the groups $\langle 2,-2,n\rangle$ and $\langle -2,2,n\rangle$, for $n > 2$, are as follows:

i. If $k = 2$, the periods are $h_2(4(n-1))$.

ii. If $k \geq 3$, the periods are the smallest non-trivial integers such that

$$4n \mid P_k(\langle 2,-2,n\rangle; x,y,z),\ h_k(4(n-1)) \mid P_k(\langle 2,-2,n\rangle; x,y,z).$$

Proof: These groups have orders $4n(n-1)$. Let us consider the group given by the presentation $\langle 2,-2,n\rangle$. We first note in the group defined by this presentation $|x|=4(n-1)=|y|, |z|=2n(n-1)$, $z=y^{4(n-1)-1}x, x=yz$ and we can deduce the following:

$$z=y^{-1}x, \underline{z}xz=y^{-1}\underline{xx}z=y^{-1}\underline{z}x^2=y^{-1}y^{-1}xx^2=$$
$$=y^{-2}x\underline{x^2}=y^{-4}x=\underline{(y^{-2})}^2x=z^{2n}x$$

i. The proof is similar the proof of Theorem 2.1 and is omitted.
ii. If $k=3$, see [4] for a proof. If $k\geq 4$, consider the recurrence relations defined by the following:

$$u_m=u_{m-k}+u_{m-(k-1)}+u_{m-(k-2)}+\cdots+u_{m-1},$$
$$u_3=1, u_4=0,\cdots, u_{k+1}=0, u_{k+2}=0;$$
$$v_m=v_{m-k}+v_{m-(k-1)}+v_{m-(k-2)}+\cdots+v_{m-1},$$
$$v_3=1, v_4=2, v_5=2^2,\cdots, v_{k+1}=2^{k-2}, v_{k+2}=\left(2+2^2+\cdots+2^{k-2}\right)+1$$

Then a routine induction suffices to show that the number of x's and z's in m th entry of the *k*-nacci sequence is given by u_m and v_m respectively. Here the start of the *k*-nacci sequence is

$$x_1=x, x_2=y, x_3=z, x_4=x^2, x_5=x^{2^2},\cdots, x_k=x^{2^{k-3}}, x_{k+1}=x^{2^{k-2}},$$
$$x_{k+2}=x^{\left(2+2^2+\cdots+2^{k-2}\right)+1}, x_{k+3}=zxx^{\left(2^2+2^3+\cdots+2^{k-1}\right)}, x_{k+4}=xzxx^{\left(2^3+2^4+\cdots+2^{k}\right)},$$
$$x_{k+5}=xz^2xx^{\left(2^4+2^5+\cdots+2^{k+1}\right)},\cdots.$$

For $m>k+3$ we can see that the *k*-nacci sequence will separate into some natural layers and each layer will be of the form

$$
x_m = \begin{cases}
z^{\left(u_m-(m-1)/2\right)n} z^{(m-1)/4} xx^{v_m-1}, & m \equiv 1 \bmod 2k+2, \\
z^{\left(u_m-1\right)n} zxx^{v_m-1}, & m \equiv 2 \bmod 2k+2, \\
z^{\left(u_m-1\right)n} zx^{v_m}, & m \equiv 3 \bmod 2k+2, \\
z^{\left(u_m-(m-4)/2\right)n} z^{(m-4)/2} x^{v_m}, & m \equiv 4 \bmod 2k+2, \\
z^{\left(u_m-(m-5)/2\right)n} z^{(m-5)/2} x^{v_m}, & m \equiv 5 \bmod 2k+2, \\
\quad \vdots, & \quad \vdots, \\
z^{\left(u_m-(m-k)/2\right)n} z^{(m-k)/2} x^{v_m}, & m \equiv k \bmod 2k+2, \\
z^{\left(u_m-(m-(k+1))/2\right)n} z^{(m-(k+1))/2} x^{v_m}, & m \equiv k+1 \bmod 2k+2, \\
z^{\left(u_m-(m-(k+2))/2\right)n} xz^{(m-(k+2))/2} x^{v_m-1}, & m \equiv k+2 \bmod 2k+2, \\
z^{\left(u_m-1\right)} zxx^{v_m-1}, & m \equiv k+3 \bmod 2k+2, \\
z^{\left(u_m-1\right)} xzxx^{v_m-2}, & m \equiv k+4 \bmod 2k+2, \\
z^{\left(u_m-(m-(k+5))/4+2\right)n} xz^{(m-(k+5))/4+2} xx^{v_m-2}, & m \equiv k+5 \bmod 2k+2, \\
z^{\left(u_m-(m-(k+6))/4+2\right)n} xz^{(m-(k+6))/4+2} xx^{v_m-2}, & m \equiv k+6 \bmod 2k+2, \\
\quad \vdots, & \quad \vdots, \\
z^{\left(u_m-(m-(2k+1))/4+2\right)n} xz^{(m-(2k+1))/4+2} xx^{v_m-2}, & m \equiv 2k+1 \bmod 2k+2, \\
z^{\left(u_m-(m-(2k+2))/4+2\right)n} xz^{(m-(2k+2))/4+2} xx^{v_m-2}, & m \equiv 2k+1 \bmod 2k+2.
\end{cases}
$$

Letting $P = P_k\left(\langle 2,-2,n \rangle; x, y, z\right)$ we have:

$$
\begin{aligned}
x_{P+3} &= z^{\left(u_{P+3}-1\right)n} zx^{v_{P+3}}, \\
x_{P+4} &= z^{\left(u_{P+4}-(P+4-4)/2\right)n} z^{(P+4-4)/2} x^{v_{P+4}}, \\
x_{P+5} &= z^{\left(u_{P+5}-(P+5-5)/2\right)n} z^{(P+5-5)/2} x^{v_{P+5}}, \\
\vdots, &\qquad \vdots, \\
x_{P+k} &= z^{\left(u_{P+k}-(P+k-k)/2\right)n} z^{(P+k-k)/2} x^{v_{P+k}}, \\
x_{P+k+1} &= z^{\left(u_{P+k+1}-(P+k+1-(k+1))/2\right)n} z^{(P+k+1-(k+1))/2} x^{v_{P+k+1}}, \\
x_{P+k+2} &= z^{\left(u_{P+k+2}-(P+k+2-(k+2))/2\right)n} xz^{(P+k+2-(k+2))/2} x^{v_{P+k+2}-1}.
\end{aligned}
$$

So we need $h_k(|x|)\,|\,P$ that is $h_k(4(n-1))\,|\,P$, where $h_k(m)$ is the k-step Wall number of the positive integer m. Using Lemma 1.1 and Corollary 1.1 the above equalities give

$$x_{P+3}=z^{(1-1)n}zx^0=z,$$
$$x_{p+4}=z^{(0-(P/2))n}z^{(P)/2}x^2=z^{(P/2)(1-n)}x^2,$$
$$x_{P+5}=z^{(0-(P/2))n}z^{(P)/2}x^4,$$
$$\vdots, \qquad \vdots,$$
$$x_{P+k}=z^{(0-(P/2))n}z^{(P)/2}x^{2^{k-3}},$$
$$x_{P+k+1}=z^{(0-(P/2))n}z^{(P)/2}x^{2^{k-2}},$$
$$x_{P+k+2}=z^{(0-(P)/2)n}xz^{(P)/2}x^{(2+2^2+\cdots+2^{k-2})+1}=xz^{(P/2)(1-n)}x^{(2+2^2+\cdots+2^{k-2})+1}.$$

So we will also need $2n\,|\,P/2$ if $x_{P+4}=x^2, x_{P+5}=x^{2^2},\cdots,x_{P+k}=x^{2^{k-3}}, x_{P+k+1}=x^{2^{k-2}}$ and $x_{P+k+2}=x^{(2+2^2+\cdots+2^{k-2})+1}$. So all we need is P to be the smallest number satisfying

$$4n\,|\,P,$$
$$h_k(4(n-1))\,|\,P.$$

The proof for the group $\langle -2,2,n\rangle$ is similar and is omitted.

Theorem 2.3: The periods of the k-nacci sequences in the groups $\langle -n,2,2\rangle$ and $\langle 2,-n,2\rangle$, for $n>2$, are $2k+2$.

Proof: These groups have orders $4n$. Let us consider the group given by the presentation $\langle -n,2,2\rangle$. We first note that in the group defined by this presentation z^2 is central and $|y|=4, |z|=4$ and $|x|=2n$ then $x^{-n}=x^n$.

If $k=2$, we have the sequence

$$x_1 = x,\ x_2 = y,\ x_3 = z,\ x_4 = yz,\ x_5 = zyz,\ x_6 = y\underline{zz}yz =$$
$$= z^2\underline{yy}z = \underline{z^2y^2}z = z,\ x_7 = zy\underline{zz} = zyz^2 = x\underline{yyz^2} = x,$$
$$x_8 = \underline{zz^3}y = y,\ x_9 = xy = z\cdots$$

which has period 6.

If $k = 3$, see [4] for a proof. If $k \geq 4$, the first k elements of the sequence are

$$x_1 = x, x_2 = y, x_3 = z, x_4 = z^2, x_5 = z^4, \cdots, x_k = z^{2^{k-3}}.$$

Thus, using the above information, the sequence reduces to,

$$x_1 = x,\ x_2 = y,\ x_3 = z,\ x_4 = z^2,\ x_5 = 1, \cdots, 1$$

where $x_j = 1$ for $5 \leq j \leq k$. Thus,

$$x_{k+1} = \prod_{i=1}^{k} x_i = z^{2^{k-2}} = 1,\ x_{k+2} = \prod_{i=2}^{k+1} x_i = yz^3,\ x_{k+3} = \prod_{i=3}^{k+2} x_i = zyz,$$
$$x_{k+4} = \prod_{i=4}^{k+3} x_i = z,\ x_{k+5} = \prod_{i=5}^{k+4} x_i = y\underline{z^3z}yzz = \underline{yy}\underline{zz} = 1, \cdots.$$

It follows that $x_{k+j} = 1$ for $5 \leq j \leq k+1$. We also have,

$$x_{k+k+2} = \prod_{i=k+2}^{k+k+1} x_i = y\underline{z^3z}yzz = \underline{yy}\underline{zz} = 1,$$
$$x_{k+k+3} = \prod_{i=k+3}^{k+k+2} x_i = \underline{zy}z^2 = x\underline{yyz^2} = x,$$
$$x_{k+k+4} = \prod_{i=k+4}^{k+k+3} x_i = zx = y,\ x_{k+k+5} = \prod_{i=k+5}^{k+k+4} x_i = xy = z.$$

Since the elements succeeding $x_{2k+3}, x_{2k+4}, x_{2k+5}$, depend on x, y and z for their values, the cycle begins again with the $2k+3^{\text{nd}}$ element; that is, $x_1 = x_{2k+3}, x_2 = x_{2k+4}, x_3 = x_{2k+5}, \cdots$. Thus, $P_k(\langle -n,2,2\rangle; x, y, z) = 2k+2$.

The proof for the group $\langle 2, -n, 2\rangle$ is similar and is omitted.

Corollary 2.1: The periods of the *k*-nacci sequences in the group $\langle 2,2,-n\rangle$, for $n > 2$, are as follows:

i. The periods of the *k*-nacci sequences in the group $\langle 2,2,-n\rangle$ are $2k+2$ with respect to the generating set $\{z,x,y\}$. That is $P_k(\langle 2,2,-n\rangle;z,x,y)=2k+2$.

ii. The periods of the *k*-nacci sequences in the group $\langle 2,2,-n\rangle$ with respect to the generating set $\{x,y,z\}$ are as follows:

i'. $P_2(\langle 2,2,-n\rangle;x,y,z)=6$.

ii'. $P_{3,4}(\langle 2,2,-n\rangle;x,y,z)=\begin{cases} n(k+1), & n \text{ is even,} \\ 2n(k+1) & n \text{ is odd.} \end{cases}$

iii'. Let $k \geq 5$.

1. If there is no $t \in [3, k-2]$ such that t is an odd factor of n then,

$$P_k(\langle 2,2,-n\rangle;x,y,z)=\begin{cases} n(k+1), & n \text{ is even,} \\ 2n(k+1) & n \text{ is odd.} \end{cases}$$

2. Let α be the biggest odd factor of n in $[3, k-2]$, then two cases occur:

i''. If $\alpha.3^j \notin [3, k-2]$ for $j \in N$, then

$$P_k(\langle 2,2,-n\rangle;x,y,z)=\begin{cases} \alpha(n(k+1)), & n \text{ is even,} \\ \alpha(2n(k+1)) & n \text{ is odd.} \end{cases}$$

ii''. If β is the biggest odd number which is in $[3, k-2]$ and $\beta=\alpha 3^j$ for $j \in N$, then

$$P_k(\langle 2,2,-n\rangle;x,y,z)=\begin{cases} \beta(n(k+1)), & n \text{ is even,} \\ \beta(2n(k+1)) & n \text{ is odd.} \end{cases}$$

Proof: We first note that the order of z is $2n$, the orders of x and y are 4 and the order of the group is $4n$.

i. The proof is similar to the proof of the Theorem 2.3 and is omitted.

ii. i'. If $k=2$, we have the sequence

$$x_1=x,\ x_2=y,\ x_3=z,\ x_4=yz=x,\ x_5=zx=y^3\underline{xx}=\underline{y^3y^2}=y,$$

$$x_6=xy,\ x_7=yxy,\ x_8=x\underline{yy}xy=\underline{xx^2xy}=y,$$

$$x_9=yx\underline{yy}=y^3x=z,\ x_{10}=yz=x,\cdots.$$

Thus, $P_2\left(\langle 2,2,-n\rangle;x,y,z\right)=6$.

Since $\langle 2,2,-n\rangle\cong\langle 2,2,n\rangle$, the proof follows from the results for $\langle 2,2,n\rangle$. If $k=3$, see [4] and if $k\geq 4$, see [9] for a proof.

Corollary 2.2: The periods of the *k*-nacci sequences in the group $(-2,2,n),(2,-2,n)$ and $(2,2,-n)$, for $n>2$, are as follows:

i. The periods of the *k*-nacci sequences in the groups $(-2,2,n),(2,-2,n)$ and $(2,2,-n)$ are $2k+2$ with respect to the generating set $\{z,x,y\}$. That is $P_k\left(G_n;z,x,y\right)=2k+2$.

ii. The periods of the *k*-nacci sequences in the groups $(-2,2,n),(2,-2,n)$ and $(2,2,-n)$ with respect to the generating set $\{x,y,z\}$ are follows:

i'. $P_2\left(G_n;x,y,z\right)=6$.

ii'. $$P_{3,4}\left(G_n;x,y,z\right)=\begin{cases} n\left(\dfrac{k+1}{2}\right), & n\equiv 0 \bmod 4,\\ n(k+1), & n\equiv 2 \bmod 4,\\ 2n(k+1), & \text{otherwise.}\end{cases}$$

iii'. Let $k\geq 5$.

1. If there is no $t\in[3,k-2]$ such that t is a odd factor of n then,

$$P_k(G_n;x,y,z)=\begin{cases} n\left(\frac{k+1}{2}\right), & n\equiv 0 \bmod 4, \\ n(\mathrm{k}+1), & n\equiv 2 \bmod 4, \\ 2n(k+1), & \text{otherwise.} \end{cases}$$

2. Let α be the biggest odd factor of n in $[3, k-2]$, then two cases occur:

i''. If $\alpha 3^j \notin [3, k-2]$ for $j \in N$, then

$$P_k(G_n;x,y,z)=\begin{cases} \alpha\left(n\left(\frac{k+1}{2}\right)\right), & n\equiv 0 \bmod 4, \\ \alpha(n(k+1)), & n\equiv 2 \bmod 4, \\ \alpha(2n(k+1)), & \text{otherwise.} \end{cases}$$

ii''. If β is the biggest odd number which is in $[3, k-2]$ and $\beta=\alpha 3^j$ for $j \in N$, then

$$P_k(G_n;x,y,z)=\begin{cases} \beta\left(n\left(\frac{k+1}{2}\right)\right), & n\equiv 0 \bmod 4, \\ \beta(n(k+1)), & n\equiv 2 \bmod 4, \\ \beta(2n(k+1)), & \text{otherwise.} \end{cases}$$

Here G_n is one of the groups $(-2,2,n)$, $(2,-2,n)$ and $(2,2,-n)$.

Proof: We first note the order of z is n, the orders of x and y are 2 in the group G_n and the order of the group G_n is $2n$.

i. The proof is similar to the proof of Theorem 2.3 and is omitted.

ii. i'. If $k=2$, we have the sequence

$$x_1 = x, x_2 = y, x_3 = z, x_4 = yz = x, x_5 = zx = y\underline{xx} = y,$$
$$x_6 = xy, x_7 = yxy, x_8 = x\underline{yy}xy = \underline{xx^2xy} = y,$$
$$x_9 = yx\underline{yy} = yx = z, x_{10} = yz = x, \cdots.$$

Thus, $P_2(G_n; x, y, z) = 6$.

Since $G_n \cong (2,2,n)$, the proofs follow from the results for $(2,2,n)$. If $k = 3$, see [4] and if $k \geq 4$, see [13] for a proof.

Corollary 2.3: The periods of the k-nacci sequences in the groups $(-2,n,2)$, $(2,-n,2)$, $(2,n,-2)$, $(-n,2,2)$, $(n,-2,2)$ and $(n,2,-2)$, for $n > 2$, are $P_k(G_n; x, y, z) = 2k+2$. Where G_n is one of the groups $(-2,n,2)$, $(2,-n,2)$, $(2,n,-2)$, $(-n,2,2)$, $(n,-2,2)$ and $(n,2,-2)$.

Proof: Since $(-2,n,2) \cong (2,-n,2) \cong (2,n,-2) \cong (2,n,2)$ and $(-n,2,2) \cong (n,-2,2) \cong (n,2,-2) \cong (n,2,2)$, the proofs follow from the results for $(2,n,2)$ and $(n,2,2)$. If $k = 3$, see [4] and if $k \geq 4$ and $k = 2$, see [13] for a proof.

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