Every toroidal graph without 4- and 6-cycles is acyclically 5-choosable [1]

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Abstract

A proper vertex coloring of a graph $G = (V, E)$ is acyclic if G contains no bicolored cycle. A graph G is acyclically L-list colorable if for a given list assignment $L = \{L(v) : v \in V\}$, there exists a proper acyclic coloring ϕ of G such that $\phi(v) \in L(v)$ for all $v \in V(G)$. If G is acyclically L-list colorable for any list assignment with $|L(v)| = k$ for all $v \in V$, then G is acyclically k-choosable. In this paper it is proved that every toroidal graph without 4- and 6-cycles is acyclically 5-choosable.

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1 Introduction

All graphs considered in this paper are finite simple *toroidal* graphs. A graph G is *toroidal* (or *planar*) if G can be drawn on the *torus* (or on the *plane*) so that the edges meet only at the vertices of the graph. A face f is called a *2-cell* if any simple closed curve inside f can be continuously contracted to a single point. An embedding of G is called a *2-cell embedding* if all the faces are *2-cell*. We assume that all graphs under consideration admit *2-cell embeddings* on the torus.

Let $G = (V, E, F)$ denote a *toroidal* graph, with V, E and F being the set of vertices, edges and faces of G, respectively. A *proper coloring* of a graph G is a mapping ϕ from $V(G)$ to the set of colors $\{1, 2, 3, \cdots, k\}$ such that $\phi(x) \neq \phi(y)$ for every edge xy of G. A *proper vertex coloring* of a graph is *acyclic* if there is no bicolored cycle in G. The *acyclic chromatic number*, denoted by $\chi_a(G)$, of a graph G is the smallest integer k such that G has an *acyclic coloring* using k colors.

The *acyclic colorings* of graphs were introduced by Grünbaum in [10] and studied by Mitchem [14], Albertson and Berman [1], and Kostochka [13]. In 1979, Borodin [3] proved Grünbaum's conjecture that every planar

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graph is acyclically 5-colorable. This result is best possible in the sense that there exist infinitely many planar graphs G such that $\chi_a(G) = 5$. In 1973, a 4-regular planar graph which is not acyclically 4-colorable was first obtained by Grünbaum in [10].

Borodin, Kostochka and Woodall [5] studied the acyclic chromatic number of planar graphs with a given girth, conditionally improved the upper bound by showing that if G is a planar graph of girth g then $\chi_a(G) \leq 4$ if $g \geq 5$, $\chi_a(G) \leq 3$ if $g \geq 7$. We recall that the girth of a graph is the length of its shortest cycle.

A graph $G = (V, E)$ is L-list colorable if for a given list assignment $L = \{L(v) : v \in V(G)\}$, there exists a *proper coloring* c of G such that $c(v) \in L(v)$ for all $v \in V$. If G is L-list colorable for every list assignment L with $|L(v)| \geq k$ for all $v \in V$, then G is said to be k-choosable. A graph G is acyclically L-list colorable if for a given list assignment $L = \{L(v) : v \in V\}$, there exists a proper acyclic coloring ϕ of G such that $\phi(v) \in L(v)$ for all $v \in V(G)$. If G is acyclically L-list colorable for any list assignment with $|L(v)| = k$ for all $v \in V$, then G is acyclically k-choosable. The acyclic list chromatic number of G, $\chi_a^l(G)$, is the smallest integer k such that G is acyclically k-choosable.

Borodin et al. [4] first investigated the acyclic list coloring of planar graphs. They showed that every planar graph is acyclically 7-choosable and put forward the following challenging conjecture:

Conjecture 1 *Every planar graph is acyclically 5-choosable.*

If Conjecture 1 is true, it would imply both Borodin's acyclic 5 color theorem [3] and Thomassen's list 5 color theorem [19] about planar graphs.

In the course of studying the maximum average degree of graphs, Montassier, Ochem and Raspaud [17] showed that if G is a planar graph of girth g then $\chi_a^l(G) \leq 3$ if $g \geq 8$, $\chi_a^l(G) \leq 4$ if $g \geq 6$ and $\chi_a^l(G) \leq 5$ if $g \geq 5$. Wang and Chen [20] proved that every planar graph without 4-cycles is acyclically 6-choosable. Some sufficient conditions for a planar graph to be acyclically 4-choosable or 3-choosable were established in [6, 8, 9, 11, 12, 15, 16]. Montassier, Raspaud and Wang [18] proved that every planar graph G without 4-cycles and 5-cycles, or without 4-cycles and 6-cycles is acyclically 5-choosable. Chen and Wang [7] prove that every planar graph without 4-cycles and without two 3-cycles at distance less than 3 is acyclically 5-choosable.

Let $\mathcal{G}$ denote the set of *toroidal* graphs without 4-cycles and 6-cycles. In this article, we focus on the acyclic choosability of graphs in $\mathcal{G}$. More precisely, we prove the following result.

Theorem 1 *Every toroidal graph without* 4- *and* 6-*cycles is acyclically* 5-*choosable.*

2 Notation

We use $b(f)$ to denote the boundary walk of a face f and write $f = [v_1v_2v_3\dots v_n]$ if $v_1, v_2, v_3, \dots, v_n$ are the vertices of $b(f)$ in a cyclic order. A face f is incident with all vertices and edges on $b(f)$. Let $d_G(x)$, or simply $d(x)$, denote the degree of x in G. A vertex (resp. face) of degree k is called a *k-vertex* (resp. *k-face*). If $r \leq k$ or $1 \leq k \leq r$, then a k-vertex (resp. k-face) is called an r^+- or r^--*vertex* (resp. r^+- or r^--*face*), respectively. A k-cycle is a cycle with k edges.

For a vertex $v \in V(G)$, let $n_i(v)$ denote the number of i-vertices adjacent to v for $i \geq 1$, and let $m_3(v)$ be the number of 3-faces incident with v. For a face $f \in F(G)$, let $n_i(f)$ denote the number of i-vertices incident with f for $i \geq 2$, and let $m_3(f)$ be the number of 3-faces adjacent to f.

A 3-face $f = [v_1v_2v_3]$ is called an (a_1, a_2, a_3)-face if the degree of the vertex v_i is a_i for $i = 1, 2, 3$. A 3-vertex is called *light* if it is incident with a 3-face. If a vertex v is adjacent to a 3-vertex u such that the edge uv is not incident with any 3-face, then we call u a *pendent* 3-vertex of v. A *pendent light* 3-vertex is a *light* and *pendent* 3-vertex. If v is a *pendent light* 3-vertex incident with an (a_1, a_2, a_3)-face, then v is called a *pendent light* (a_1, a_2, a_3)-vertex. A triangle is synonymous with a 3-cycle.

3 Structural properties

Suppose that H is a counterexample with the smallest number of vertices to Theorem 1. We first investigate the structural properties of H, and then use Euler's formula and the discharging technique to derive a contradiction. The following Lemma 1 holds for H, where the proofs of (C4), (C5.3), (C6.3), (C7.2) and (C8) were provided in [7] (see Lemmas 2-6 of [7]), and the proofs of the others can be found in Lemma 1 of [18].

Lemma 1 *A minimal counterexample H to Theorem 1 satisfies the following.*

(C1) *H contains no 1-vertices.*
(C2) *A 2-vertex is not adjacent to a vertex of degree at most* 4.
(C3) *Let v be a* 3*-vertex. Then*
  (C3.1) *If v is adjacent to a* 3*-vertex, then v is not adjacent to other* 4^-*-vertices;*
  (C3.2) *v is not adjacent to any pendent light* 3*-vertex.*
(C4) *Every* 4*-vertex is adjacent to at most one pendent light* 3*-vertex and a pendent light* 3*-vertex of a* 4*-vertex must be a pendent light* $(3, 5^+, 5^+)$*-vertex.*
(C5) *Let v be a* 5*-vertex. Then*

(C5.1) *v is adjacent to at most one 2-vertex;*

(C5.2) *If $n_2(v)$=1, then v is not adjacent to any pendent light 3-vertex;*

(C5.3) *If $n_2(v)$=1 and v is incident with a 3-face f, then $n_3(f) = 0$.*

(C6) *Let v be a 6-vertex. Then*

(C6.1) *v is adjacent to at most four 2-vertices;*

(C6.2) *If $n_2(v)$=4, then v is not adjacent to any 3-vertex;*

(C6.3) *If $n_2(v)$=4, then v is not incident with any 3-face.*

(C7) *Let v be a 7-vertex. Then*

(C7.1) *v is adjacent to at most five 2-vertices;*

(C7.2) *If $n_2(v)$=5, then $n_3(v) = 0$ and v is not incident with any 3-face.*

(C8) *Let v be an 8-vertex. Then v is adjacent to at most six 2-vertices.*

(C9) *There does not exist a 3-face $[xyz]$ with $d(x) \le d(y) \le d(z)$ such that one of the following holds:*

(C9.1) *$d(x)$=2;*

(C9.2) *$d(x)$=$d(y)$=3 and $d(z) \le 5$;*

(C9.3) *$d(x)$=3 and $d(y)$=$d(z)$=4.*

(C10) *There does not exist a 5-face $[v_1v_2\cdots v_5]$ such that $d(v_1)$=2, $d(v_2)$=5, and $d(v_3)$=3.*

Observation 1 *Let H be a graph described above. Then we have:*

(O1) H has no 4-faces, no two adjacent 3-faces, and no 3-face adjacent to 5-faces;
(O2) $n_2(f) \le \lfloor \frac{d(f)}{2} \rfloor$;
(O3) $n_3(f) \le \lfloor \frac{2d(f)}{3} \rfloor$;
(O4) $2n_2(f) + n_3(f) + 1 \le d(f)$ if $n_3(f) \ge 1$;
(O5) $m_3(f) + 2n_2(f) \le d(f) - \lceil \frac{1}{2} n_3(f) \rceil$;
(O6) $m_3(f) + n_3(f) \le d(f)$.

Proof. (O1) to (O4) were trivial by Lemma 1 and the properties of H. As to (O5) and (O6), combining the following two facts with (O2) and (O3) will get the proof of the two results. Suppose tu, uv and vw are three consecutive edges on the boundary of a 5^+-face f and f_{uv} is a 3-face, then at most two of t, u, v and w are 3-vertices by (C3.1) and (C3.2). Another fact is that for any 3-vertex v on $b(f)$, at most one of its two incident edges on $b(f)$ is adjacent to a 3-face by (O1). ∎

4 Proof of Theorem 1

By contradiction, suppose that G is a minimal counterexample to Theorem 1. That is, G is a non-acyclically 5-choosable *toroidal* graph with the smallest number of vertices. Let L be a list assignment of G with $|L(v)| = 5$ for all $v \in V(G)$ such that G is not acyclically L-colorable.

We define a weight function ω on $V \cup F$ by letting $\omega(x) = d_G(x) - 4$ for each $x \in V \cup F$. By Euler's formula for *toroidal* graphs, we have $\sum_{x \in V \cup F} \omega(x) = 0$. If we obtain a new nonnegative weight $\omega^*(x)$ for all $x \in V \cup F$ and some positive weight for some $x \in V \cup F$ by transferring weights from one element to another, this leads to the following obvious contradiction,

$$0 = \sum_{x \in V \cup F} \omega(x) = \sum_{x \in V \cup F} \omega^*(x) > 0,$$

that will complete the proof of Theorem 1.

Our transferring rules are as follows:

(R1) Every 5^+-vertex v gives $\frac{1}{2}$ to each adjacent 2-vertex.

(R2) Every 5^+-vertex v gives $\frac{w(v) - \frac{1}{2} n_2(v)}{n_3(v)}$ to each adjacent 3-vertex.

Let $\beta(v)$ denote the total weight of a 3-vertex v obtained from its adjacent 5^+-vertices.

(R3) Every 5^+-face f gives $\frac{1}{2}$ to each incident 2-vertex.

(R4) Every 5^+-face f gives $\frac{1}{3}$ to each incident 3-vertex.

(R5) Every 7^+-face f gives $\frac{1}{3}$ to each adjacent 3-face.

(R6) Every 3-vertex v donates $\beta(v)/3$ to each of its incident 5-faces.

For an edge $e \in b(f)$, we use f_e to denote the face adjacent to f by sharing the common edge e.

During a discharging procedure, let $\tau(x \to y)$ denote the amount of weights transferred from x to y. We first obtain the following claim.

Claim 1 *Let G be a graph described above. Then the following hold.*

(C11) *If v is a pendent light 3-vertex, then v is adjacent to at least two 5^+-vertices;*
(C12) *Every 8^+-vertex gives at least $\frac{1}{2}$ to each adjacent 3-vertex;*
(C13) *Let u be a pendent light 3-vertex of v_1, and suppose that $N(u) = \{v_1, v_2, v_3\}$. Then for $i \in \{1, 2, 3\}$, $\tau(v_i \to u) \geq \frac{1}{5}$ if $d(v_i) = 5$, $\tau(v_i \to u) \geq \frac{1}{6}$ if $d(v_i) = 6$, and $\tau(v_i \to u) \geq \frac{1}{3}$ if $d(v_i) = 7$.*

Proof. Let v be a *pendent light* 3-vertex of u. By (C3.2), we see that $d(u) \geq 4$. If $d(u) = 4$, then (C11) is true by (C4). If $d(u) \geq 5$, (C11) is also true by (C9.3).

Since every 5^+-vertex transfers charge only to its adjacent 2-veritces or 3-vertices, by $d(v) \geq n_2(v) + n_3(v)$, we have $w(v) - \frac{1}{2}(n_2(v) + n_3(v)) \geq d(v) - 4 - \frac{1}{2}d(v) = \frac{1}{2}d(v) - 4 \geq 0$ while $d(v) \geq 8$. Therefore, $\frac{\omega(v) - \frac{1}{2}n_2(v)}{n_3(v)} \geq \frac{1}{2}$ while $n_3(v) \geq 1$, and hence (C12) holds.

Now let us prove (C13). If $d(v_1) = 5$, then $n_2(v_1) = 0$ by (C5.1) and (C5.2), so $\tau(v_1 \to u) = \frac{w(v_1) - \frac{1}{2}n_2(v_1)}{n_3(v_1)} \geq \frac{1}{5}$ by (R1) and (R2). For $i = 2, 3$, if $d(v_i) = 5$, then $n_2(v_i) = 0$ by (C5.1) and (C5.3), and hence $\tau(v_i \to u) \geq \frac{1}{5}$.

Suppose that $d(v_i) = 6$, $i \in \{1, 2, 3\}$. Then, $n_2(v_i) \leq 3$ by (C6.2) and (C6.3). By (R2), it is easy to deduce that $\tau(v_i \to u) = \frac{w(v_i) - \frac{1}{2}n_2(v_i)}{n_3(v_i)} \geq \frac{2 - \frac{1}{2}n_2(v_i)}{6 - n_2(v_i)} = \frac{1}{2} - \frac{1}{6 - n_2(v_i)} \geq \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

Suppose that $d(v_i) = 7$, $i \in \{1, 2, 3\}$. Then we have $n_2(v_i) \leq 4$ by (C7.2). By (R2), similarly, we may derive that $\tau(v_i \to u) = \frac{w(v_i) - \frac{1}{2}n_2(v_i)}{n_3(v_i)} \geq \frac{3 - \frac{1}{2}n_2(v_i)}{7 - n_2(v_i)} = \frac{1}{2} - \frac{1}{14 - 2n_2(v_i)} \geq \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$. ∎

By (C3.2), it is easy to obtain the following claim.

Claim 2 *Suppose tu, uv and vw are three consecutive edges on the boundary of a 5^+-face f and $d(u) = d(v) = 3$. Then $d(f_{tu}) \geq 4$ and $d(f_{vw}) \geq 4$.*

Now, we proceed to calculate $\omega^*(x)$ for $x \in V(G) \cup F(G)$.

Let v be a k-vertex of G. Then $k \geq 2$ by (C1).

If $k = 2$, then v is incident with two 5^+-faces by (C9.1) and (O1). Moreover, v is adjacent to two 5^+-vertices by (C2). By (R1) and (R3), we derive that $\omega^*(v) = 2 - 4 + 2 \times \frac{1}{2} + 2 \times \frac{1}{2} = 0$.

If $k = 3$, then $\omega(v) = -1$. It is easy to see that $m_3(v) \leq 1$ by (O1). Namely, v is incident to at least two 5^+-faces. If $m_3(v) = 1$, then v is a pendent light 3-vertex and there is no 5-face incident to v by (O1). Moreover, by (C11), at least two vertices in $N(v)$ are of degree at least 5. So, by (C13) and (R4), $\omega^*(v) = -1 + 2 \times \frac{1}{3} + 2 \times \frac{1}{6} = 0$. Now suppose that $m_3(v) = 0$. Then v is incident to three 5^+-faces. By (R4), we have that $\omega^*(v) \geq -1 + 3 \times \frac{1}{3} = 0$.

If $k = 4$, it is clear that $\omega^*(v) = \omega(v) = d(v) - 4 = 0$.

Suppose that $k \geq 5$. If $n_3(v) \geq 1$, then by (R1) and (R2) we obtain that $\omega^*(v) \geq \omega(v) - \frac{1}{2}n_2(v) - n_3(v) \times \frac{w(v) - \frac{1}{2}n_2(v)}{n_3(v)} = 0$. In what follows, we suppose that $n_3(v) = 0$. If $k = 5$, then v is adjacent to at most one 2-vertex by (C5.1) and thus $\omega^*(v) \geq 1 - \frac{1}{2} = \frac{1}{2}$. If $6 \leq k \leq 7$, then $n_2(v) \leq k - 2$ by (C6.1), (C7.1) and thus $\omega^*(v) \geq k - 4 - \frac{1}{2}n_2(v) \geq k - 4 - \frac{1}{2}(k - 2) = \frac{1}{2}k - 3 \geq 0$. If $k \geq 8$, then $\omega^*(v) \geq k - 4 - \frac{1}{2}n_2(v) \geq k - 4 - \frac{1}{2}k \geq 0$.

Let f be an h-face of G. The following proof is divided into four cases according to the value of h.

Case 1. $h = 3$. Then $w(f) = -1$. By (O1), f is adjacent to three 7^+-faces. Thus, $\omega^*(f) = -1 + 3 \times \frac{1}{3} = 0$ by (R5).

Suppose that $h \geq 5$. We write $f = [x_1 x_2 \cdots x_h]$.

Case 2. $h = 5$. Then $w(f) = 1$.

By (O1) and (O2), we have that $m_3(f) = 0$ and $n_2(f) \leq 2$, respectively. Note that every 3-vertex incident to f which gets charges from its adjacent 5^+-vertices sends charges to its incident 5-faces. If $n_2(f) = 0$, $n_3(f) \leq 3$ by (O3). So by (R4), we have that

$$w^*(f) \geq w(f) - 3 \times \frac{1}{3} = 0. \quad (1)$$

If $n_2(f) = 2$, then $n_3(f) = 0$ by (O4), and hence by (R3),

$$w^*(f) = w(f) - 2 \times \frac{1}{2} = 0. \quad (2)$$

Suppose that $n_2(f) = 1$. Then $n_3(f) \leq 2$ by (O4). If $n_3(f) \leq 1$, then

$$w^*(f) \geq w(f) - \frac{1}{2} - \frac{1}{3} = \frac{1}{6} > 0. \quad (3)$$

So, we further suppose that $n_3(f) = 2$.

Without loss of generality, we suppose that $d(x_1) = 2$ and $d(x_3) = d(x_4) = 3$. Let y_3 be the neighbor of x_3 different from x_2 and x_4, and y_4 the neighbor of x_4 different from x_3 and x_5. (C10) asserts that $d(x_2) \geq 6$ and $d(x_5) \geq 6$. If $d(x_2) \geq 8$, then x_2 gives at least $\frac{1}{2}$ to x_3 by (C12). If $d(x_2) = 7$, then x_2 is adjacent to at most five 2-vertices by (C7) and hence x_2 gives x_3 at least $(3 - \frac{5}{2})/2 = \frac{1}{4}$. Assume that $d(x_2) = 6$. It follows from (C6.1) and (C6.2) that $n_2(x_2) \leq 3$ and thus x_2 gives x_3 at least $(2 - \frac{3}{2})/3 = \frac{1}{6}$. Moreover, we see that $d(y_3) \geq 5$ by (C3.1). If $d(y_3) \geq 6$, then by a similar discussion as above, y_3 gives a weight at least $\frac{1}{6}$ to x_3. If $d(y_3) = 5$, then $n_2(y_3) \leq 1$ by (C5.1) and y_3 gives x_3 at least $(1 - \frac{1}{2})/4 = \frac{1}{8}$. So we always have that $\beta(x_3) \geq \frac{1}{6} + \frac{1}{8} = \frac{7}{24}$ by (R1) and (R2). The same argument works for the vertex x_4. Therefore by (R3), (R4) and (R6)

$$w^*(f) = 1 - \frac{1}{2} - 2 \times \frac{1}{3} + \beta(x_3)/3 + \beta(x_4)/3 \geq -\frac{1}{6} + 2 \times \frac{7}{24 \times 3} = \frac{1}{36} > 0. \quad (4)$$

Case 3. $h = 7$. Then $w(f) = 3$.

By (O2), we have $n_2(f) \leq 3$. If $n_2(f) = 3$, then $n_3(f) = 0$ by (O4), $m_3(f) \leq 1$ by (O5). So by (R3), (R4) and (R5),

$$w^*(f) \geq 3 - 3 \times \frac{1}{2} - \frac{1}{3} = \frac{7}{6} > 0. \quad (5)$$

If $n_2(f) = 2$, then $n_3(f) \leq 2$ by (O4) and $m_3(f) \leq 3$ by (O5). Thus by (R3), (R4) and (R5), we have

$$w^*(f) \geq 3 - 2 \times \frac{1}{2} - 5 \times \frac{1}{3} = \frac{1}{3} > 0. \tag{6}$$

If $n_2(f) = 1$, then $n_3(f) \leq 4$ by (O4) and $m_3(f) \leq 5$ by (O5). But it is easy to observe that $n_3(f) \leq 3$ by (C3.1). When $n_3(f) = 3$, we have $m_3(f) \leq 3$ by (O5) and

$$w^*(f) \geq 3 - \frac{1}{2} - 6 \times \frac{1}{3} = \frac{1}{2} > 0. \tag{7}$$

When $n_3(f) = 2$, we have $m_3(f) \leq 4$ by (O5) and

$$w^*(f) \geq 3 - \frac{1}{2} - 6 \times \frac{1}{3} = \frac{1}{2} > 0. \tag{8}$$

When $n_3(f) \leq 1$, by (R3), (R4) and (R5), we have

$$w^*(f) \geq 3 - \frac{1}{2} - 6 \times \frac{1}{3} = \frac{1}{2} > 0. \tag{9}$$

Finally suppose that $n_2(f) = 0$. Since $n_3(f) + m_3(f) \leq 7$ by (O6), we have

$$w^*(f) \geq 3 - 7 \times \frac{1}{3} = \frac{2}{3} > 0. \tag{10}$$

Case 4. $h \geq 8$. Then $w(f) = h - 4 \geq 4$.

Applying (O3) and (O5), we can establish the following estimate:

$$\begin{aligned} w^*(f) &= h - 4 - (\tfrac{1}{3}m_3(f) + \tfrac{1}{3}n_3(f) + \tfrac{1}{2}n_2(f)) \\ &\geq h - 4 - [\tfrac{1}{3}(h - 2n_2(f) - \lceil\tfrac{1}{2}n_3(f)\rceil) + \tfrac{1}{2}n_2(f) + \tfrac{1}{3}n_3(f)] \\ &= h - 4 - \tfrac{1}{3}h + \tfrac{1}{6}n_2(f) + \tfrac{1}{3}\lceil\tfrac{1}{2}n_3(f)\rceil - \tfrac{1}{3}n_3(f) \\ &\geq h - 4 - \tfrac{1}{3}h + \tfrac{1}{6}n_2(f) - \tfrac{1}{6}n_3(f) \geq h - 4 - \tfrac{1}{3}h - \tfrac{1}{6}n_3(f) \\ &\geq \tfrac{2}{3}h - 4 - \tfrac{1}{6}\lfloor\tfrac{2}{3}h\rfloor \\ &\geq \tfrac{5}{9}h - 4 > 0. \end{aligned} \tag{11}$$

Now, we get that $\omega^*(x) \geq 0$ for each $x \in V(G) \cup F(G)$. It follows that $0 = \sum_{x \in V \cup F} \omega(x) = \sum_{x \in V \cup F} \omega^*(x) \geq 0$. If there exists an element $x \in V(G) \cup F(G)$ such that $\omega^*(x) > 0$, we are done. Otherwise, in the following, we assume that $\omega^*(x)_{x \in V(G) \cup F(G)} = 0$.

Claim 3 *Each face in G has degree 3, or 5.*

Proof. By the proof for the case $h = 7$ or 8^+, the claim follows.

Claim 4 *G contains no 3-face.*

Proof. If G contains a 3-face, by Claim 3 and (O1), $G = C_3$, a contradiction to (C2).

Claim 5 *G contains no 5-face.*

Proof. Assume G contains a 5-face, by Claim 3 and Claim 4, the number of 5-faces in G is at least two, otherwise $G = C_5$, which is a contradiction to (C2). Then by (1), (2) (3) and (4) in the proof of Case 2, for any 5-face f, we only have the following two possible cases, $n_2(f) = 0$ or $n_2(f) = 2$. We will further show that both these two cases are impossible.

If $n_2(f) = 2$, then $n_3(f) = 0$ by (O4) and thus $w^*(f) = 0$ by (2). We assume that $b(f) = [v_1v_2v_3v_4v_5]$ and $d(v_1) = d(v_3) = 2$. Then $d(v_i) \geq 5$ ($i = 2, 4, 5$) by (C2). Since any 2-vertex is incident with two 5^+-faces by (C9.1), $d(f_{v_1v_2}) = d(f_{v_2v_3}) = 5$ by Claim 3 and Claim 4. However, a 6-cycle is established, which is a contradiction.

Now suppose $n_2(f) = 0$, then $n_3(f) \leq 3$ by (O4). If $n_3(f) \leq 2$, then $w^*(f) \geq 1 - \frac{1}{3} \times 2 > 0$. Consider the case that $n_3(f) = 3$. By (C3.1), we may assume that $d(v_1) = d(v_2) = d(v_4) = 3$. Then $d(v_3) \geq 5$ and $d(v_5) \geq 5$ by (C3.1). In this situation, by (R2), (R4) and (R6), $w^*(f) = 1 - \frac{1}{3} \times 3 + \frac{\beta(v_1)}{3} + \frac{\beta(v_2)}{3} + \frac{\beta(v_4)}{3} = \sum_{i \in \{1,2,4\}} \frac{\beta(v_i)}{3}$, note that $\beta(v_i) \geq 0$ by (R2). Moreover, applying the same analysis as used in Case 2 for $n_2(f) = 1$ and $n_3(f) = 2$, we can get $\beta(v_i) > 0$ for $i \in \{1, 2, 4\}$ and thus we complete the proof of Theorem 1. ∎

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