The Exterior of a Graph or Tree

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ABSTRACT

The eccentricity $e(v)$ of a vertex v in a connected graph G is the distance between v and a vertex furthest from v. The center $C(G)$ is the subgraph induced by those vertices whose eccentricity is the radius of G, denoted rad$G$, and the periphery $P(G)$ is the subgraph induced by those vertices with eccentricity equal to the diameter of G, denoted diam$G$. The annulus $Ann(G)$ is the subgraph induced by those vertices with eccentricities strictly between the radius and diameter of G. In a graph G where rad$G$ < diam$G$, the interior of G is the subgraph $Int(G)$ induced by the vertices v with $e(v)$ < diam$G$. Otherwise, if rad$G$ = diam$G$, then $Int(G) = G$. Another subgraph for a connected graph G with rad$G$ < diam$G$, called the exterior of G, is defined as the subgraph $Ext(G)$ induced by the vertices v with rad$G$ < $e(v)$. As with the interior, if rad$G$ = diam$G$, then $Ext(G) = G$. In this paper, the annulus, interior, and exterior subgraphs in trees are characterized.

Key Words: distance, eccentricity, trees, annulus, interior, exterior

1. Introduction

In a connected graph G, the distance from a vertex v to a vertex u is commonly denoted as $d(v, u)$. For each vertex v of G, the *eccentricity* $e(v) = \max\{d(v, u) \mid u \in V(G)\}$. The *radius* is defined as rad$G = \min\{e(v) \mid v \in V(G)\}$, and the *diameter* is diam$G = \max\{e(v) \mid v \in V(G)\}$. If a graph is disconnected, the eccentricities are defined to be ∞; however, each connected subgraph is called a component, and by restricting the focus to a single component, the eccentricities, radius and diameter can be defined for that component.

From these two extreme eccentricity values, two subgraphs were created: the *center* $C(G)$ is the subgraph of G induced by the vertices v with $e(v)$ = rad$G$ and the *periphery* $P(G)$ is the subgraph induced by the vertices v with $e(v)$ = diam$G$. These subgraphs have been studied extensively (see [1], [4], and [6]). In particular, Hedetniemi (see [4]) proved that every graph is the center of some connected graph; while Jordan [10] proved that the center of a tree is either K_1 or K_2. If the center is K_1, then the tree is called a *central* tree; otherwise the tree is called *bicentral*. Bielak and Syslo [1] proved that a graph G is the periphery of some connected graph if and only if no vertex of G has eccentricity 1 or G is complete, and that the periphery of a tree consists only of end-vertices. If rad$G$ = diam$G$, then G is called *self-centered.*

More recently, subgraphs of connected graphs G induced by vertices with intermediate eccentricity values were investigated. For instance, in [7] the interior and annulus for a graph were introduced and characterized, and in [8] and [9], the annulus was studied further. In a graph G where rad$G$ < diam$G$, the *interior* of G is the subgraph $Int(G)$ induced by the vertices v with $e(v)$ < diam$G$. Otherwise, if rad$G$ = diam$G$, then $Int(G) = G$. (In [5] the interior of a graph was defined differently, and in general, the two definitions do not describe the same subgraph.) Second, in a graph G with rad$G$ < diam$G$ - 1, the *annulus* of G is the subgraph $Ann(G)$ induced by the vertices v with rad$G$ < $e(v)$ < diam$G$. If rad$G \geq$ diam$G$ - 1, then G is said to have no annulus.

A third intermediate distance-dependent subgraph for a connected graph G with rad$G$ < diam$G$, called the *exterior* of G, can be defined as the subgraph $Ext(G)$ induced by the vertices v with rad$G$ < $e(v)$. As with the interior, if rad$G$ = diam$G$, then $Ext(G) = G$.

In many of the characterizations, the graph constructed to include the given graph as the desired subgraph has many additional edges and thus, a small diameter. Since trees have fewer edges and larger diameters than other graphs with the same vertex set, a natural question arises: Which graphs can be the interior, annulus, or exterior of a tree?

2. The Interior of a Tree

In [7], it was shown that every graph G is the interior of some connected graph not isomorphic to G. In addition, since it is sometimes desirable to distinguish the interior of a graph from the center, the following result was proved for connected graphs.

Theorem A: Let G be a connected graph. Then there exists a connected graph H such that $Int(H) = G$ and $C(H) \neq G$ if and only if G is not complete.

It is not known whether a similar result is true if G is disconnected; however, the following construction from [7] shows that every disconnected graph G is the interior of some connected graph H with $G = Int(H) = C(H)$ (see Figure 1). Let $G = F_1 \cup F_2$ where F_1 is one component of G. Join a new vertex u to every vertex of G, join two more vertices v_1 and w_1 to every vertex in F_1, and join two new vertices v_2 and w_2 to every vertex of F_2. In addition, join a sixth vertex w_3 to w_1 and w_2, and finally, add the two edges v_1v_2 and w_1w_2 to complete the graph H.

Thus, each vertex of $F_1 \cup F_2$ has eccentricity 2 while the added vertices have eccentricities 3. Also, the classic construction by Hedetniemi [see 4] showing that every graph G (and in particular, each disconnected graph) is the center of some connected graph H (by adding four new vertices u_1, u_2, v_1, v_2 so that for $i = 1, 2$, every vertex of G is joined to v_i, and u_i is joined to v_i (see Figure 2)) is an example of a connected graph H such that $C(H) = G$ and $Int(H) \neq G$. The question remains:

H:

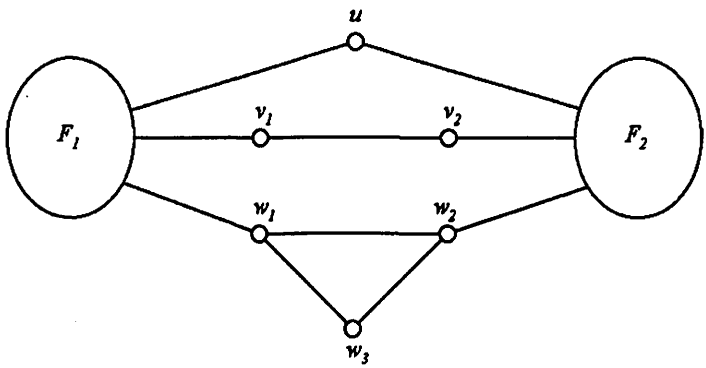

Figure 1

Open Question: For which disconnected graphs G (if any) does there exist a connected graph H such that $Int(H) = G$ and $C(H) \neq G$?

A complete characterization for the interiors of trees can be proved:

H :

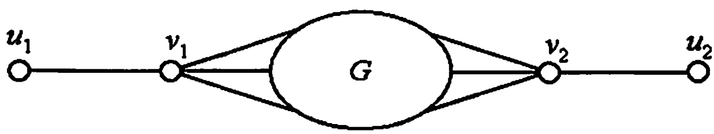

Figure 2

Theorem 1: A graph G is the interior of infinitely many trees if and only if G is a tree.

Proof: Let G be a tree and let u and v be peripheral vertices such that $d(u, v) =$ diam$G$. (Note that $u = v$ if $G = K_1$). For some integer $n \geq 1$, form tree H by adding vertices $w_1, w_2, \ldots, w_n$, and x so that each w_i is adjacent only to u and x is adjacent only to v. Now, $d(w_i, x) =$ diam$H =$ diam$G + 2$, and $e_G(y) \leq e_H(y) \leq e_G(y) + 1 \leq$ diam$G + 1$ for every vertex y of G. Thus, $V(P(H)) = \{x, w_1, w_2, \ldots, w_n\}$ and $Int(H) = G$ for infinitely many trees.

To prove the converse, suppose that a graph G is the interior of some tree H. First, since the interior is a subgraph of the tree, G must be acyclic. Second, since peripheral vertices of a tree are end-vertices, they are not cut-vertices, and their removal leaves a connected subgraph. Thus, the interior must be connected and G must be a tree. □

Theorem 2: Let T be a tree. Then there exist infinitely many trees H such that $Int(H) = T$ and $C(H) \neq T$ if and only if T is not complete (i.e. K_1 or K_2).

Proof: First, let T be a tree other than K_1 and K_2. By Theorem 1, T is the interior for infinitely many trees; however, since the center of a tree is either K_1 or K_2, it follows that $C(H) \neq T$.

For the converse, suppose that T is either K_1 or K_2 and that there is a tree H with $Int(H) = T$. It is enough to show that $C(H) = T$. If $T = K_1 = \{w\}$ is the interior of tree H, then $\{w\}$ must be the center since $V(C(H)) \subseteq V(Int(H))$. On the other hand, if $T = K_2 = \{u, v\}$ is the interior of tree H, then every peripheral vertex of H is an end-vertex adjacent to either u or v. Thus, diam$H = 3$ and $e_H(u) = e_H(v) =$ rad$H = 2$, again forcing T to be the center of H. □

2. The Exterior of a Tree

Similar to the interior of a graph G, the exterior $Ext(G)$ is defined as the subgraph induced by the vertices u with $e(u) >$ rad$G$ if G is not self-centered; while $Ext(G) = G$ if G is self-centered. Note that the vertices in the center and the exterior partition the vertex set of G when G is not self-centered.

Recall that a graph G is the periphery of some connected graph if and only if G is complete or no vertex of G has eccentricity 1. A construction from the proof can be extended to the exterior as well: If no vertex of G has eccentricity 1, then G is the periphery (and exterior) for the graph $H = K_1 + G$. The graph K_1 can be replaced with a copy of K_n for $n > 1$, providing an infinite family of graphs H that have G as the periphery (and exterior). This is not true if G is complete, as the next lemma shows.

Lemma 1: If G is complete and $G = P(H)$ for a connected graph H, then $H = G$.

Proof: Let G be complete and let H be a connected graph such that $G = P(H)$. If v is a vertex in $V(G)$, then v is a peripheral vertex of H and $e_H(v) = d(v, w)$ for some vertex w in $P(H) = G$. Thus, diam$G = e_H(v) = 1$, since G is complete. Since H is connected, $0 <$ rad$H \leq 1$, and every vertex of H has eccentricity 1. Thus, $H = P(H) = G$. □

Thus, a characterization for those graphs that are the exterior for some graph is the following.

Theorem 3: A graph G is the exterior of some graph if G is complete, and G is the exterior of infinitely many connected graphs if no vertex of G has eccentricity 1.

The next two lemmas will be used to prove the more interesting characterization for those graphs that are the exterior of a connected graph when the exterior and periphery are not equal.

Lemma 2: If a disconnected graph G with isolated vertices is the exterior of a connected graph H with an annulus, then none of the isolated vertices of G are peripheral vertices of H.

Proof: Let G be a disconnected graph with isolated vertices that is the exterior for a connected graph H with an annulus. Suppose that u is an isolated vertex of G that is also a peripheral vertex of H. If v is a vertex of H adjacent to u, then $v \in C(H)$ and $v \notin V(G)$. Since $e_H(v) =$ rad$H$, it follows that $e_H(u) =$ diam$H =$ rad$H + 1$. However, this means that H has no annulus – a contradiction. □

Lemma 3: In a graph (or a component of a disconnected graph) of diameter 4, every vertex with eccentricity 2 is adjacent to some vertex of eccentricity 3.

Proof: Let G be a graph of diameter 4 with u being a peripheral vertex and v being a vertex of eccentricity 2. Suppose that v is only adjacent to other vertices of eccentricity 2. Then a shortest v-u path passes through a second vertex of eccentricity 2, at least one vertex of eccentricity 3, and finally, vertex u. Therefore, $d(v, u) \geq 3$, but this is impossible since $e(v) = 2$. Thus, v must be adjacent to some vertex of eccentricity 3. □

Theorem 4: A graph G is the exterior, but not the periphery, of an infinite family of connected graphs H if and only if

(1) G is connected with rad$G \geq 3$ and diam$G \geq 4$, or

(2) G is disconnected with exactly one component C with diam$C \geq 4$ and all

other components isolated vertices, or

(3) G is disconnected with at least two components that are not isolated vertices.

Proof: We begin by showing that a graph G satisfying one of (1) - (3) is the exterior, but not the periphery of an infinite family of connected graphs H. First, for case (1), suppose that G is a connected graph with rad$G \geq 3$ and diam$G \geq 4$, and let u and v be peripheral vertices of G such that $d(u, v) =$ diam$G$. Also, let S be the (possibly empty) set of vertices w such that $d(u, w) = d(v, w) = 2$. For some integer $n \geq 1$, join every vertex in a copy of the graph K_n to every vertex of $G - (S \cup \{u, v\})$ and call this graph H. Note that each vertex w in S is adjacent to a vertex x on a shortest w-v path such that $d_G(x, u) \geq 3$. Thus, x is adjacent to the new vertices in K_n and each of the vertices in K_n has an eccentricity of 2 in H. In addition, the vertices u and v have an eccentricity of 4 in H, and the remaining vertices of G have eccentricity 3 in H. This last statement follows from the fact that if y is a vertex of $G - (S \cup \{u, v\})$, then either $d_H(u, y) = 3$ or $d_H(v, y) = 3$; and if $y \in S$, then, since rad$G \geq 3$, there exists a vertex $z \in V(G)$ such that $d_G(y, z) = 3$. Since the same shortest y-z path remains in H, the eccentricity $e_H(y) = 3$. Thus, G is the exterior of H and not the periphery.

For case (2), let G be a disconnected graph whose components are isolated vertices except for one component C where diam $C \geq 4$. If rad$C \geq 3$, then the construction for H from case (1) can be used. If, however, the radius is 2, then H is constructed as above, except that the vertices of K_n are not joined to the vertices of eccentricity 2. It is easy to see that since u and v are adjacent to vertices of eccentricity at least 3, and since every vertex of eccentricity 2 is adjacent to some vertex of eccentricity 3 (by Lemma 3), they are distance 2 from the added vertices of H. Thus, each of the new vertices in K_n has an eccentricity of 2 in H, the vertices u and v have an eccentricity of 4, and the remaining vertices of G have eccentricity 3; and again, G is the exterior of H and not the periphery. For case (3), suppose that G is disconnected with at least two components C_1 and C_2 that are not isolated vertices. To construct an infinite family of graphs that have G as their exteriors, let u be a vertex of C_1 and let v be a vertex of C_2. As before, join every vertex in a copy of the graph K_n, for any $n \geq 1$, to every vertex of $G - \{u, v\}$ and call this graph H. Note that each of the new vertices in K_n has an eccentricity of 2 in H, the vertices u and v have an eccentricity of 4, and the remaining vertices of G have eccentricity 3. Thus, G is the exterior of H and not the periphery.

To prove the converse, let G be a graph that does not satisfy conditions (1) - (3) and suppose that G is the exterior of some connected graph H with $Ext(H) \neq P(H)$. The graph G must fall into one of the following categories:

(a) If G is connected, then rad$G \leq 2$ or diam$G \leq 3$ [i.e. either, for some vertex

w, the eccentricity $e(w) \leq 2$, or for every vertex, the eccentricity is at most 3], or

(b) If G is disconnected, then G consists only of isolated vertices, or G has exactly one component C that is not an isolated vertex with diam$C \leq 3$.

It remains to show that no graph can satisfy these conditions and be the exterior of a graph whose exterior and periphery are not equal.

For category (a), suppose first that G is connected with some vertex w having $e_G(w) \leq 2$. If $e_H(w) \leq 2$ also, then any central vertex x must have a smaller eccentricity. This means that $e_H(x) = 1$, the vertex x is adjacent to every other vertex of H, the diameter of H is 2, and G is the periphery of H - a contradiction. On the other hand, if $e_H(w) = n \geq 3$, then all vertices y in H such that $d(w, y) = n$ are not in $V(G)$ and must be central vertices. This forces $e_H(w) \leq e_H(y) =$ rad$H$, which contradicts the fact that w is in the exterior and not in the center of H. Second, suppose that $e_G(v) = 3$ for every vertex in G. Then diam$G = 3$, and one of the following three situations must occur. If $e_H(v) \leq 2$ for every vertex in G, then the central vertices in H must have eccentricity 1 and $G = P(H)$ – a contradiction. If diam$H = 3$, then the central vertices must have eccentricity 2, and, again, $G = P(H)$. Finally, if some vertex v of G has $e_H(v) = n > 3$, then there is a vertex u such that $d(u, v) = n$; however, since $d_H(v, w) \leq 3$ for all w in G, the vertex u is a central vertex and $e_H(v) \leq e_H(u) =$ rad$H$ – a contradiction. Thus, no graph that is the exterior and not the periphery of a connected graph satisfies the conditions in category (a).

Now consider category (b). If G is disconnected and has only isolated vertices for its components, then since $V(G) = V(P(H))$, some isolated vertex of G will be a peripheral vertex of H, which contradicts Lemma 2. On the other hand, consider G with exactly one component C that is not an isolated vertex such that diam$C \leq 3$. If u and v are peripheral vertices of H such that $d(u, v) =$ diam$H$, then u and v must both be in component C by Lemma 2. This forces diam$H \leq$ diam$C \leq 3$; however, then diam$H$ - 1 $\leq$ rad$H \leq$ diam$H$, and $G = P(H)$ – a contradiction. Thus, no graph that is the exterior and not the periphery of a connected graph satisfies the conditions in category (b). Therefore, if a graph is the exterior of a connected graph and not its periphery, it must satisfy one of the conditions (1) - (3). □

In Theorem 3, the graphs H had small diameters because many vertices were joined to the added central vertices, creating many cycles. This same construction method will not work with trees; however, there is a characterization for those graphs which are the exterior of a tree whose exterior is different from its periphery.

Recall that for a tree, the center is either one vertex (central tree) or two adjacent vertices (bicentral tree) and each central vertex is a cut-vertex if the tree is non-trivial. Clearly, the exterior of a non-trivial tree is a subgraph of the tree, so it must be a forest. With the aid of a lemma, the next theorem characterizes those graphs that are exteriors of non-trivial trees.

Lemma 4: In a tree with diameter d and vertices u and v such that $d(u, v) = d$, if w is a vertex on the u-v path with $d(u, w) \geq d/2$, then $e(w) \leq d(u, w)$.

Proof: Let T be a tree with diameter d and vertices u and v such that $d(u, v) = d$. Also, let w be a vertex on the u-v path P with $d(u, w) \geq d/2$. Consider a vertex z. Then w is on a u-z path P_u or w is on a v-z path P_v, or possibly both. (The vertex w must be on at least one of the two paths P_u or P_v; otherwise, $P \cup P_u \cup P_v$ would contain a cycle -- a contradiction.) Suppose that w is on a u-z path. Then, $d(u, z) = d(u, w) + d(w, z)$. Since $d(u, z) \leq d$ and $d(u, w) \geq d/2$, the distance $d(w, z) \leq d/2 \leq d(u, w)$. On the other hand, suppose that w is on a v-z path. Then, $d \geq d(v, z)$, which can be written as $d(u, w) + d(w, v) \geq d(v, w) + d(w, z)$. Therefore, $d(u, w) \geq d(w, z)$, and the result follows. □

Theorem 5: A graph F is the exterior of a non-trivial tree if and only if F is a forest that has at least two non-trivial components with diameters d_1 and d_2 such that d_1 is the largest diameter among all of the components and such that $d_1/2 \leq d_2$. Furthermore, the diameter d of a central tree can be any even integer such that $2(\lceil d_1/2 \rceil + 1) \leq d \leq 2(d_2 + 1)$ and the diameter f of a bicentral tree can be any odd integer such that $2\lceil d_1/2 \rceil + 3 \leq f \leq 2d_2 + 3$.

Proof: Let F be a forest with $k \geq 2$ components $C_1, C_2, \ldots, C_k$ ordered so that their diameters, denoted by $d_1, d_2, \ldots, d_k$, satisfy the condition that $d_1 \geq d_2 \geq \ldots \geq d_k$. Consider the two cases of a central or bicentral tree with exterior F.

Case 1: For the central case, let d be an even integer such that $2(\lceil d_1/2 \rceil + 1) \leq d \leq 2(d_2 + 1)$ and let u_i and v_i be vertices in C_i such that $d(u_i, v_i) = \min\{(d/2 - 1, d_i\}$ for $1 \leq i \leq k$ (with $u_i = v_i$ if C_i is an isolated vertex). Let w be the vertex on the u_1-v_1 path that satisfies $d(w, u_1) = d/2 - 1$. A central tree T with diameter d and exterior F is formed by joining one new vertex x to w and to each of the vertices $v_2, v_3, \ldots, v_k$. To see this, note that $e_T(x) = d(x, u_1) = d/2$. For every vertex of F not in C_1, its distance to u_1 is at least $d/2 + 1$, and for every vertex in C_1, its distance to u_2 is at least $d/2 + 1$. Thus, F is the exterior of the central tree T with diameter d where $2(\lceil d_1/2 \rceil + 1) \leq d \leq 2(d_2 + 1)$.

Case 2: For the bicentral case, let f be an odd integer such that $2\lceil d_1/2 \rceil + 3 \leq f \leq$

$2d_2 + 3$ and let u_i and v_i be vertices in C_i such that $d(u_i, v_i) = \min\{(f-3)/2, d_i\}$ for $1 \le i \le k$ (with $u_i = v_i$ if C_i is an isolated vertex). Let w be the vertex on the u_1-v_1 path that satisfies $d(w, u_1) = (f-3)/2$. A bicentral tree T with diameter f and exterior F is constructed by joining a new vertex x to w and new vertex y to each of the vertices $x, v_2, v_3, \ldots, v_k$. From the construction, the diameter is $d(u_1, u_2) = f$, as well as $e_T(x) = d(x, u_2) = (f+1)/2$ and $e_T(y) = d(x, u_1) = (f+1)/2$. In addition, $d(v, u_2) > (f+1)/2$ for $v \in V(C_1)$ and $d(v, u_1) > (f+1)/2$ for $v \in V(F - C_1)$. Thus, F is the exterior of the bicentral tree T with diameter f where $2\lceil d_1/2 \rceil + 3 \le f \le 2d_2 + 3$.

To prove the converse, suppose that F is the exterior of some tree T. If T is either K_1 or K_2, then $F = T$. If $V(T) \ge 3$, it must be shown that F is not connected and that $d_2 \ge d_1/2$. First, since the central vertices of a non-trivial tree are all cut-vertices and not end-vertices, their removal forces $F = Ext(T) = T - V(C(T))$ to be disconnected. Therefore, the forest F has at least two components $C_1, C_2, \ldots, C_k$ ordered so that their diameters, denoted by $d_1, d_2, \ldots, d_k$, satisfy the condition that $d_1 \ge d_2 \ge \ldots \ge d_k$. Let u_i and v_i be vertices in C_i such that $d(u_i, v_i) = d_i$ for $1 \le i \le k$ (with $u_i = v_i$ if C_i is an isolated vertex). Finally, suppose that $d_2 < d_1/2$. Since T has no cycles, only one vertex w of C_1 is adjacent to a central vertex. Without loss of generality, assume that $d(u_1, w) \ge d(v_1, w)$. By Lemma 4, it is easy to see that $n = d(u_1, w) = e_{C1}(w)$ and that $d_2 < d_1/2 \le n$. Thus, the integer $d_2 + 1 \le n$. To reach a contradiction, consider the following cases based on the two possibilities for the center of T.

<u>Case 1:</u> Suppose that $C(T) = \{y\}$. Since $d(w, x) \le n$ for any vertex x in C_1, by Lemma 4, and since $d(w, x) \le d_2 + 2$ for any vertex x of $T - C_1$ (if y is adjacent to a peripheral vertex of C_2, for instance), then $e_T(w) \le max\{n, d_2 + 2\}$. If $e_T(w) \le n$, then $e(y) < e_T(w) \le n$; however, $e(y) \ge d(y, u_1) = n + 1$, a contradiction. If $e_T(w) > n$, then $e_T(w) \le d_2 + 2$. Since $d_2 < n < d_2 + 2$, the integer $n = d_2 + 1$. However, this forces $e(y) \ge d(y, u_1) = n + 1 = d_2 + 2 \ge e_T(w)$, causing a contradiction. Therefore, $C(T)$ cannot be a single vertex.

<u>Case 2:</u> Suppose that $C(T)$ is the subgraph induced by two adjacent vertices y and z. Exactly one of y and z is adjacent to w. Let it be y. Then, $e(z) \ge d(z, u_1) = n + 2$. Also, $d(y, x) \le d(y, u_1) = n + 1$ for all vertices x in C_1, by Lemma 4, and $d(y, x) \le d_2 + 2$ for all vertices x in $T - C_1$ (if z is adjacent to a peripheral vertex of C_2, for instance, and y is not). Thus, $e(y) \le max\{n + 1, d_2 + 2\}$. However, since $d_2 + 1 \le n$, it follows that $d_2 + 2 \le n + 1$, and $e(y) \le n + 1$. Thus, $e(y) < e(z)$, which contradicts the fact that both are central vertices of T, and $C(T) \ne K_2$.

Therefore, $d_1/2 \le d_2$ and this completes the proof. □

It is important to note that in Theorem 4, an infinite family of graphs was found for each exterior graph since the center could contain many vertices . However, for trees, an infinite family cannot be found for each graph because the center is isomorphic to only K_1 or K_2. It is worth mentioning though, that in the statement of Theorem 5, even though the values for d are only even and the values for f are only odd, the constructions in the proof of Theorem 5 give all possible values for the diameters d or f between the stated bounds. This follows from a result in [2] which states that if a tree T has just one central vertex, then diam$T$ = 2rad$T$ (and is even), and if the tree T is bicentral, then diam$T$ = 2rad$T$ -1 (and is odd).

Since the periphery of a tree T is the entire tree if T is K_1 or K_2, or a set of isolated vertices otherwise, the following characterization of graphs that are the exterior and not the periphery of a tree is an immediate corollary to Theorem 5.

Corollary 1: A graph is the exterior of a tree, and not the periphery, if and only if the graph is a forest that has at least two non-trivial components with diameters d_1 and d_2 such that d_1 is the largest diameter among all of the components and such that $d_1/2 \leq d_2$.

3. The Annulus of a Tree

When rad$G$ < diam$G$ - 1, the annulus, $Ann(G)$, of a connected graph G is defined as the subgraph induced by those vertices v with rad$G < e(v) <$ diam$G$. If rad$G$ = diam$G$ - 1, then the graph has no annulus. Those graphs that are the annulus of a connected graph were characterized in [7]:

Theorem C: For every nontrivial graph G, there exists a connected graph H such that $Ann(H) = G$ if and only if G has no vertices of eccentricity 1.

The question still remains: Which graphs G can be the annulus of a tree? Of course, the graph G must be disconnected, since central vertices of trees are cut-vertices and G must be a forest, since it is a subgraph of an acyclic graph. In order to complete the characterization for the annulus of a tree, two ideas must be introduced. First, vertex v is an *eccentric vertex* of a graph if there is some vertex u of the graph such that $d(u, v) = e(u)$. Second, in a tree, the subgraph induced by the set of eccentric vertices is the periphery for the tree (see [3]).

Theorem 6: A graph is the annulus of a tree if and only if it is the exterior of a non-self-centered tree.

Proof: Let G be the annulus of a tree T. This forces T to be non-self-centered. Remove the end-vertices that are peripheral vertices for T to form T'. Since the peripheral vertices are the eccentric vertices for T, then $e_{T'}(u) = e_T(u) - 1$ for every vertex u in T'. Therefore, $C(T) = C(T')$, the graph G is the exterior of T', and since T is not self-centered, neither is T'.

For the converse, let G be the exterior of a non-self-centered tree T. Form T' by doing the following: for each peripheral vertex u of T, add a new vertex $v(u)$ and then join u to $v(u)$. These new vertices will have an eccentricity equal to $\text{diam}T' = \text{diam}T + 2$. For vertex u in T, the eccentricity $e_{T'}(u) = e_T(u) + 1 \leq \text{diam}T + 1$. Thus, $C(T) = C(T')$ and G is the annulus of T'. □

Note that the added vertices in the construction for the second portion of the proof can be replaced by n copies of K_1 for $n > 1$ to form an infinite family of trees having G as the annulus.

Corollary 2: A graph is the annulus of an infinite family of trees if and only if the graph is a forest that has at least two components with diameters d_1 and d_2 such that d_1 is the largest diameter among all of the components and such that $d_2 \geq d_1/2$.

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REFERENCES

[1] H. Bielak and M.M. Syslo, Peripheral vertices in graphs, Studia Sci. Math. Hungar. 18 (1983) 269-275.

[2] F. Buckley and M. Lewinter, A Friendly Introduction to Graph Theory, Prentice Hall, Upper Saddle River, New Jersey (2002).

[3] F. Buckley and M. Lewinter, Minimal graph embeddings, eccentric vertices, and the peripherian, Proceedings of the Fifth Caribbean Conference on Combinatorics and Computing. University of the West Indies, (1988) 72-84.

[4] F. Buckley, Z. Miller, and P.J. Slater, On graphs containing a given graph as center, J. Graph Theory 5 (1981) 427-434.

[5] G. Chartrand, D. Erwin, G.L. Johns, and P. Zhang, On boundary vertices in graphs, The Journal of Combinatorial Mathematics and Combinatorial Computing 48 (2004) 39-53.

[6] G. Chartrand, G.L. Johns, and O.R. Oellermann, On peripheral vertices in graphs, Topics in Combinatorics and Graph Theory. Physica-Verlag, Heidelberg, Germany (1990) 194-199.

[7] G. Chartrand, G.L. Johns, S. Tian, and S.J. Winters, The interior and annulus of a graph, Congressus Numerantium 102 (1994) 57-62.

[8] G.L. Johns, Critical supergraphs of distance-related subgraphs, Congressus Numerantium 120 (1996) 97-102.

[9] G.L. Johns and S.J. Winters, Improving the bounds for the annular and marginal appendage numbers, Congressus Numerantium 159 (2002) 177-182.

[10] C. Jordan. Sur les assemblages des lignes, J. Reine Angew Math. 70 (1869) 185-190.