On friendly index sets of cycles with parallel chords

Sin-Min Lee
Department of Computer Science
San Jose State University
San Jose, CA 95192, USA

Ho Kuen Ng
Department of Mathematics
San Jose State University
San Jose, CA 95192, USA

Abstract Let G be a graph with vertex set V(G) and edge set E(G), and let A be an abelian group. A labeling $f : V(G) \to A$ induces an edge labeling $f^* : E(G) \to A$ defined by $f^*(xy) = f(x) + f(y)$, for each edge $xy \in E(G)$. For $i \in A$, let $v_f(i) = \text{card}\{v \in V(G) : f(v) = i\}$ and $e_f(i) = \text{card}\{e \in E(G) : f^*(e) = i\}$. Let $c(f) = \{|e_f(i) - e_f(j)| : (i, j) \in A \times A\}$. A labeling f of a graph G is said to be A-friendly if $| v_f(i) - v_f(j) | \leq 1$ for all $(i, j) \in A \times A$. If c(f) is a (0, 1)-matrix for an A-friendly labeling f, then f is said to be A-cordial. When $A = Z_2$, the ***friendly index set*** of the graph G, FI(G), is defined as $\{|e_f(0) - e_f(1)| :$ the vertex labeling f is Z_2-friendly$\}$. In [13] the friendly index set of cycles are completely determined. In this paper we describe the friendly index sets of cycles with parallel chords. We show that for a cycle with an arbitrary non-empty set of parallel chords, the numbers in its friendly index set form an arithmetic progression with common difference 2.

Key words: vertex labeling, friendly labeling, cordiality, friendly index set, cycle with parallel chords, arithmetic progression.

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1. Introduction

Let G be a graph with vertex set V(G) and edge set E(G). Let A be an abelian group. A labeling $f: V(G) \to A$ induces an edge labeling $f^* : E(G) \to A$ defined by $f^*(xy) = f(x) + f(y)$, for each edge $xy \in E(G)$. For $i \in A$, let $v_f(i) =$

card$\{v \in V(G) : f(v) = i\}$ and $e_f(i) = $ card$\{e \in E(G) : f^*(e) = i\}$. Let $c(f) = \{|e_f(i) - e_f(j)| : (i, j) \in A \times A\}$. A labeling f of a graph G is said to be A-friendly if $|v_f(i) - v_f(j)| \leq 1$ for all $(i, j) \in A \times A$. If c(f) is a (0, 1)-matrix for an A-friendly labeling f, then f is said to be A-cordial.

The notion of A-cordial labelings was first introduced by Hovey [10], who generalized the concept of cordial graphs of Cahit [2]. Cahit considered $A = \mathbf{Z}_2$ and he proved the following: every tree is cordial; K_n is cordial if and only if $n \leq 3$; $K_{m,n}$ is cordial for all m and n; the wheel W_n is cordial if and only if $n \neq 3 \pmod 4$; C_n is cordial if and only if $n \neq 2 \pmod 4$; and an Eulerian graph is not cordial if its size is congruent to 2 (mod 4). Benson and Lee [1] showed a large class of cordial regular windmill graphs which include the friendship graphs as a subclass.

Lee and Liu [12] investigated cordial complete k-partite graphs. Kuo, Chang and Kwong [11] determined all m and n for which mK_n is cordial. In 1989, the first author, Ho and Shee [9] completely characterized cordial generalized Petersen graphs. Ho, Lee and Shee [8] investigated the construction of cordial graphs by Cartesian product and composition. Seoud and Abdel [18] proved certain cylinder graphs are cordial. Several constructions of cordial graphs were proposed in [16, 17, 18, 19, 20]. For more details of known results and open problems on cordial graphs, see [4, 7].

In this paper, we will exclusively focus on $A = Z_2$, and drop the reference to the group. In [6] the following concept was introduced.

Definition 1. The friendly index set FI(G) of a graph G is defined as $\{|e_f(0) - e_f(1)| :$ the vertex labeling f is friendly$\}$.

When the context is clear, we will drop the subscript f.

Note that if 0 or 1 is in FI(G), then G is cordial. Thus the concept of friendly index sets could be viewed as a generalization of cordiality.

Cairnie and Edwards [5] have determined the computational complexity of cordial labeling and Z_k-cordial labeling. They proved that to decide whether a graph admits a cordial labeling is NP-complete. Even the restricted problem of deciding whether a connected graph of diameter 2 has a cordial labeling is NP-complete. Thus in general it is difficult to determine the friendly index sets of graphs.

In [13, 14, 15] the friendly index sets of a few classes of graphs, in particular, complete bipartite graphs and cycles are determined. The following result was established.

Theorem 1. For any graph with q edges, the friendly index set $FI(G) \subseteq \{0, 2, 4, \ldots, q\}$ if q is even, and $FI(G) \subseteq \{1, 3, \ldots, q\}$ if q is odd.

Example 1. Figure 1 illustrates the friendly index set of wheel W_4.

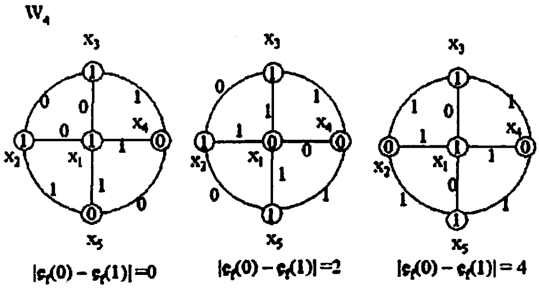

Figure 1.

The authors [13] showed that

Theorem 2. The friendly index set of a cycle is given as follows:

(i) $FI(C_{2n}) = \{0, 4, 8, \dots, 2n\}$ if n is even.

$FI(C_{2n}) = \{2, 6, 10, \dots, 2n\}$ if n is odd.

(ii) $FI(C_{2n+1}) = \{1, 3, 5, \dots, 2n - 1\}$.

2. Cycles with a full set of parallel chords

Recently Elumalai considered the gracefulness and cordiality of the following graphs in his dissertation [7].

Definition 2. A cycle with a full set of parallel chords is the graph G obtained from the cycle C_n: $v_0v_1v_2\dots v_{n-1}v_0$ ($n \geq 6$) by adding the chords v_1v_{n-1}, v_2v_{n-2}, ..., $v_{(n-2)/2}v_{(n+2)/2}$ if n is even, or adding the chords v_2v_{n-1}, v_3v_{n-2}, ..., $v_{(n-1)/2}v_{(n+3)/2}$ if n is odd.

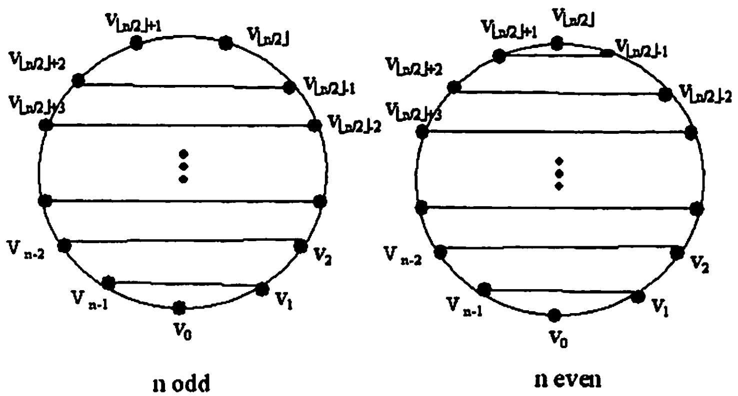

Figure 2.

The graph G will be denoted by PC_n. Observe that PC_n has n vertices and (3n – 3)/2 edges if n is odd, and (3n – 2)/2 edges if n is even. In this section, we give explicit constructions that exhibit the friendly index sets of all cycles with a full set of parallel chords.

In [13], the authors proposed the following
Conjecture. The numbers in FI(T) for any tree T forms an arithmetic progression.

We observe the same phenomenon for cycles and cycles with parallel chords.

We first consider odd values of n. Let n = 2m + 1. If m = 1, PC_3 is simply a cycle. We may assume that $m \geq 2$.

From [13], we have the following Lemma.
Lemma 1. Any vertex labeling (not necessarily friendly) of a cycle must have e(1) equal to an even number.

Lemma 2. $FI(PC_{2m+1}) \subseteq \{3m-2, 3m-4, 3m-6, \dots\}$.
Proof. PC_{2m+1} has 3m edges. By Theorem 1, $FI(PC_{2m+1}) \subseteq \{3m, 3m-2, 3m-4, 3m-6, \dots\}$. If $3m \in FI(PC_{2m+1})$, then either all edge labels are 1 or all edge labels are 0. Applying Lemma 1 to the cycle C_{2m+1}, we see that the former is impossible. The latter vertex labeling is not friendly.□

Lemma 3. Each of the non-negative integers 3m – 2, 3m – 4, 3m – 6, ... can be attained as |e(1) – e(0)| using a friendly vertex labeling for PC_{2m+1}.
Proof. We first label the vertices $v_0, v_1, v_2, \dots, v_m$ by 0, and the vertices $v_{m+1}, v_{m+2}, \dots, v_{2m}$ by 1. This vertex labeling is obviously friendly with v(1) – v(0) = –1. All the parallel chords have label 1. The edges (v_0, v_{2m}) and (v_m, v_{m+1}) have label 1, and all other edges of the cycle have label 0. Thus e(1) – e(0) = (m – 1) + 2 – (2m – 1) = –m + 2.

Interchange the vertex labels of v_m and v_{m+1}. The vertex labeling is still friendly with v(1) – v(0) = –1. The parallel chords are not affected, while the cycle C_{2m+1} has two more 1-edges and two fewer 0-edges, making e(1) – e(0) = –m + 6.

Now interchange the vertex labels of v_m and v_{m+1}, and the vertex labels of v_{m-1} and v_{m+2}. The vertex labeling is still friendly with v(1) – v(0) = –1. The parallel chords are not affected, while the cycle C_{2m+1} has two more 1-edges and two fewer 0-edges, making e(1) – e(0) = –m + 10.

Continue this process, interchanging an additional pair of vertices each time, keeping the vertex labeling friendly and increasing e(1) – e(0) by 4. Stop when the vertices v_2 and v_{2m-1} have been interchanged. At this point, e(1) – e(0) = 3m

− 2. Note that in each of the above labelings, v_0, v_1, and v_{2m} are always labeled 0, 0, and 1 respectively.

So far, we have constructed m friendly vertex labelings, each with v(1) − v(0) = −1. For each of these labelings except the last one, change the vertex label at v_1 from 0 to 1. Each vertex labeling remains friendly, with v(1) − v(0) = 1 now. The only edges that will be affected are (v_1, v_0), (v_1, v_2), and (v_1, v_{2m}), with labels changed from 0, 0, and 1 to 1, 1, and 0 respectively. Thus the value of e(1) − e(0) is increased by 2 for each of these (m − 1) labelings.

To conclude, the values of e(1) − e(0) are −m + 2, −m + 4, −m + 6, −m + 8, … , 3m − 6, 3m − 4, and 3m − 2. Taking absolute values finishes the proof.□

<u>**Theorem 3.**</u> $FI(PC_{2m+1}) = \{3m - 2, 3m - 4, 3m - 6, \dots \}$. In other words, $FI(PC_{2m+1}) = \{3m - 2, 3m - 4, 3m - 6, \dots, 0\}$ if m is even, and $FI(PC_{2m+1}) = \{3m - 2, 3m - 4, 3m - 6, \dots, 1\}$ if m is odd.

<u>**Example 2.**</u> $FI(PC_5) = \{4, 2, 0\}$.

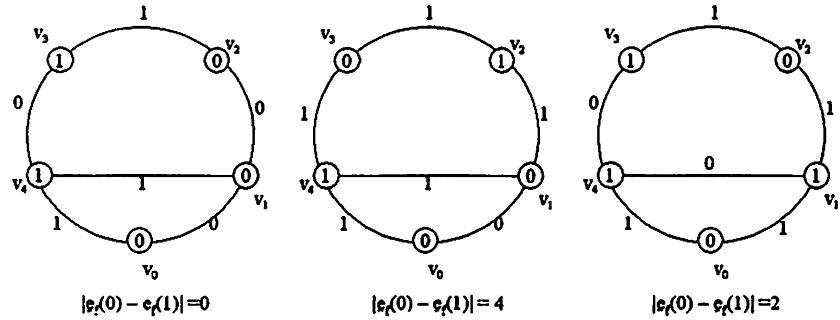

Figure 3.

<u>**Example 3.**</u> $FI(PC_7) = \{7, 5, 3, 1\}$.

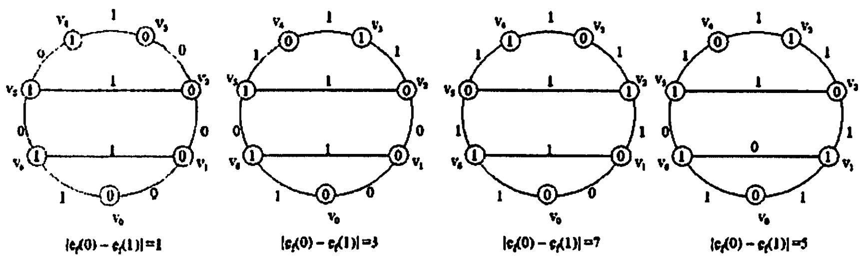

Figure 4.

Now we consider even values of n. We first take care of the case PC_4. Since there are 5 edges, $FI(PC_4) \subseteq \{1, 3, 5\}$. By Lemma 1, e(1) ≠ 5. If e(0) = 5, the vertex labeling would not be friendly. Thus $5 \notin FI(PC_4)$. By labeling the vertices consecutively by 1, 0, 1, 0, we have $e(1) - e(0) = 4 - 1 = 3$. By labeling the vertices consecutively by 1, 1, 0, 0, we have $e(1) - e(0) = 3 - 2 = 1$. Thus $FI(PC_4) = \{1, 3\}$. From now on in this section, we assume $n = 2m$, where $m \geq 3$.

__Lemma 4.__ If $m \geq 3$, $FI(PC_{2m}) \subseteq \{3m - 5, 3m - 7, 3m - 9, \dots\}$.
Proof. PC_{2m} has $3m - 1$ edges. By Theorem 1, $FI(PC_{2m}) \subseteq \{3m - 1, 3m - 3, 3m - 5, 3m - 7, 3m - 9, \dots\}$. If $3m - 1 \in FI(PC_{2m})$, then either all edge labels are 1 or all edge labels are 0. Applying Lemma 1 to the cycle with vertices v_0, v_1, and v_{2m-1}, we see that the former is impossible. The latter vertex labeling is not friendly. If $3m - 3 \in FI(PC_{2m})$, then either $e(1) = 1$ or $e(0) = 1$. The former is impossible, by Lemma 1, no matter where the sole 1-edge is. For the latter case, if the sole 0-edge is in the cycle C_{2m}, then C_{2m} has an odd number of 1-edges, contradicting Lemma 1. Note that since $m \geq 3$, there are at least two parallel chords. If the sole 0-edge is a parallel chord, then this and the next parallel chords, and the two edges of C_{2m} joining these two parallel chords, form a 4-cycle with exactly one 0-edge, again contradicting Lemma 1.□

__Lemma 5.__ Each of the non-negative integers $3m - 5, 3m - 7, 3m - 9, \dots$ can be attained as $|e(1) - e(0)|$ using a friendly vertex labeling for PC_{2m}.
Proof. We first label the vertices $v_0, v_1, v_2, \dots, v_{m-1}$ by 0, and the vertices v_m, $v_{m+1}, \dots, v_{2m-1}$ by 1. This vertex labeling is obviously friendly with $v(1) - v(0) = 0$. All the parallel chords have label 1. The edges (v_0, v_{2m-1}) and (v_{m-1}, v_m) have label 1, and all other edges of the cycle have label 0. Thus $e(1) - e(0) = (m - 1) + 2 - (2m - 2) = -m + 3$.

Interchange the vertex labels of v_{m-1} and v_{m+1}. The vertex labeling is still friendly. The parallel chords are not affected, while the cycle C_{2m} has two more 1-edges and two fewer 0-edges, making $e(1) - e(0) = -m + 7$.

Now interchange the vertex labels of v_{m-1} and v_{m+1}, and the vertex labels of v_{m-2} and v_{m+2}. The vertex labeling is still friendly. The parallel chords are not affected, while the cycle C_{2m} has two more 1-edges and two fewer 0-edges, making $e(1) - e(0) = -m + 11$.

Continue this process, interchanging an additional pair of vertices each time, keeping the vertex labeling friendly and increasing $e(1) - e(0)$ by 4. Stop when the vertices v_2 and v_{2m-2} have been interchanged. At this point, $e(1) - e(0) = 3m - 5$. Note that in each of the above labelings, v_0, v_1, and v_{2m-1} are always labeled 0, 0, and 1 respectively.

So far, we have constructed $(m - 1)$ friendly vertex labelings. For each of these labelings except the last one, interchange the vertex labels at v_0 and v_{2m-1}. Each vertex labeling remains friendly, with $v(1) - v(0) = 0$. The only edges that will be affected are (v_0, v_1), (v_0, v_{2m-1}), (v_{2m-1}, v_{2m-2}), and (v_1, v_{2m-1}), with labels changed from 0, 1, 0, and 1 to 1, 1, 1, and 0 respectively. Thus the value of $e(1) - e(0)$ is increased by 2 for each of these $(m - 2)$ labelings.

To conclude, the values of e(1) – e(0) are –m + 3, –m + 5, –m + 7, –m + 9, … , 3m – 9, 3m – 7, and 3m – 5. Taking absolute values finishes the proof.□

<u>**Theorem 4.**</u> $FI(PC_4) = \{1, 3\}$. For $m \geq 3$, $FI(PC_{2m}) = \{3m - 5, 3m - 7, 3m - 9, \ldots\}$. In other words, for $m \geq 3$, $FI(PC_{2m}) = \{3m - 5, 3m - 7, 3m - 9, \ldots, 1\}$ if m is even, and $FI(PC_{2m}) = \{3m - 5, 3m - 7, 3m - 9, \ldots, 0\}$ if m is odd.

<u>**Example 4.**</u> $FI(PC_6) = \{4, 2, 0\}$.

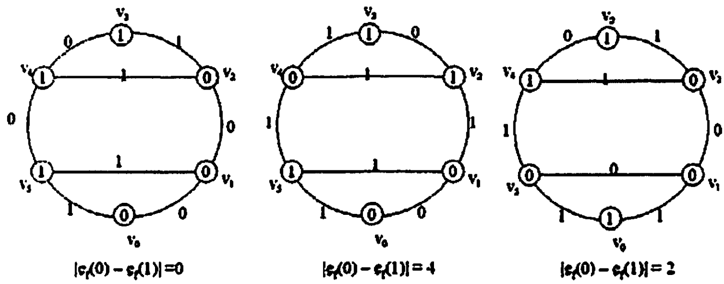

Figure 5.

<u>**Example 5.**</u> $FI(PC_8) = \{7, 5, 3, 1\}$.

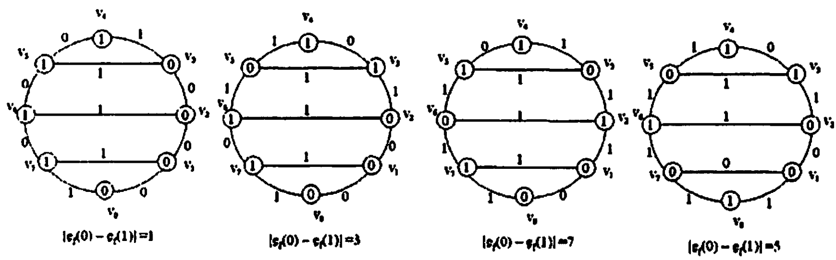

Figure 6.

3. Cycles with arbitrary parallel chords

Now we generalize the results of the previous section. We define a parallel chord of C_n to be an edge of the form (v_i, v_{n-i}), where $i < n - i$, that is not an edge of C_n. Consider the class PC(n, p), which consists of all graphs that are formed by the cycle C_n with p parallel chords, where $1 \leq p \leq \lfloor n/2 \rfloor - 1$. By an abuse in the notation, we will also use PC(n, p) to denote a graph in this class, and FI(PC(n, p)) to denote the friendly index set of such a graph. This slight abuse makes the notation much cleaner, because we really do not need to identify the exact positions of the parallel chords. Note that $p = \lfloor n/2 \rfloor - 1$ in the previous section, i.e., we had a full set of parallel chords. In other words, $PC_n = PC(n, \lfloor n/2 \rfloor - 1)$.

Consider the parallel chord (v_i, v_{n-i}), where i is smallest. We will call this the lowest parallel chord.

First consider odd values of n. Let $n = 2m + 1$, with $m \geq 2$.

Lemma 6. $FI(PC(2m + 1, p)) \subseteq \{2m + p - 1, 2m + p - 3, 2m + p - 5, \ldots\}$.
Proof. $PC(2m + 1, p)$ has $2m + p + 1$ edges. The rest of the proof is similar to that of Lemma 2.□

Lemma 7. Each of the non-negative integers $2m + p - 1, 2m + p - 3, 2m + p - 5, \ldots$ can be attained as $|e(1) - e(0)|$ using a friendly vertex labeling for $PC(2m + 1, p)$.
Proof. As in Lemma 3, we first label the vertices $v_0, v_1, v_2, \ldots, v_m$ by 0, and the vertices $v_{m+1}, v_{m+2}, \ldots, v_{2m}$ by 1. This vertex labeling is obviously friendly with $v(1) - v(0) = -1$, and with all the parallel chords labeled 1. Perform the same interchanges as in the proof of Lemma 3 to obtain $e(1) - e(0) = -2m + p + 3, -2m + p + 7, -2m + p + 11, \ldots, 2m + p - 1$.

Let the lowest parallel chord be (v_i, v_{2m+1-i}). Start from the first vertex labeling again, i.e., with the vertices $v_0, v_1, v_2, \ldots, v_m$ labeled by 0, and the vertices $v_{m+1}, v_{m+2}, \ldots, v_{2m}$ labeled by 1. Note that v_i has label 0. Change this vertex label to 1. The vertex labeling remains friendly, with $v(1) - v(0) = 1$ now. The edges (v_i, v_{i-1}), (v_i, v_{i+1}), and (v_i, v_{n-i}) are changed from 0, 0, and 1 to 1, 1, and 0 respectively. The graph has one more 1-edge and one fewer 0-edge, making $e(1) - e(0) = -2m + p + 5$. If $m = 2$, we are done.

If $m \geq 3$, perform the same interchanges again. This produces new friendly vertex labelings. The value of $e(1) - e(0)$ generally increases by 4 after each interchange. However the interchange ending at the vertices v_{i+1} and v_{2m-i}, and the interchange ending at the vertices v_i and v_{2m+1-i} do not affect the value of $e(1) - e(0)$ since the vertex labels at v_i and v_{2m+1-i} are the same. At the end, the value of $e(1) - e(0)$ is $2m + p - 7$. Note that if $i = 1$, we could get the extra value of $e(1) - e(0) = 2m + p - 3$. But we will take care of this case below even if $i \neq 1$.

Now we show that $e(1) - e(0) = 2m + p - 3$ can be attained using a friendly vertex labeling for $PC(2m + 1, p)$. Label the vertices of the cycle C_{2m+1}, namely $v_{2m+1-i}, v_{2m+2-i}, \ldots, v_{2m}, v_0, v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_{2m-1-i}$, and v_{2m-i} by 0, 1, 0, 1, 0, 1, …, 0, 1, 0, 1, 0 respectively. In the cycle C_{2m+1}, only the edge (v_{2m+1-i}, v_{2m-i}) has label 0, while all the other edges have label 1. The lowest parallel chord (v_i, v_{2m+1-i}) has label 0, since both vertices have label 0. All the other parallel chords have label 1. Thus $e(1) - e(0) = 2m + (p - 1) - 1 - 1 = 2m + p - 3$.

To conclude, the values of e(1) – e(0) are –2m + p + 3, –2m + p + 5, –2m + p + 7, –2m + p + 9, ... , 2m + p – 7, 2m + p – 5, 2m + p – 3, 2m + p – 1. Taking absolute values finishes the proof.□

Theorem 5. FI(PC(2m + 1, p)) = {2m + p – 1, 2m + p – 3, 2m + p – 5, ... }. In other words, FI(PC(2m + 1, p)) = {2m + p – 1, 2m + p – 3, 2m + p – 5, ... , 1} if p is even, and FI(PC(2m + 1, p)) = {2m + p – 1, 2m + p – 3, 2m + p – 5, ... , 0} if p is odd.

Example 6. FI(PC(9, 2)) = {9, 7, 5, 3, 1}.

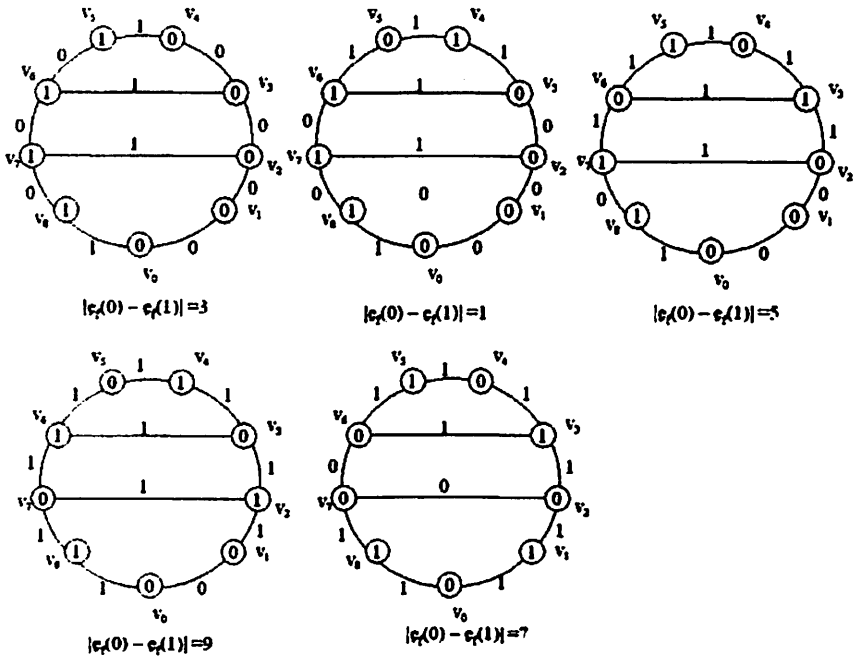

Figure 7.

Example 7. FI(PC(11, 1)) = {10, 8, 6, 4, 2, 0}.

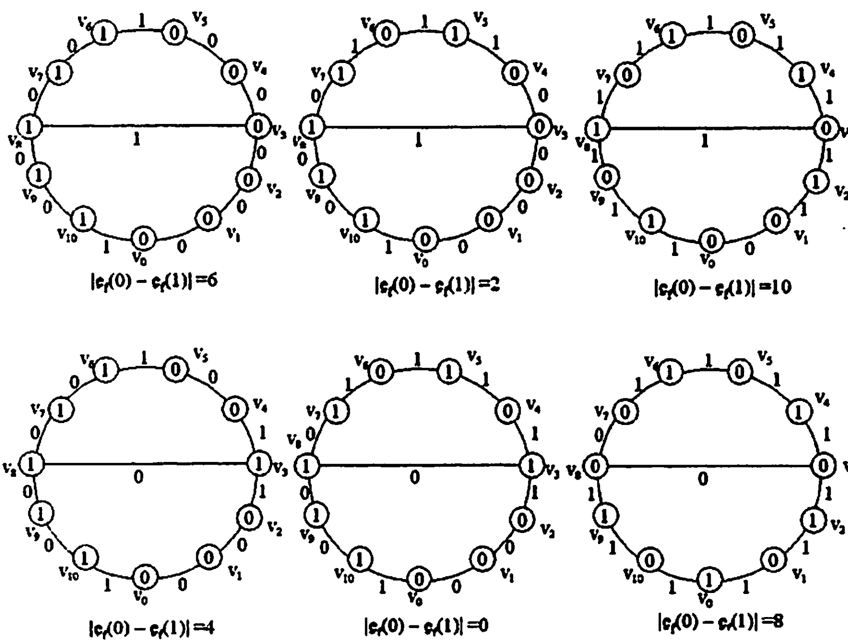

Figure 8.

Now we consider even values of n. Let $n = 2m$, where $m \geq 2$.

__Lemma 8.__ $FI(PC(2m, 1)) \subseteq \{2m-1, 2m-3, 2m-5, \ldots\}$.
Proof. $PC(2m, 1)$ has $2m + 1$ edges. By Theorem 1, $FI(PC(2m, 1)) \subseteq \{2m+1, 2m-1, 2m-3, 2m-5, \ldots\}$. If $2m + 1 \in FI(PC(2m, 1))$, then either all edge labels are 1 or all edge labels are 0. Applying Lemma 1 to a cycle containing the parallel chord, we see that the former is impossible. The latter vertex labeling is not friendly.□

__Lemma 9.__ Each of the non-negative integers $2m-1, 2m-3, 2m-5, \ldots$ can be attained as $|e(1) - e(0)|$ using a friendly vertex labeling for $PC(2m, 1)$.
Proof. As in Lemma 5, we first label the vertices $v_0, v_1, v_2, \ldots, v_{m-1}$ by 0, and the vertices $v_m, v_{m+1}, \ldots, v_{2m-1}$ by 1. This vertex labeling is obviously friendly with $v(1) - v(0) = 0$, and with the sole parallel chord labeled 1. Perform the same interchanges as in the proof of Lemma 5 to obtain $e(1) - e(0) = -2m + 5, -2m + 9, -2m + 13, \ldots, 2m - 3$.

Start from the first vertex labeling again, i.e., with the vertices $v_0, v_1, v_2, \ldots, v_{m-1}$ labeled by 0, and the vertices $v_m, v_{m+1}, \ldots, v_{2m-1}$ labeled by 1. The sole parallel chord has label 1, and thus its two adjacent vertex labels are complementary. Interchange the 1-vertex label of the parallel chord with the 0-vertex label at v_0. There are four or five edges that are affected, depending on whether the parallel chord is (v_1, v_{2m-1}). It is easy to see that after the interchange, PC(2m, 1) has one more 1-edge and one fewer 0-edge, making $e(1) - e(0) = -2m + 7$.

Now perform the same interchanges again. This produces new friendly vertex labelings. The value of $e(1) - e(0)$ generally increases by 4 after each interchange. However the interchange ending at the two vertices above the parallel chord, and the interchange ending at the two vertices of the parallel chord do not affect the value of $e(1) - e(0)$ since the labels at the two vertices forming the parallel chord are the same. At the end, the value of $e(1) - e(0)$ is $2m - 9$. Note that if the parallel chord is (v_1, v_{2m-1}), we could get the extra value of $e(1) - e(0) = 2m - 5$. But this is immaterial, as we will point out below.

Finally we show that $e(1) - e(0) = 2m - 1$ can be attained using a friendly vertex labeling for PC(2m, 1). Label the vertices $v_0, v_1, \ldots, v_{2m-2}, v_{2m-1}$ by 0, 1, ..., 0, 1 respectively. All the edges of the cycle have label 1, while the parallel chord has label 0.

To conclude, the values of $e(1) - e(0)$ are $-2m + 5, -2m + 7, -2m + 9, -2m + 11, \ldots, 2m - 9, 2m - 7, 2m - 3$, and $2m - 1$. Taking absolute values finishes the proof. Note that although the value $2m - 5$ for $e(1) - e(0)$ might be missing in our construction, it does not affect our result which only involves absolute values, since $-2m + 5$ is attainable.□

<u>**Theorem 6.**</u> FI(PC(2m, 1)) = {2m – 1, 2m – 3, 2m – 5, ... , 1}.

<u>**Example 8.**</u> FI(PC(8, 1)) = {7, 5, 3, 1}.

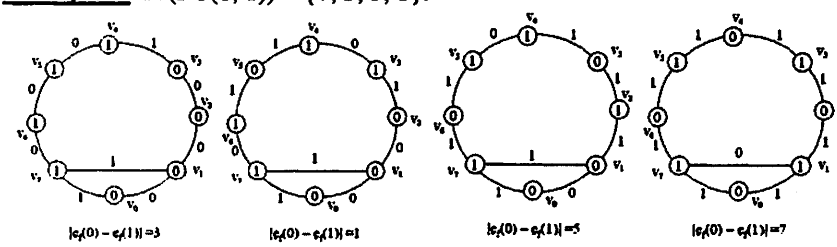

Figure 9.

<u>**Example 9.**</u> FI(PC(10, 1)) = {9, 7, 5, 3, 1}.

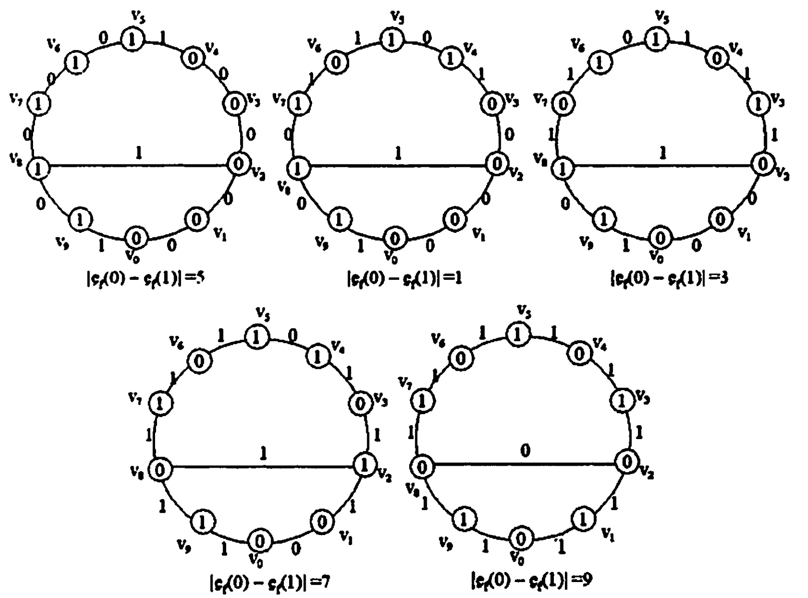

Figure 10.

Finally we consider PC(2m, p), where $p \geq 2$. In order for there to be at least two parallel chords, m must be at least 3.

Lemma 10. Let $m \geq 3$ and $p \geq 2$. $FI(PC(2m, p)) \subseteq \{2m + p - 4, 2m + p - 6, 2m + p - 8, \ldots\}$.
Proof. PC(2m, p) has $2m + p$ edges. The rest of the proof is similar to that of Lemma 4.□

Lemma 11. Let $m \geq 3$ and $p \geq 2$. Each of the non-negative integers $2m + p - 4$, $2m + p - 6$, $2m + p - 8$, … can be attained as $|e(1) - e(0)|$ using a friendly vertex labeling for PC(2m, p).
Proof. The first part of this proof is essentially the same as that for Lemma 9. We have friendly vertex labelings with $e(1) - e(0) = -2m + p + 4, -2m + p + 8, -2m + p + 12, \ldots, 2m + p - 4$.

Let the lowest parallel chord be (v_i, v_{2m-i}). Start from the first vertex labeling again, i.e., with the vertices $v_0, v_1, v_2, \ldots, v_{m-1}$ labeled by 0, and the vertices $v_m, v_{m+1}, \ldots, v_{2m-1}$ labeled by 1. The lowest parallel chord has label 1, with v_i labeled by 0 and v_{2m-i} labeled by 1. Interchange the 1-vertex label at v_{2m-i} with the 0-vertex label at v_0. There are four or five edges that are affected, depending on whether the lowest parallel chord is (v_1, v_{2m-1}). It is easy to see that after the interchange, PC(2m, p) has one more 1-edge and one fewer 0-edge, making $e(1) - e(0) = -2m + p + 6$.

The second part of this proof is again essentially the same as that for Lemma 9. Interchanges produce new friendly vertex labelings. The value of $e(1) - e(0)$ generally increases by 4 after each interchange. However the interchange ending at the vertices v_{i+1} and v_{2m-i-1}, and the interchange ending at the vertices v_i and v_{2m-i} do not affect the value of $e(1) - e(0)$ since the vertex labels at v_i and v_{2m-i} are the same. At the end, the value of $e(1) - e(0)$ is $2m + p - 10$. Note that if $i = 1$, we could get the extra value of $e(1) - e(0) = 2m + p - 6$. But we will take care of this case below even if $i \neq 1$.

Now we show that $e(1) - e(0) = 2m + p - 6$ can be attained using a friendly vertex labeling for PC(2m, p). Label both vertices v_i and v_{2m-i} by 0. We have an odd number of vertices in the path $v_{2m-i+1}, v_{2m-i+2}, \ldots, v_{2m-1}, v_0, v_1, \ldots, v_{i-2}, v_{i-1}$. Label them consecutively by 1, 0, 1, 0, ... , 1, 0, and 1. Label the vertices $v_{i+1}, v_{i+2}, \ldots, v_{m-1}$ alternately by 1 and 0. Label the vertices $v_{2m-i-1}, v_{2m-i-2}, \ldots, v_{m+1}$ alternately by 0 and 1. Label the vertex v_m by 1. This vertex labeling is friendly with $v(1) - v(0) = 0$. The lowest parallel chord (v_i, v_{2m-i}) has label 0. All the other parallel chords have label 1, because (v_i, v_{2m-i}) is the lowest. Exactly one of the two edges incident at v_m has label 0. The edge (v_{2m-i-1}, v_{2m-i}) has label 0. All other edges of the cycle C_{2m} have label 1. Thus $e(1) - e(0) = (p - 1) + (2m - 2) - 1 - 2 = 2m + p - 6$.

To conclude, the values of $e(1) - e(0)$ are $-2m + p + 4, -2m + p + 6, -2m + p + 8, -2m + p + 10, \ldots, 2m + p - 10, 2m + p - 8, 2m + p - 6$, and $2m + p - 4$. Taking absolute values finishes the proof. □

<u>**Theorem 7.**</u> Let $m \geq 3$ and $p \geq 2$. $FI(PC(2m, p)) = \{2m + p - 4, 2m + p - 6, 2m + p - 8, \ldots\}$. In other words, $FI(PC(2m, p)) = \{2m + p - 4, 2m + p - 6, 2m + p - 8, \ldots, 0\}$ if p is even, and $FI(PC(2m, p)) = \{2m + p - 4, 2m + p - 6, 2m + p - 8, \ldots, 1\}$ if p is odd.

<u>**Example 10.**</u> $FI(PC(8, 2)) = \{6, 4, 2, 0\}$

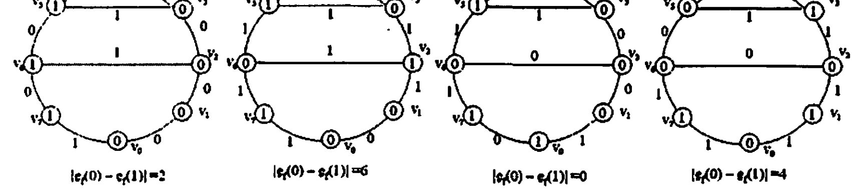

Figure 11.

<u>**Example 11.**</u> $FI(PC(10, 3)) = \{9, 7, 5, 3, 1\}$.

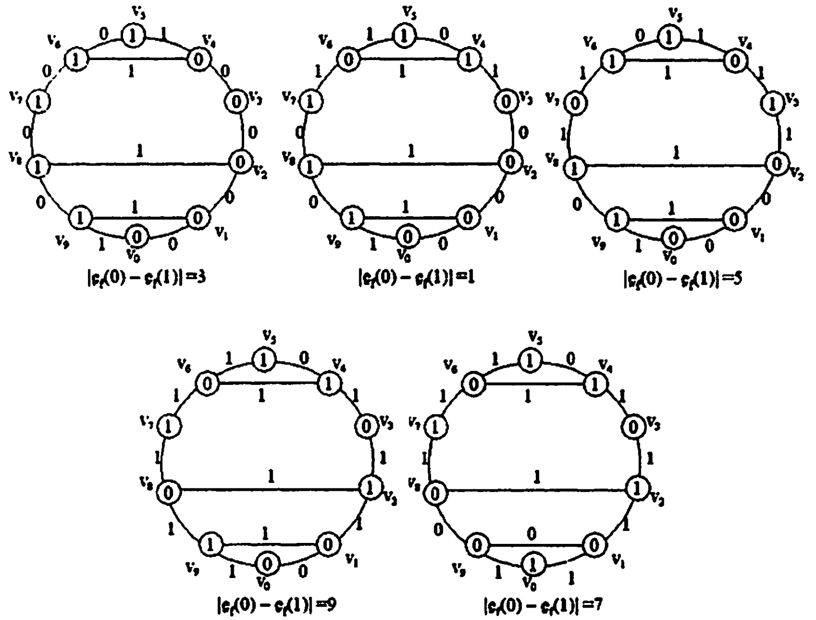

Figure 12.

4. Counter-examples

In this paper, we show that for a cycle with an arbitrary non-empty set of parallel chords, the numbers in its friendly index set form an arithmetic progression with common difference 2. We end this paper with a couple of comments, indicating that certain conditions in the theorems cannot be waived.

If the set of parallel chords is empty, i.e., we have a cycle, Theorem 2 shows that the numbers in the friendly index set still form an arithmetic progression. But if the cycle is even, the common difference is 4, not 2.

If the chords are not parallel, the numbers in the friendly index set might not form an arithmetic progression. See [15] on the friendly index sets of Möbius ladders.

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