# ON THE NUMBER OF GENERALIZED DYCK PATHS

MITSUNORI IMAOKA[1], ISAO TAKATA[2] AND YU FUJIWARA[3]

ABSTRACT. It is known that the number of Dyck paths is given by a Catalan number. Dyck paths are represented as plane lattice paths which start at the origin $O$ and end at the point $P_n = (n, n)$ repeating $(1,0)$ or $(0,1)$ steps without going above the diagonal line $OP_n$. Therefore, it is reasonable to ask of any positive integers $a$ and $b$ what number of lattice paths start at $O$ and end at point $A = (a, b)$ repeating the same steps without going above the diagonal line $OA$. In this article, we show a formula to represent the number of such generalized Dyck paths.

## 1. INTRODUCTION

For a positive integer $n$, a Dyck $n$–path is usually defined as a plane lattice path which starts at the origin $O = (0,0)$ and ends at $(2n, 0)$ repeating $(1,1)$ or $(1,-1)$ steps without ever going below the $x$–axis. It is a common result that the number of the Dyck $n$–paths is equal to the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{1}$$

An equivalent definition of a Dyck $n$–path is given as a plane lattice path which starts at the origin $O$ and ends at the point $P_n = (n,n)$ repeating $(1,0)$ or $(0,1)$ steps without going above the diagonal line $OP_n$.

For any positive integers $r$ and $k$, a plane lattice path which starts at $O$ and ends at the point $P_{r,k} = (rk, r)$ by repeating $(1,0)$ or $(0,1)$ steps without going above the diagonal line $OP_{r,k}$ of the $(rk \times r)$ rectangle is called a $k$–Catalan path, which has appeared in Goulden and Jackson [2] and been given noteworthy interpretations by Pak [6], Mansour and Sun [5] and Heubach, Li and Mansour [3]. For any fixed integer $r \geq 1$, the total number $C_r^{k+1}$ of the $k$–Catalan paths is named as the $(k+1)$–ary number in [3], and given as

$$C_r^{k+1} = \frac{1}{(k+1)r+1}\binom{(k+1)r+1}{r}. \tag{2}$$

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Next, we consider a plane lattice path which starts at $O$ and ends at a point $A = (a, b)$ repeating $(1, 0)$ or $(0, 1)$ steps without going above the diagonal line $OA$ for any positive integers $a$ and $b$. In this article, we shall refer to such a lattice path as a Dyck path of type $(a, b)$, and analyze the number of such generalized Dyck paths. Duchon [1] has called the Dyck paths of type $(a, b)$ the rational slope Dyck paths, and given good estimations of the number of them. He has studied the rational slope Dyck paths as special cases of "generalized Dyck words", and his estimations of the number of them have been obtained applying 2 kinds of "conjugations of words." In the lower bound estimation, he has used one of the conjugations of words, which corresponds to our action of a cyclic group on a set of lattice paths to be introduced in the next section. We shall elaborate on the effects of the actions to count relevant values, and give a formula representing the number of Dyck paths of type $(a, b)$ in Theorem 1.1 as seen below. Some more generalized notions of Dyck paths have been studied by Labelle and Yen [4], but our methods seem to be too specialized to apply to such further generalizations.

Now, we shall state our concrete results. For any positive integers $a$ and $b$, we set $r = \gcd\{a, b\}$, $a = rc$ and $b = rd$, and denote the number of all Dyck paths of type $(a, b)$ by $d_r(c, d)$. Thus, for any integer $k \geq 1$ and any coprime integers $i \geq 1$ and $j \geq 1$, $d_k(i, j)$ denotes the number of all Dyck paths of type $(ki, kj)$.

We set

$$s_i(c, d) = \frac{1}{i(c+d)} \binom{i(c+d)}{ic} \in \mathbb{Q} \tag{3}$$

for any integer $i \geq 1$. We notice that $s_i(c, d)$ is not an integer in general but is a rational number. So, our main result is stated as follows.

**Theorem 1.1.** *For any integer $r \geq 1$ and any coprime integers $c \geq 1$ and $d \geq 1$, we have a formula*

$$d_r(c, d) = \sum_{k=1}^{r} \sum_{i_1 j_1 + \cdots + i_k j_k = r} \frac{s_{i_1}(c, d)^{j_1}}{j_1!} \cdots \frac{s_{i_k}(c, d)^{j_k}}{j_k!},$$

*where the second sum is taken over sequences $(i_1, \cdots, i_k) \in \mathbb{N}^k$ and $(j_1, \cdots, j_k) \in \mathbb{N}^k$ satisfying both $i_1 < i_2 < \cdots < i_k$ and $i_1 j_1 + \cdots + i_k j_k = r$.*

As a special case, we have the following corollary, originally our cause for using the values $s_i(c, d)$ in (3).

**Corollary 1.2.** *For any coprime integers $a \geq 1$ and $b \geq 1$,*

$$d_1(a, b) = s_1(a, b) = \frac{1}{a+b} \binom{a+b}{a}.$$

As a result, we can give a basic proof of the formula for $C_r^{k+1} = d_r(k,1)$ in (2) as an application of Corollary 1.2, which we shall show at the end of this paper.

We can estimate the difference between the values of $d_r(c,d)$ and $s_r(c,d)$ using the terms for $(k, i_1, j_1) = (1, r, 1)$ and $(1, 1, r)$ on the right-hand side of the equation in Theorem 1.1, as follows.

**Corollary 1.3.** *For any integer $r \geq 2$ and any coprime integers $c \geq 1$ and $d \geq 1$,*

$$d_r(c,d) - s_r(c,d) \geq \frac{s_1(c,d)^r}{r!}.$$

Let $C_k$ be the Catalan number as in (1). Then, since

$$2ks_k(1,1) = (k+1)C_k = (k+1)d_k(1,1),$$

we have the following relation of the Catalan numbers as a special case $c = d = 1$ in Theorem 1.1.

**Corollary 1.4.** *For any positive integer $r$, we have*

$$C_r = \sum_{k=1}^{r} \sum_{i_1 j_1 + \cdots + i_k j_k = r} \frac{(\frac{i_1+1}{2i_1} C_{i_1})^{j_1}}{j_1!} \cdots \frac{(\frac{i_k+1}{2i_k} C_{i_k})^{j_k}}{j_k!},$$

*where the second sum is taken over sequences $(i_1, \cdots, i_k) \in \mathbb{N}^k$ and $(j_1, \cdots, j_k) \in \mathbb{N}^k$ satisfying both $i_1 < i_2 < \cdots < i_k$ and $i_1 j_1 + \cdots + i_k j_k = r$.*

From here, we organize this paper as follows. In the next section, we count various necessary values and prove Theorem 2.6 as a goal; and, in the last section, we complete the proof of Theorem 1.1 using Theorem 2.6.

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## 2. Countings

Let any coprime integers $c \geq 1$ and $d \geq 1$ be given. Then, for a positive integer $r$, we denote by $E_r$ the set of all plane lattice paths which start at the origin $O$ and end at the point $P_r = (rc, rd)$ repeating $(1,0)$ or $(0,1)$ steps. Thus, any element of $E_r$ is a lattice path in the $(rc \times rd)$ rectangle with vertices $O$, $(rc, 0)$, $P_r$ and $(0, rd)$; and, $E_r$ has $\binom{r(c+d)}{rc}$ elements. We denote the lattice points on the diagonal $OP_r$ of the rectangle by $P_i = (ic, id)$ for $0 \leq i \leq r$, where $O = P_0$.

As in the previous section, a Dyck path of type $(rc, rd)$ is an element of $E_r$ which does not go above the diagonal $OP_r$. That is, it is not ever higher than $OP_r$. We set

(4) $\quad D_r(i) = \{l \in E_r \mid l \text{ is a Dyck path of type } (rc, rd) \text{ which passes } P_i\}$

for each $i$ with $1 \leq i \leq r$, and put

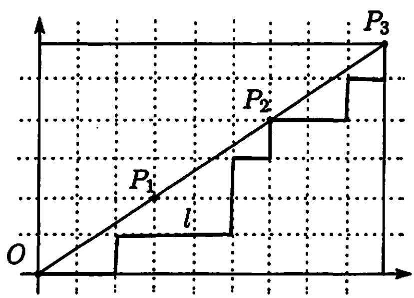
$$D_r = D_r(r).$$

Then, $D_r$ is the set of all Dyck paths of type $(rc, rd)$, and our main result (Theorem 1.1) will show a formula representing the number $d_r(c,d) = |D_r|$. Here, $|S|$ denotes the number of elements of a finite set $S$.

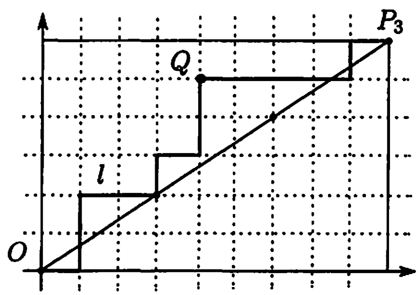
Let $F_0 = D_r - \bigcup_{1 \leq i \leq r-1} D_r(i)$. Then, $F_0$ is the set of Dyck paths of type $(rc, rd)$ which do not pass any $P_i$ for $1 \leq i \leq r-1$. Also, let $F_k$ for $1 \leq k \leq r-1$ denote the set of Dyck paths of type $(rc, rd)$ each of which passes exactly $k$ of the lattice points among $\{P_1, \cdots, P_{r-1}\}$. Then, $D_r$ is represented as the disjoint union

$$D_r = F_0 \cup F_1 \cup \cdots \cup F_{r-1}. \tag{5}$$

Now, we shall introduce the notion of a peak. For any lattice point $A$ in the $(rc \times rd)$ rectangle with the diagonal line $OP_r$, where $P_r = (rc, rd)$, $l(A)$ denotes the (Euclidean) distance between $A$ and $OP_r$ if $A$ is in the region $y \geq (d/c)x$, and $(-1)$ times the distance between $A$ and $OP_r$ if $A$ is in the region $y < (d/c)x$. Then, for any $l \in E_r$, we say that a lattice point $Q$ on $l$ is a *peak* of $l$ if $Q \neq O$ and $Q$ is the lattice point which has the maximal distance to $OP_r$. Thus, if $l \in D_r$, then the set of peaks of $l$ is a subset of $\{P_1, \cdots, P_r\}$.




A Dyck path $l$ of type $(9,6)$




A peak $Q$ of a lattice path $l$

Fig. 1. A Dyck path and a peak

Let $G_m = \{1, \sigma, \sigma^2, \cdots, \sigma^{m-1}\}$ be the cyclic group of order $m$ with generator $\sigma$. Then, $G_{r(c+d)}$ acts on $E_r$ in the following way. For any $l \in E_r$, if $l$ passes the lattice points $\{A_i \mid 0 \leq i \leq r(c+d)\}$ in an ascending order of indices, we denote it by $O = A_0 \to A_1 \to \cdots \to A_k \to \cdots \to A_{r(c+d)} = P_r$. Then, for

$0 \leq k \leq r(c+d) - 1$, we define $\sigma^k l \in E_r$ as the lattice path which passes the lattice points as

$$\sigma^k l: O \to (A_{k+1} - A_k) \to (A_{k+2} - A_k) \to \cdots \to (P_r - A_k) \\ \to (P_r - A_k + A_1) \to \cdots \to (P_r - A_k + A_{k-1}) \to P_r.$$

That is, the lattice path $\sigma^k l$ is obtained by cutting the path $O \to \cdots \to P_k$ at $A_k$, translating the second part of the path such that $A_k$ maps to the origin, and then attaching the first part of the original path at the end of the translated part. Let $o(l) = \{\sigma^i l \mid 0 \leq i \leq r(c+d) - 1\}$ be the orbit of $l \in E_r$ under the action of $G_{r(c+d)}$; and, let $G^l_{r(c+d)} = \{\sigma^i \in G_{r(c+d)} \mid \sigma^i l = l\}$ be the stabilizer to $l \in E_r$. Then, the following lemma is clear from the definitions.

**Lemma 2.1.** (i) *If $o(l) = o(l')$, then the number of peaks of $l$ is equal to that of $l'$.*
(ii) *For any $l \in E_r$, $o(l) \cap D_r \neq \emptyset$.*
(iii) *For any $l \in E_r$, $|o(l)| = r(c+d)/|G^l_{r(c+d)}|$.*

For $0 \leq k \leq r-1$, let $F_k$ be the subset of $D_r$ introduced above and satisfying (5). Then, we set

(6) $$J_k = \{l \in E_r \mid o(l) = o(l') \text{ for some } l' \in F_k\}.$$

That is, $J_k$ is the set of lattice paths in $E_r$ each of which has exactly $(k+1)$ peaks. Then, by (5) and Lemma 2.1, the set $E_r$ is represented as the disjoint union

(7) $$E_r = J_0 \cup J_1 \cup \cdots \cup J_{r-1}.$$

Now, for $0 \leq k \leq r-1$, the cyclic group $\widetilde{G}_{k+1} = \{c^i \mid 0 \leq i \leq k\}$ of order $(k+1)$ acts on $F_k$ in the following way. For any $l \in F_k$, if $l$ passes the diagonal lattice points $O = P_{j_0}, P_{j_1}, \cdots, P_{j_k}$ in an ascending order of indexes, then we set $c^i l = \sigma^{j_i} l$ for $0 \leq i \leq k$, where $\sigma$ is the generator of the cyclic group $G_{r(c+d)}$ acting on $E_r$ as above. That is, the Dyck path $c^i l$ is obtained by translating a part of $l \cup (l + P_r)$ by $-P_{j_i}$. We notice that the map $h: \widetilde{G}_{k+1} \to G_{r(c+d)}$ defined by $h(c^i) = \sigma^{j_i}$ is injective.

Let $\widetilde{G}^l_{k+1} = \{c^j \in \widetilde{G}_{k+1} \mid c^j l = l\}$ be the stabilizer to $l \in F_k$. Then, we have the following lemma.

**Lemma 2.2.** *For any $l \in F_k$, we have $|\widetilde{G}^l_{k+1}| = |G^l_{r(c+d)}|$ and*

$$|o(l) \cap F_k| = \frac{k+1}{|G^l_{r(c+d)}|}.$$

*Proof.* We fix any $l \in F_k$. Let $h : \widetilde{G}_{k+1} \to G_{r(c+d)}$ be the above injective map satisfying $c^i l = h(c^i)l$. Thus, $|\widetilde{G}^l_{k+1}| \leq |G^l_{r(c+d)}|$ holds. But, if $\sigma^j l = l$, there exists $c^i \in \widetilde{G}^l_{k+1}$ which satisfies $h(c^i) = \sigma^j$. Thus, the first equation $|\widetilde{G}^l_{k+1}| = |G^l_{r(c+d)}|$ holds.

Since the restriction of the action of $\widetilde{G}_{k+1}$ on $o(l) \cap F_k$ is transitive, we have $|o(l) \cap F_k| = |\widetilde{G}_{k+1} l| = |\widetilde{G}_{k+1}|/|\widetilde{G}^l_{k+1}| = (k+1)/|G^l_{r(c+d)}|$ as required. □

Let $J_k$ be the subset of $E_r$ given in (6) for $0 \leq k \leq r-1$ and let $c \geq 1$ and $d \geq 1$ be given coprime integers. Then, we have the following proposition.

**Proposition 2.3.** *For* $0 \leq k \leq r-1$, *we have*

$$|J_k| = \frac{r(c+d)}{k+1}|F_k|.$$

*Proof.* For any $l \in F_k$, we have $|o(l)| = r(c+d)/|G^l_{r(c+d)}|$ by Lemma 2.1(iii), and $|o(l) \cap F_k| = (k+1)/|G^l_{r(c+d)}|$ by Lemma 2.2. Thus, we have $|o(l)| = (r(c+d)/(k+1))|o(l) \cap F_k|$ for any $l \in F_k$. Since $|F_k| = \sum |o(l) \cap F_k|$ and $|J_k| = \sum |o(l)|$, where both sums are taken over all $o(l)$ with $l \in F_k$, we conclude that $|J_k| = (r(c+d)/(k+1))|F_k|$. □

Let $s_i(c,d) = (1/(i(c+d)))\binom{i(c+d)}{ic}$ be the rational number in (3). Then, the next proposition follows from Proposition 2.3.

**Proposition 2.4.** *For any integer* $r \geq 1$, *we have*

$$s_r(c,d) = \sum_{k=0}^{r-1} \frac{1}{k+1}|F_k|.$$

*Proof.* By (7) and Proposition 2.3, it follows

$$|E_r| = \sum_{k=0}^{r-1} |J_k| = \sum_{k=0}^{r-1} \frac{r(c+d)}{k+1}|F_k|.$$

Since $|E_r| = \binom{r(c+d)}{rc}$, we have

$$s_r(c,d) = \frac{1}{r(c+d)}\binom{r(c+d)}{rc} = \sum_{k=0}^{r-1} \frac{1}{k+1}|F_k|$$

as required. □

Let $D_r(i)$ be the subset of $D_r$ in (4) for any $r \geq 1$ and any $i$ with $1 \leq i \leq r$. Then, we set

(8) $\quad a_r(i_1, i_2, \cdots, i_k) = |D_r(i_1) \cap D_r(i_1+i_2) \cap \cdots \cap D_r(i_1+\cdots+i_k)|$

for any integer $k$ with $1 \le k \le r-1$ and any sequence $(i_1, \cdots, i_k) \in \mathbb{N}^k$ satisfying $i_1 + \cdots + i_k \le r-1$. Then, applying Proposition 2.4, $s_r(c,d)$ is represented by using $|D_r|$ and $a_r(i_1, i_2, \cdots, i_k)$ as follows.

**Proposition 2.5.** *For any integer $r \ge 1$, we have*

$$s_r(c,d) = |D_r| + \sum_{k=1}^{r-1} (-1)^k \frac{1}{k+1} \sum_{(i_1,\cdots,i_k)} a_r(i_1, i_2, \cdots, i_k),$$

*where the latter sum is taken over all sequences $(i_1, \cdots, i_k) \in \mathbb{N}^k$ satisfying $i_1 + \cdots + i_k \le r-1$.*

*Proof.* We denote the right-hand side of the equation of Proposition 2.5 by $H_r$, and the right-hand side of the equation of Proposition 2.4 by $K_r$. Then, it is sufficient to show the equation $K_r = H_r$. However, we can write $K_r = \sum_{l \in D_r} a_l$ and $H_r = \sum_{l \in D_r} b_l$ for some $a_l \in \mathbb{Q}$ and $b_l \in \mathbb{Q}$, where $a_l$ and $b_l$ are the contributions of $l$ to $K_r$ and $H_r$ respectively, and thus it is sufficient to show that $a_l = b_l$ for any $l \in D_r$.

Recall that $D_r$ is the disjoint union of $F_m$ for $0 \le m \le r-1$ as in (5). First, we assume that $l \in F_0$. Then, the contribution of $l$ is 1 (to the count of $|F_0|$) in $K_r$, thus $a_l = 1$. Such a path contributes only to the term $|D_r|$ in $H_r$, and the contribution is $b_l = 1 = a_l$. If $l \in F_m$ for $1 \le m \le r-1$, then $l$ contributes $1/(m+1)$ to $K_r$ as it is counted once in $|F_m|$, therefore $a_l = 1/(m+1)$. On the other hand, $l$ contributes to several terms in $H_r$. It is counted once in the term $|D_r|$, and it is also counted in $a_r(i_1, i_2, \cdots, i_k)$ for $1 \le k \le m$; each time, the path is counted for any selection of the $k$ points $P_i$ that $l$ passes. Thus the path is counted $\binom{m}{k}$ times in $a_r(i_1, i_2, \cdots, i_k)$. The overall contribution is

$$b_l = 1 + \sum_{k=1}^{m} (-1)^k \frac{1}{k+1} \binom{m}{k} = 1 - \frac{1}{m+1} \sum_{k=1}^{m} (-1)^{k+1} \binom{m+1}{k+1} = \frac{1}{m+1}.$$

Thus, $a_l = b_l$ holds in this case too, and we have completed the proof. □

Let $d_i(c,d)$ be the number of Dyck paths of type $(ic, id)$ for any integer $i \ge 1$ and given coprime integers $c \ge 1$ and $d \ge 1$. Then, using Proposition 2.5, we can prove the next theorem from which our main theorem (Theorem 1.1) will follow.

**Theorem 2.6.** *For any integer $r \ge 1$, we have*

$$s_r(c,d) = \sum_{h=1}^{r} (-1)^{h-1} \frac{1}{h} \sum_{(m_1,\cdots,m_h)} d_{m_1}(c,d) \cdots d_{m_h}(c,d),$$

*where the latter sum is taken over all sequences* $(m_1, \cdots, m_h) \in \mathbb{N}^h$ *satisfying* $m_1 + \cdots + m_h = r$.

*Proof.* We notice that the partial sum limited to $h = 1$ on the right-hand side of the required equation is equal to $d_r(c,d) = |D_r|$. Let $a_r(i_1, i_2, \cdots, i_k)$ be the integers in (8) for $1 \leq k \leq r-1$ and $(i_1, \cdots, i_k) \in \mathbb{N}^k$. Then, $a_r(i_1, i_2, \cdots, i_k)$ is the number of Dyck paths, each of which passes all the lattice points $P_i = (ic, id)$ for $i = i_1, i_1 + i_2, \cdots, i_1 + \cdots + i_k$. Hence, we have

$$a_r(i_1, i_2, \cdots, i_k) = d_{i_1}(c,d) d_{i_2}(c,d) \cdots d_{i_k}(c,d) d_{r-(i_1+\cdots+i_k)}(c,d).$$

Thus, by Proposition 2.5, we have the required representation of $s_r(c,d)$ using the values of $d_i(c,d)$. □

## 3. Proof of Main Results

In this section, we prove Theorem 1.1 by applying Theorem 2.6. Hereafter, we abbreviate the notations $d_i(c,d)$ and $s_i(c,d)$ for any fixed coprime integers $c \geq 1$ and $d \geq 1$ to $d_i$ and $s_i$, respectively. Then, we denote the generating functions of the sequences $\{d_i\}$ and $\{s_i\}$ by

$$f_d(x) = \sum_{i \geq 1} d_i x^i \text{ and } g_s(x) = \sum_{i \geq 1} s_i x^i,$$

respectively.

By Theorem 2.6, we have the equation

$$s_r = \sum_{h=1}^{r} (-1)^{h-1} \frac{1}{h} e(r,h), \tag{9}$$

where we set

$$e(r,h) = \sum_{(m_1, \cdots, m_h)} d_{m_1} \cdots d_{m_h}.$$

Here, the sum in the expression for $e(r,h)$ is taken over sequences $(m_1, \cdots, m_h) \in \mathbb{N}^h$ satisfying $m_1 + \cdots + m_h = r$. Then, $e(r,h)$ is equal to the coefficient of $x^r$ in the power series $f_d(x)^h$, and thus using (9) we have

$$\begin{aligned} g_s(x) &= \sum_{r \geq 1} \left( \sum_{h=1}^{r} (-1)^{h-1} \frac{1}{h} e(r,h) \right) x^r \\ &= \sum_{h \geq 1} (-1)^{h-1} \frac{1}{h} f_d(x)^h. \end{aligned}$$

By taking the differentials of both sides in this expression, we have

$$\begin{aligned} g_s'(x) &= f_d'(x)\sum_{h\geq 1}(-1)^{h-1}f_d(x)^{h-1} \\ &= \frac{f_d'(x)}{1+f_d(x)} = (\log(1+f_d(x)))'. \end{aligned}$$

Since $f_d(0) = g_s(0) = 0$, we have $\log(1+f_d(x)) = g_s(x)$. Hence, we have

$$1 + f_d(x) = \exp(g_s(x)) = \sum_{k\geq 0}\frac{1}{k!}g_s(x)^k.$$

By the polynomial expansion of the power series, it follows

$$g_s(x)^k = \sum_{r\geq 1}\left(\sum_{i_1j_1+\cdots+i_kj_k=r}\frac{k!}{j_1!\cdots j_k!}s_{i_1}^{j_1}\cdots s_{i_k}^{j_k}\right)x^r,$$

where the second sum is taken over all sequences $(i_1,\cdots,i_k)\in\mathbb{N}^k$ and $(j_1,\cdots,j_k)\in\mathbb{N}^k$ satisfying both $i_1<\cdots<i_k$ and $i_1j_1+\cdots+i_kj_k=r$. Thus, we have

$$\begin{aligned} f_d(x) &= \sum_{k\geq 1}\frac{1}{k!}g_s(x)^k \\ &= \sum_{k\geq 1}\sum_{r\geq 1}\left(\sum_{i_1j_1+\cdots+i_kj_k=r}\frac{1}{j_1!}\cdots\frac{1}{j_k!}s_{i_1}^{j_1}\cdots s_{i_k}^{j_k}\right)x^r \\ &= \sum_{r\geq 1}\left(\sum_{k=1}^{r}\sum_{i_1j_1+\cdots+i_kj_k=r}\frac{s_{i_1}^{j_1}}{j_1!}\cdots\frac{s_{i_k}^{j_k}}{j_k!}\right)x^r. \end{aligned}$$

Hence, we obtain the required expression

$$d_r = \sum_{k=1}^{r}\sum_{i_1j_1+\cdots i_kj_k=r}\frac{s_{i_1}^{j_1}}{j_1!}\cdots\frac{s_{i_k}^{j_k}}{j_k!},$$

and we have completed the proof of Theorem 1.1.

Corollaries 1.2, 1.3 and 1.4 are clear from Theorem 1.1.

Lastly, as an application of Corollary 1.2, we shall prove the formula

$$d_r(k,1) = \frac{1}{(k+1)r+1}\binom{(k+1)r+1}{r}, \tag{10}$$

as remarked in the first section.

Since the formula is true for $r = 1$, we assume that $r \geq 2$. We consider a $((rk+1)\times r)$ rectangle with vertices $A = (-1,0)$, $B = (rk,0)$, $C = (rk,r)$ and $D = (-1,r)$; and, we consider the Dyck paths in this rectangle, that is, the lattice paths starting at $A = (-1,0)$ and ending at $C = (rk,r)$, repeating $(1,0)$

or $(0,1)$ steps, and not entering the region $y > (r/(rk+1))(x+1)$. Temporarily, we call such a Dyck path an *e-path*. Since $(rk+1)$ and $r$ are coprime, the number of e-paths is equal to $d_1(rk+1, r)$ which is equal to the right-hand side of the required equation in (10) by Corollary 1.2.

We now show that there are no lattice points in the interior of the triangle $AOC$ defined by $\min\{(1/k)x, 0\} < y \leq (r/(rk+1))(x+1)$ and $-1 < x < rk$ using Pick's Theorem. Note that there are no lattice points in the interior of the segment $AC$ since $(rk+1)$ and $r$ are coprime. On the segment $OC$, there are exactly $(r+1)$ lattice points $(ki, i)$ for $0 \leq i \leq r$. Thus, the number $b$ of the lattice points on the boundary of the triangle $AOC$ is equal to $(r+2)$. The area $S$ of the triangle $AOC$ is obviously $r/2$, and thus the number $i$ of the lattice points interior to the triangle $AOC$ is equal to 0 since we have $S = i + b/2 - 1$ by Pick's Theorem [7].

Therefore, any e-path goes first from $(-1,0)$ to $O$ since $r \geq 2$, and then continues on a Dyck path of type $(rk, r)$ from $O$ to $C$. Thus, the set of e-paths is bijective to the set of Dyck paths of type $(rk, r)$, and we have $d_r(k,1) = d_1(rk+1, r)$. In this way we can prove (10) only using Corollary 1.2.

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[1]DEPARTMENT OF MATHEMATICS EDUCATION, GRADUATE SCHOOL OF EDUCATION, HIROSHIMA UNIVERSITY, 1-1-1 KAGAMIYAMA HIGASHI–HIROSHIMA 739–8524, JAPAN

*E-mail address*: imaoka@hiroshima-u.ac.jp

[2]DEPARTMENT OF ARTS AND SCIENCE, AKASHI NATIONAL COLLEGE OF TECHNOLOGY, 679–3 NISHIOKA, UOZUMI, AKASHI 674–8501, JAPAN

*E-mail address*: takata@akashi.ac.jp

[3]GRADUATE SCHOOL OF EDUCATION, HIROSHIMA UNIVERSITY, 1-1-1 KAGAMIYAMA HIGASHI–HIROSHIMA 739–8524, JAPAN

*E-mail address*: m082470@hiroshima-u.ac.jp