A Note on the Maximum Number of Edges of a Spanning Eulerian Subgraph*

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Abstract A graph G is supereulerian if G has a spanning eulerian subgraph. We use SL to denote the families of supereulerian graphs. In 1995, Zhi-Hong Chen and Hong-Jian Lai presented the following open problem [2, problem 8.8] : Determine

$$L = \underset{G \in SL - \{K_1\}}{minmax} \{\frac{|E(H)|}{|E(G)|} : H \text{ is a spanning eulerian subgraph of } G\}$$

For a graph G, O(G) denotes the set of all odd-degree vertices of G. Let G be a simple graph and $|O(G)| = 2k$. In this note, we show that if $G \in SL$ and $k \leq 2$, then $L \geq 2/3$.
Keywords: Number of edges; Spanning eulerian subgraph; Supereulerian

We use [1] for terminology and notations not defined here, and consider finite simple graphs only. For a graph G, let $O(G)$ denote the set of all odd-degree vertices of G. An eulerian graph G is a connected graph with $O(G) = \emptyset$. A graph G is supereulerian if G has a spanning eulerian subgraph. We use SL to denote the families of supereulerian graphs.

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In 1995, Zhi-Hong Chen and Hong-Jian Lai presented the following open problem [2, problem 8.8] :

Problem. Determine

$$L = \min_{G \in SL - \{K_1\}} \max \{\frac{|E(H)|}{|E(G)|} : H \text{ is a spanning eulerian subgraph of } G\}$$

P. A. Catlin once thought that L could be 2/3. In [3], we presented infinite families of graphs to show L should be less than 2/3.

In the present paper, we prove the following result.

Theorem Let G be a simple graph with $|O(G)| = 2k$. If $G \in SL$ and $k \le 2$, then $L \ge 2/3$.

Proof: Suppose that H is a spanning eulerian subgraph of G with maximum number of edges. Therefore, every nontrivial connected component of the graph $G - E(H)$ is a tree. According to $k = 1$, and $k = 2$, respectively, we distinguish two cases to complete the proof of the theorem. Suppose that u and v are the vertices of G, we use $P(u-v)$ to denote the path from u to v in G.

The case of $k = 1$:

We assume that the vertices u and v are the vertices of odd-degree in G. Since H is a spanning eulerian subgraph of G, we have $O(G) = O(G - E(H))$. Hence, $O(G - E(H))$ has two vertices of odd-degree only. It follows that $O(G - E(H))$ has a nontrivial connected component P only, where P is a path from u to v.

Suppose that $|V(P)| = n$. Since H is a spanning eulerian subgraph of G, for each vertex w in $V(P)$, there exist at least two edges wv', wv'' in $E(H)$(See Fig. 1). Note that G is a simple graph. If v'(or v'') $\in V(P)$, then the length of the path $P_1(w - v')$ (or $P_1(w - v'')) \subset P$ greater than 1. Thus, the graph $H_1 = H - wv' + P_1(w - v')$ is also a spanning eulerian subgraph of G, and $|E(H_1)| > |E(H)|$, contrary to the assumption of H. Therefore, $|E(H)| \ge 2n$.

Let $E'' = \{wv', wv'' | w \in V(P), wv', wv'' \in E(H)\}$ and $E' = E(G) - (E'' \cup E(P))$

Thus, we have that

$$|E(H)| = 2n + |E'|, |E(G)| = 2n + |E'| + n - 1,$$

$$\frac{|E(H)|}{|E(G)|} = \frac{2n + |E'|}{2n + n - 1 + |E'|} \geq \frac{2n}{3n - 1} > \frac{2}{3}$$

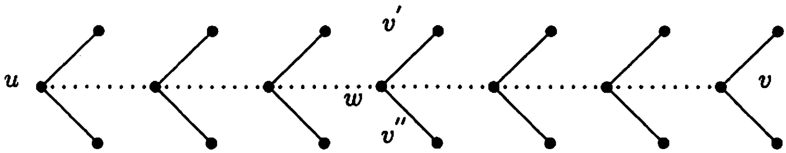


Figure 1, The path $P(u - v)$, and the heavy edges belong to $E(H)$.

The case of $k = 2$:

Suppose that the u_1, v_1, u_2, v_2 are the vertices of odd-degree of G. Without loss of generality, we may assume that nontrivial components of the graph $G - E(H)$ are two paths or two paths with a vertex in common.

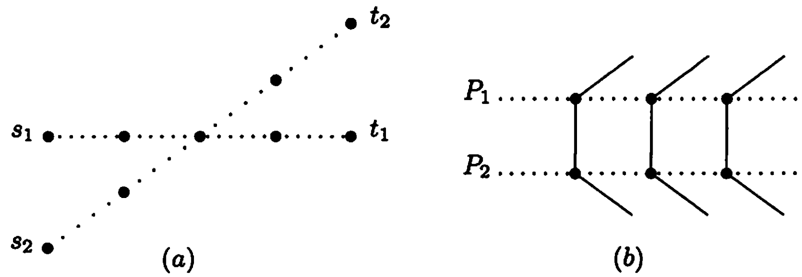
Case 2.1. Suppose that $G - E(H)$ has a nontrivial connected component T only.

Since T is a tree and $O(T) = \{s_1, t_1, s_2, t_2\}, T = P_1 \cup P_2$, P_i is a path from s_i to $t_i, i = 1, 2$, and the paths P_1, P_2 have one vertex in common only. (Figure 2,(a))

Define $E'' = \{wv', wv'' | w \in V(T), wv', wv'' \in E(H)\}$.

Let $E' = E(G) - (E'' \cup E(T))$.

Analogously, for each vertex in $V(P)$, there exist at least two edges $wv', wv'' \in E(H)$. Since G is simple, if there exist distinct $w_1, w_2 \in V(T)$ and w_1w_2 is an edge in $E(H)$, it cannot be an edge in T. Therefore, the distance of w_1 and w_2 in T is at least 2. Let P_3 denote the only path in T connecting w_1 and w_2. Then $H - w_1w_2 + E(P_3)$ is a spanning eulerian subgraph of G with more edge than H, contrary to the choice of H. Thus for any distinct $w_1, w_2 \in V(T)$, $\{w_1v_1', w_1v_1''\} \cap \{w_2v_2', w_2v_2''\} = \emptyset$.



(a) (b)

Figure 2. the heavy edges belong to $E(H)$.

Suppose that $|V(P_1)| = n_1$, $|V(P_2)| = n_2$. By a similar argument to the case $k = 1$, we have the following:

$$\frac{|E(H)|}{|E(G)|} = \frac{2(n_1+n_2-1)+|E'|}{2(n_1+n_2-1)+n_1-1+n_2-1+|E'|} \geq \frac{2(n_1+n_2)-2}{3(n_1+n_2)-4} > \frac{2}{3}$$

.

Case 2.2. Suppose that $G - E(H)$ has two nontrivial connected components P_1, P_2, where P_i is a path from s_i to t_i, $i = 1, 2$.(See Fig. 2,(b))

Set $E_1 = \{uv|uv \in E(H), u \in V(P_1), v \in V(P_2)\}$ and $|E_1| = y$. Since H is a spanning eulerian subgraph of G, for each vertex $w \in V(P_1) \cup V(P_2)$, there exist at least two edges $wv', wv'' \in E(H)$. But each edge $e \in E_1$ counts twice, therefore $|E(H)| \geq 2(n_1+n_2) - y$, using an analogous argument in the case $k = 1$, we obtain

$$\frac{|E(H)|}{|E(G)|} \geq \frac{2(n_1+n_2)-y}{2(n_1+n_2)-y+n_1-1+n_2-1} \geq \frac{2(n_1+n_2)-y}{3(n_1+n_2)-y-2}.$$

If $y \leq 4$,then

$$\frac{2(n_1+n_2)-y}{3(n_1+n_2)-y-2} \geq \frac{2}{3}$$

In the following, we assume that $|E_1| = y \geq 5$.

The subgraph of G induced by E_1 is denoted by $G[E_1]$ and $G[E_1]$ is a bipartite graph with bipartition (X, Y). Let $K = G[E_1] = (X, Y; E_1)$, where $X \subset V(P_1), Y \subset V(P_2)$.

If $|X| \leq 2$ and $|Y| \leq 2$, then $|E_1| \leq |E(K_{2,2})| = 4$, contrary to the assumption that $|E_1| = y \geq 5$. Therefore, $|X| + |Y| \geq 5$.

Case 2.2.1. If $y = 5$ and $|X| + |Y| = 5$,then $|X| = 2$ and $|Y| = 3$ (or $|X| = 3$ and $|Y| = 2$). Suppose $v_1, v_3 \in V(P_1)$, and $v_2, v_4, v_5 \in V(P_2)$, then there are two cases only:

(1) v_1 is incident with two edges in E_1 and v_3 is incident with three edges in E_1(See Fig. 3(a)).

(2) v_3 is incident with two edges in E_1 and v_1 is incident with three edges in E_1(See Fig.3(b)).

Without loss of generality, we may assume that $v_1v_3 \in E(P_1)$, $v_2v_4, v_4v_5 \in E(P_2)$ in Figure3(a),(b). In Figure3(a), the edge set $\{v_1v_2, v_1v_4, v_2v_3, v_3v_4, v_3v_5\} \subset E(H)$ and the edge set $\{v_1v_3, v_2v_4, v_4v_5\}$ is not in $E(H)$.

Define $H_1 = H - \{v_1v_2, v_3v_5\} + \{v_1v_3, v_2v_4, v_4v_5\}$.

Since the cycles $(v_1v_2v_3v_1)$ and $(v_3v_4v_5v_3) \subset H \cup H_1$, it follows that H_1 is a connected graph. Since the edge set $\{v_1v_3, v_2v_4, v_4v_5\}$ is not in $E(H)$ and the edge set $\{v_1v_2, v_3v_5\} \subset E(H)$, it follows that H_1 is also a spanning eulerian subgraph of G. But $|E(H_1)| > |E(H)|$, a contradiction.

In Figure 3(b), define $H_1 = H - \{v_1v_5, v_2v_3\} + \{v_1v_3, v_2v_4, v_4v_5\}$.

By the same reason, H_1 is also a spanning eulerian subgraph of G, but $|E(H_1)| > |E(H)|$.

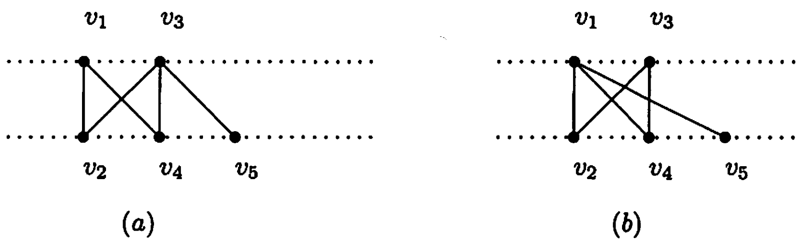


Figure 3, the heavy edges belong to $E(H)$.

Case 2.2.2. Let $y = 5$, and $|X| \geq 4$ (or$|Y| \geq 4$)(See Fig. 4(a))

Let $P_1[v_i, v_j]$ (respectively, $P_2[u_i, u_j]$) denote the only (v_i, v_j)-path in P_1(respectively, the only (u_i, u_j)-path in P_2). Since $|X| \geq 4$, the length of the path $P_1[v_i, v_j] \geq 3$. Hence in Figure 4(a), $H - \{v_iu_r, v_ju_s\} + P_1[v_i, v_j] + P_2[u_r, u_s]$ is a spanning eulerian subgraph with more edges than H, contrary to the choice of H(See Fig. 4(a)).

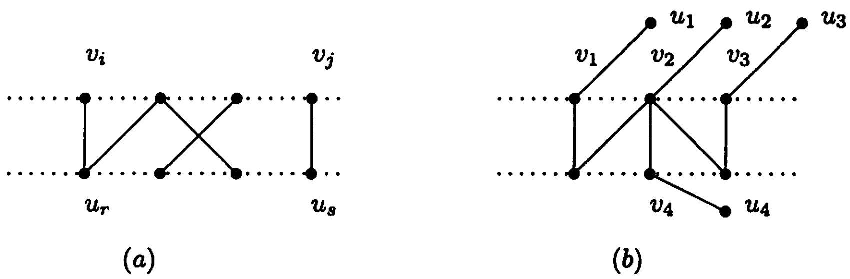


Figure 4, the heavy edges belong to $E(H)$.

Case 2.2.3. Let $y \geq 5$, and $|X| = |Y| = 3$.Note that $K = G[E_1] = (X, Y; E_1)$, where $X \subseteq V(P_1), Y \subseteq V(P_2)$.

If $|O(K)| \geq 4$, we assume that $v_1, v_2, v_3, v_4 \in O(K)$. Since H is a spanning eulerin subgraph, there exist at least four edges $v_1u_1, v_2u_2, v_3u_3, v_4u_4$ are in $E(H) - E(K)$, where u_i is not in $V(P_1 \cup V(P_2)$.(See Fig. 4(b))

Let $E'' = \{wv', wv'' | w \in (V(P_1) - X) \cup (V(P_2) - Y), wv', wv'' \in E(H)\}$ and let $E' = E(G) - E'' - E(P_1) - E(P_2) - E_1 - \{v_1u_1, v_2u_2, v_3u_3, v_4u_4\}$. Thus,

$$\frac{|E(H)|}{|E(G)|} = \frac{2(n_1 + n_2 - 6) + y + 4 + |E'|}{2(n_1 + n_2 - 6) + y + 4 + n_1 - 1 + n_2 - 1 + |E'|}$$

$$\geq \frac{2(n_1 + n_2) + y - 8}{3(n_1 + n_2) + y - 10}.$$

When $y \geq 4$,

$$\frac{2(n_1 + n_2) + y - 8}{3(n_1 + n_2) + y - 10} \geq \frac{2}{3}.$$

Hence,

$$\frac{|E(H)|}{|E(G)|} \geq \frac{2}{3}$$

If $|O(K)| \leq 2$, we would obtain a new spanning eulerian subgraph H_1 with $|E(H_1)| > |E(H)|$, contrary to the assumption of H. We distinguish two cases to show the claim. We may assume that v_i, v_j, v_k are in X and u_r, u_s, u_t are in Y .

(i) $|O(K)| = 0$. Note that $d_K(v) \leq 3$ and $y \geq 5$, we have for any $v \in V(K), d_K(v) = 2$.Therefore, $K = C_6$.(See Fig.5(a))

(ii) $|O(K)| = 2$. Note again that $d_K(v) \leq 3$ and $|X| = |Y| = 3$. Therefore, there are two even-degree vertices in X, and there are two even-degree vertices in Y. Hence the subgraph K is one of two graphs in Fig.5(b),(c).

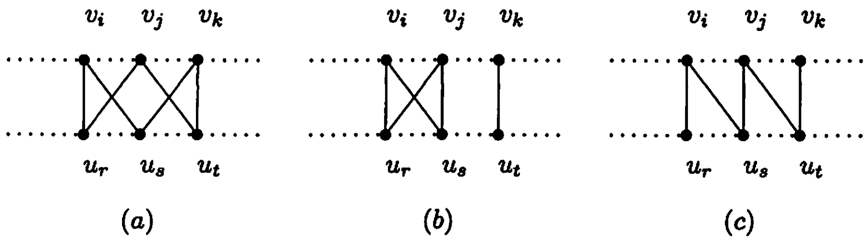


Figure 5, the heavy edges belong to $E(H)$.

In every case, $H - \{v_iu_r, v_ku_t\} + P_1[v_i, v_k] + P_2[u_r, u_t]$ is a spanning eulerian subgraph with more edges than H, contrary to the choice of H. This complete the proof of the theorem.

Remark The bound $L \geq \frac{2}{3}$ in theorem is best possible. Suppose that H is a spanning eulerian subgraph of G with maximum number of edges.

For the case of $k = 2$, Let $G = K_4$,then

$$\frac{|E(H)|}{|E(G)|} = \frac{2}{3}.$$

For the case of $k = 1$, Let G be the graph in Fig.6, then

$$\frac{|E(H)|}{|E(G)|} = \frac{2n}{3n-1} \; and \; \lim_{n \to \infty} \frac{2n}{3n-1} = \frac{2}{3}.$$

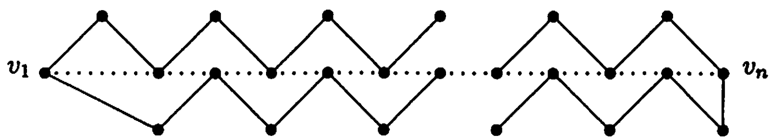


Figure 6, The path $P(v_1 - v_n)$, and the heavy edges belong to $E(H)$.

Therefore, the results in Theorem cannot be improved.

References

[1] J.A.Bondy and U.S.R.Murty, Graph Theory with Applications,American Esevier, New York,1976.

[2] Zhi-Hong Chen and Hong-Jian Lai, Reduction techniques for supereulerian graphs and related topics- An update, Combinatorics and Graph Theory 95' Vol. World Scientific, Singapore, (1995)53-69.

[3] Dengxin Li, Deying Li and Jingzhong Mao, On Maximum number of edges in a spaninnig eulerian subgraph, Discrete Mathematics 274 (2004)299-302.

[4] P.A.Catlin, A Reduction method to find spanning eulerian subgraphs, J.Graph Theory 12 (1988) 29-45.