Restricted Vertex Connectivity of Harary Graphs *

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Abstract A vertex cut that separates the connected graph into components such that every vertex in these components has at least g neighbors is an R^g-vertex-cut. R^g-vertex-connectivity, denote by $\kappa^g(G)$, is the cardinality of a minimum R^g-vertex-cut of G. In this paper, we will determine κ^g and characterize the R^g-vertex-atom-part for the first and second type Harary graphs.

Keywords: R^g-vertex connectivity; Harary graph; R^g-vertex-atom-part

1 Introduction

A network can be modelled as a graph $G = (V, E)$. A classic measure of network reliability is the vertex connectivity $\kappa(G)$. In general, the larger $\kappa(G)$ is, the more reliable the network is. It is well known that $\kappa(G) \leq \lambda(G) \leq \delta(G)$, where $\lambda(G)$ is the edge connectivity, and $\delta(G)$ is the minimum degree of G. Hence a graph G is called *maximally edge connected* or *λ-optimal* if $\lambda(G) = \delta(G)$ and *maximally vertex connected* if $\kappa(G) = \delta(G)$. However, $\kappa(G)$ is a worst case measure and thus underestimates the resilience of the network [11]. To overcome such shortcoming, Harary [6] introduced the concept of *conditional connectivity* by putting some requirements on the connected components. The *R^g-vertex-connectivity* and *g-extraconnectivity* are in this trend.

A subset $F \subset V(G)$ is called an *R^g-vertex-set* of G if each vertex $\nu \in V(G) - F$ has at least g neighbors in $G - F$. An *R^g-vertex-cut* of a connected graph G is an *R^g-vertex-set* F such that $G - F$ is disconnected. The *R^g-vertex-connectivity* of G, denoted by $\kappa^g(G)$, is the cardinality of

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a minimum R^g*-vertex-cut* of G. The idea behind this concept is that the probability that the failures concentrate around a vertex is small. For example, suppose G is a graph of order n which has t vertices of minimum degree g . If there are g faulty vertices in G, then the probability that these g vertices are exactly the neighbor set of some vertex is $t/\binom{n}{g}$, which is very small when n is large. While in the definition of R^g*-vertex-set*, the requirement that there are at least g good neighbors around each vertex takes such resilience into account.

The set of vertices adjacent to a vertex v is called the neighborhood of v and denoted by $N(v)$. A vertex in the neighborhood of v is a neighbor of v. For a subset $S \subset V(G)$, $N(S)$ denotes the vertex set in which every vertex has at least one neighbor in S. The degree of a vertex v is $d(v) = |N(v)|$ and the *minimum degree* $\delta = \delta(G)$ (respectively, *maximum degree* $\Delta = \Delta(G)$) of G is the minimum degree (respectively, maximum degree) over all vertices of G. If $S \subset V(G)$, then $G[S]$ stands for the *subgraph induced* by S.

Let $S(G) = \{T: |T| = \kappa^g(G),\ T$ is an R^g*-vertex-cut* of G, $\kappa^g(G)$ is R^g*-vertex connectivity* of $G\}$. For some $C \in S(G)$, if P is one of the components of $G[V(G) - C]$, then P is called an R^g*-vertex-part* related with C. If an R^g*-vertex-part* P has the property $|V(P)| = \min\{\min\{|V(H)|$: H is an R^g*-vertex-part* relative with C $\}$; $C \in S(G)\}$, then the R^g*-vertex-part* P is called an R^g*-vertex-atom-part*.

Harary graphs play an important role in optimal designing of networks since they are most reliable in some sense [3, 4, 12]. A Harary graph $H_{n,d}$ has vertex set $\{0, 1, ..., n-1\}$. According to the parities of n and d, there are three types of Harary graphs. In the following, additions are all taken module n.

Type 1. When d is even, suppose $d = 2k$. Two vertices i and j of $H_{n,2k}$ are adjacent if and only if $|i - j| \leq k$.

Type 2. When d is odd and n is even, suppose $d = 2k+1$. Then $H_{n,d}$ of the second type is obtained from $H_{n,2k}$ by adding edges $\{(i, i + \frac{n}{2}) : i = 0, 1, \ldots, \frac{n}{2} - 1\}$.

Type 3. When d and n are both odd, suppose $d = 2k+1$. Then $H_{n,d}$ of the third type is obtained from $H_{n,2k}$ by adding edges $\{(i, i + (n+1)/2) : i = 0, 1, \ldots, (n-3)/2\} \cup \{(0, (n-1)/2)\}$.

2 R^g-vertex Connectivity of the First Type Harary Graphs

It is well known that the Type1 Harary graph has both vertex and edge connectivity $\delta = 2k$ [9].

Lemma 2.1. *Let $G = H_{n,2k}$ be a Harary graph of the first type. Then for any nonnegative integer g, G has an R^g-vertex-cut if and only if $g \le k$ and $n \ge 2k + 2(g+1)$.*

Proof. Let $P = G[\{i, i+1, \cdots, i+g\}]$ for some $i \in \{0, 1, 2, \cdots, n-1\}$. Since $g \le k$, we see that every vertex in P has at least g neighbors. As $N(P) = \{i-1, i-2, \cdots, i-k, (i+g)+1, (i+g)+2, \cdots, (i+g)+k\}$, $G[V(G) - N(P)]$ is disconnected. Since $n \ge 2k + 2(g+1)$, we have $|V(G) - N(P) - V(P)| = n - 2k - (g+1) \ge 2k + 2(g+1) - 2k - (g+1) \ge g+1$. This implies that $G[V(G) - V(P) - N(P)]$ has at least *g+1* vertices and every vertex in it has at least g neighbors. Thus $N(P)$ is an R^g-vertex-cut.

We prove the converse by way of contradiction.
Case 1. If $g > k$, then G dose not contain any R^g-vertex-cut. Otherwise, assume C is an R^g-vertex-cut of G and H is an R^g-vertex-part related with C. Suppose $H = G[\{n_1, n_2, \cdots, n_l\}]$, $n_i \equiv n_{i-1} + s \pmod{|V(G)|}$ for some $i \in \{2, \cdots, l\}$ and $s \in \{1, 2, \cdots, k\}$. It is easy to see that the vertices n_1 and n_l have at most k neighbors in H, a contradiction.

Case 2. $n < 2k + 2(g+1)$. Assume C is an R^g-vertex-cut of G. Since $\kappa(G) = 2k$, we have $|C| \ge \kappa(G) = 2k$ and $|V(G) - C| < 2k + 2(g+1) - 2k = 2(g+1)$. As $G - C$ has at least two components, this implies that there is a component of $G - C$ which has at most g vertices. This is impossible. □

Lemma 2.2. *Let $G = H_{n,2k}$ be a Harary graph of the first type. Let g be a nonnegative integer with $g \le k$ and $n \ge 2k + 2(g+1)$, and let $S \subset V(G)$ be a minimum R^g-vertex-cut. Then every component of $G - S$ is the subgraph induced by some consecutive vertices.*

Proof. By contradiction. Assume there is a component P of $G - S$ such that its vertex order is not contiguous. Decompose P into t maximal contiguous parts, say $P_1, P_2, \cdots, P_t$ such that $G[P_i \cup P_{i+1}]$ is connected and the gaps between P_i and P_{i+1} are denoted by g_i for all $1 \le i \le t-1$. Let $P' = V(P) \cup \{g_1, \cdots, g_t\}$. Then $|N(P)| > |N(P')|$ and $V(G) - P' - N(P') = V(G) - P - N(P)$. This means that $N(P')$ is a smaller R^g-vertex-cut, a contradiction. □

Lemma 2.3. *Let $G = H_{n,2k}$ be a Harary graph of the first type. Let g be a nonnegative integer with $g \leq k$ and $n \geq 2k + 2(g+1)$, and let $S \subset V(G)$ be a minimal R^g-vertex-cut of G. Then $G - S$ has exactly two components.*

Proof. Let $P_1, P_2, \cdots, P_t$, $t \geq 3$ be the components of $G - S$ such that $|V(P_i)| \geq g + 1$ and every vertex in them has at least g neighbors. By Lemma 2.2, P_i has contiguous vertex order for all $1 \leq i \leq t$. Since $\delta(G[P_1]) \geq g$ and $\delta(G[V(G) - P_1 - N(P_1)]) \geq g$, $N(P_1)$ is an R^g-vertex-cut of G. Then $|N(P_1)| < S$, a contradiction. □

Theorem 2.4. *Let $G = H_{n,2k}$ be a Harary graph of the first type. Then, for any nonnegative integer g with $g \leq k$ and $n \geq 2k + 2(g+1)$, $\kappa^g = 2k$ and each R^g-vertex-atom-part is isomorphic to the clique induced by the vertex set $\{i, i+1, \cdots, i+g\}$ for some $i \in \{0, 1, 2, \cdots, n-1\}$.*

Proof. By Lemma 2.1, if $g \leq k$ and $n \geq 2k + 2(g+1)$, then G has an R^g-vertex-cut. Let $P = G[\{i, i+1, \cdots, (i+g)\}]$ for some $i \in \{0, 1, 2, \cdots, n-1\}$. By a similar argument as the proof of Lemma 2.1, we have that $N(P)$ is an R^g-vertex-cut of G with $|N(P)| = 2k$, which implies that $\kappa^g(G) \leq 2k$. Since $\kappa^g(G) \geq \kappa(G) = 2k$, it follows that $\kappa^g(G) = 2k$.
Obviously, an R^g-vertex-atom-part has at least $g + 1$ vertices and contiguous vertex order by Lemma 2.2. As P is an R^g-vertex-part related with $N(P)$ with $|V(P)| = g + 1$, thus every R^g-vertex-atom-part has exactly $g + 1$ vertices and is isomorphic to $G[\{i, i+1, \cdots, i+g\}]$ for some $i \in \{0, 1, 2, \cdots, n-1\}$. □

3 $R^g(G)$-vertex-cut of Harary Graph of the Second Type

3.1 $\kappa^0(G)$ of Harary Graph of the Second Type

The following theorem can be found in Harary [7].

Theorem 3.1. *$\kappa(H_{n,d}) = d$, and hence the minimum number of edges in a k-connected graph on n vertices is $\lceil \frac{kn}{2} \rceil$.*

From the Theorem 3.1, it is easy to see when $g = 0$, the second type Harary graphs have $\kappa^0(G) = d$.

3.2 $\kappa^1(G)$ of Harary Graph of the Second Type

It is easy to see that $\kappa^1(H_{8,3}) = 4$ and $H_{10,5}$ has no R^1-vertex-cut. So in the following, we assume that all the second type Harary graphs are not isomorphic to $H_{8,3}$ and $H_{10,5}$.

Lemma 3.2. *Let $G = H_{n,d}$ be a Harary graph of the second type and $G \ncong H_{10,5}$, $H_{8,3}$. Then there is an R^1-vertex-cut of G if and only if $n - d \geq 5$.*

Proof. Let $e = \{i, i+1\}$ for some $i \in \{0, 1, 2, \cdots, n-1\}$. As $N(e) = \{i-1, i-2, \cdots, i - \frac{d-1}{2}\} \cup \{(i+1)+1, (i+1)+2, \cdots, (i+1) + \frac{d-1}{2}\} \cup \{i + \frac{n}{2}, (i+1) + \frac{n}{2}\}$, and $|N(e)| = \frac{d-1}{2} + \frac{d-1}{2} + 2 = d+1$, $G - N(e)$ is disconnected. Since $n - d \geq 5$, then $|V(G - N(e) - V(e))| = n - (d+1) - 2 = n - d - 3 \geq 2$.

Case 1. $n - d > 5$. From above, we have $|V(G - N(e) - V(e))| = n - (d+1) - 2 = n - d - 3 > 2$. Since n is even, d is odd, $d + 1$ is even, then $n - (d+1) - 2$ is even, thus $|V(G - N(e) - V(e))| \geq 4$. By the construction of the second type Harary graphs, there must be $\frac{|V(G-N(e)-V(e))|}{2}$ vertices between $(i+1) + \frac{n}{2}$ and i, and $\frac{|V(G-N(e)-V(e))|}{2}$ vertices between $i + 1$ and $i + \frac{n}{2}$. Clearly, all $\frac{|V(G-N(e)-V(e))|}{2}$ vertices are connected.

Case 2. $n - d = 5$. Since $G \ncong H_{10,5}$, $H_{8,3}$, we have $d \geq 7$ $n \geq 12$, and $|V(G - N(e) - V(e))| = n - (d+1) - 2 = n - d - 3 = 2$. There must be one vertex labeled larger than $i + \frac{n}{2}$ and one vertex smaller than $i + \frac{n}{2} - 1$ (since $d \geq 7$). It follows that there is one edge between the two vertices, this implies that $G - N(e) - V(e)$ is a connected component.

We prove the converse by way of contradiction. As n is even, d is odd. If $n - d < 5$, then $n - d = 3$ or $n - d = 1$.

Case 1. $n - d = 1$. Then $G = H_{n,d}$ is a complete graph, it follows that G has no R^1-vertex-cut, which is a contradiction.

Case 2. $n - d = 3$. From the condition of the Lemma we know that G has an R^1-vertex-cut. Assume S is an R^1-vertex-cut of G, then $G - S$ has at least two components. Because every vertex in each component has at least one neighbor, each component has at least two vertices. By Lemma 3.1, we have $|S| \geq d$. We thus have $n - d \geq n - |S| \geq 2(g+1) \geq 4$, a contradiction. □

Theorem 3.3. *Let $G = H_{n,d}$ be a Harary graph of the second type and $G \ncong H_{10,5}$, $H_{8,3}$. If $n-d \geq 5$, then $\kappa^1(G) = d+1$ and each R^1-vertex-atom-part of G is isomorphic to a K_2 induced by the vertex set $\{i, i+1\}$ or $\{i, i+\frac{n}{2}\}$ for some $i \in \{0, 1, 2, \cdots, n-1\}$.*

Proof. By Lemma 3.2, if $n-d \geq 5$, then G has an R^1-vertex-cut, and by the proof of Lemma 3.2, we know that $N(e_1)$ is an R^1-vertex-cut and $|N(e_1)| = d+1$, where $e_1 = (i, i+1)$. Let S be an R^1-vertex-cut of G with S is minimal, and let P be the smallest component of $G-S$. Then we will prove $|N(P)| \geq |N(e_1)|$, so that $\kappa^1(G) = |N(e_1)| = d+1$.

Case 1. $P \cong e_j$, $e_j = (i, j)$, $2 \leq j < \frac{n}{2}$. As $S = N(e_j) = \{i-1, i-2, \cdots, i-\frac{d-1}{2}\} \cup \{i+j+1, i+j+2, \cdots, i+j+\frac{d-1}{2}\} \cup \{i+1, i+2, \cdots, i+j-1\} \cup \{i+\frac{n}{2}, i+j+\frac{n}{2}\}$, we have $|S| = |N(e_j)| = \frac{d-1}{2} + \frac{d-1}{2} + j - 1 + 2 + 2 = d + j \geq d+2$. It implies that $|N(e_j)| > |N(e_1)|$, a contradiction.

Case 2. $P \cong e_{\frac{n}{2}}$, $e_{\frac{n}{2}} = (i, \frac{n}{2})$. As $S = N(e_{\frac{n}{2}}) = \{i-1, i-2, \cdots, i-\frac{d-1}{2}\} \cup \{i+1, i+2, \cdots, i+\frac{d-1}{2}\} \cup \{(i+\frac{n}{2})-1, (i+\frac{n}{2})-2, \cdots, (i+\frac{n}{2})-\frac{d-1}{2}\} \cup \{(i+\frac{n}{2})+1, (i+\frac{n}{2})+2, \cdots, (i+\frac{n}{2})+\frac{d-1}{2}\}$, we have $|S| = |N(e_{\frac{n}{2}})| = 4 \times \frac{d-1}{2} = 2(d-1)$. Since $|N(e_1)| = d+1$, then $|N(e_{\frac{n}{2}})| - |N(e_1)| = 2(d-1) - (d+1) = d-3 \geq 0$ and with equality holds when $d = 3$. For any $d \geq 5$, $|N(e_{\frac{n}{2}})| - |N(e_1)| \geq 2 > 0$, contradicting the fact that S is minimal.

Case 3. $P \cong G[\{n_1, n_2, \cdots, n_l\} \cup \{m_1, m_2, \cdots, m_j\}]$, $|V(P)| \geq 3$, where $l \geq 2$, $1 < s \leq l$, $n_s - n_{s-1} \leq \frac{d-1}{2}$, $\{m_1, m_2, \cdots, m_j\} \subseteq \{n_1 + \frac{n}{2}, n_1 + 1 + \frac{n}{2}, n_1 + 2 + \frac{n}{2} \cdots, n_l + \frac{n}{2}\}$, $0 \leq j \leq l$.

Subcase 1. $j > 0$. As $S = N(P) \supseteq \{n_1 - 1, n_1 - 2, \cdots, n_1 - \frac{d-1}{2}\} \cup \{n_l + 1, n_l + 2, \cdots, n_l + \frac{d-1}{2}\} \cup \{(m_1 - 1, m_1 - 2, \cdots, m_1 - \frac{d-1}{2}\} \cup \{m_j + 1, m_j + 2, \cdots, m_j + \frac{d-1}{2}\}$, we have $|S| = |N(P)| \geq 4 \times \frac{d-1}{2} = 2(d-1)$. Then $|N(P)| - |N(e_1)| = 2(d-1) - (d+1) = d-3 \geq 0$ and with equality holds when $d = 3$. For any $d \geq 5$, $|N(P)| - |N(e_1)| \geq 2 > 0$, a contradiction.

Subcase 2. $j = 0$, then $l \geq 3$. As $S = N(P) = \{n_1 - 1, n_1 - 2, \cdots, n_1 - \frac{d-1}{2}\} \cup \{n_l + 1, n_l + 2, \cdots, n_l + \frac{d-1}{2}\} \cup \{n_1 + \frac{n}{2}, n_2 + \frac{n}{2}, \cdots, n_l + \frac{n}{2}\} \cup \{\{n_1, n_1 + 1, n_1 + 2, \cdots, n_l\} - \{n_1, n_2, \cdots, n_l\}\}$, we have $|S| = |N(P)| = \frac{d-1}{2} + \frac{d-1}{2} + l = d - 1 + l > d+1$. Then $|N(P)| - |N(e_1)| > (d+1) - (d+1) = 0$, a contradiction.

From above, we know that $N(e_1)$ is an R^1-vertex-cut and for any smallest component P, $|N(P)| \geq |N(e_1)|$, and only when $d = 3$, we have $|N(e_{\frac{n}{2}})| = |N(e_1)|$, that is to say when $d > 3$, the R^1-vertex-atom-part of G is isomorphic to a K_2 induced by the vertex set $\{i, i+1\}$ for some $i \in \{0, 1, 2, \cdots, n-1\}$, and when $d = 3$, the R^1-vertex-atom-part of G is isomorphic to a K_2 induced by the vertex set $\{i, i+\frac{n}{2}\}$ or $\{i, i+1\}$ for some $i \in \{0, 1, 2, \cdots, n-1\}$.

□

3.3 $\kappa^g(G)$ of Harary Graph of the Second Type for $2 \le g \le \frac{d-1}{2}$

Lemma 3.4. *Let $G = H_{n,d}$ be a Harary graph of the second type. For any integer g with $2 \le g \le \frac{d-1}{2}$, and $n \ge 3d+1$, G has an R^g-vertex-cut.*

Proof. Let $P_1 = G[i, i+1, \cdots, i+g]$ for some $i \in \{0, 1, 2, \cdots, n-1\}$. Then $|V(P_1)| = g+1$. Clearly P_1 is connected and every vertex in P_1 has g neighbors in P_1. As $N(P_1) = \{i-1, i-2, \cdots, i-\frac{d-1}{2}\} \cup \{(i+g)+1, (i+g)+2, \cdots, (i+g)+\frac{d-1}{2}\} \cup \{i+\frac{n}{2}, (i+1)+\frac{n}{2}, \cdots, (i+g)+\frac{n}{2}\}$, we have $|N(P_1)| = \frac{d-1}{2} + \frac{d-1}{2} + (g+1) = d+g$. Since $n \ge 3d+1$, then $|V(G) - N(P_1) - V(P_1)| = n-(g+1)-(d+g) = n-g-1-d-g = n-d-2g-1 \ge 3d+1-d-2g-1 = 2d-2g \ge 2(2g+1)-2g = 4g+2-2g = 2g+2 = 2(g+1)$, by the construction of the second type of Harary graphs, there must be $\frac{|V(G)-V(P_1)-N(P_1)|}{2}$ vertices labeled consecutively between $(i+g)+\frac{d-1}{2}$ and $i+\frac{n}{2}$, and another $\frac{|V(G)-V(P_1)-N(P_1)|}{2}$ vertices labeled consecutively between $(i+g)+\frac{n}{2}$ and $i-\frac{d-1}{2}$. Since $|V(G)-N(P_1)-V(P_1)| \ge 2(g+1)$, then $\frac{|V(G)-V(P_1)-N(P_1)|}{2} \ge g+1$. Let P_2 be the induced subgraph by the vertices set which are labeled consecutively between $(i+g)+\frac{d-1}{2}$ and $i+\frac{n}{2}$, and let P_3 be the induced subgraph by the vertices set which are labeled consecutively between $(i+g)+\frac{n}{2}$ and $i-\frac{d-1}{2}$. Clearly $|V(P_i)| \ge g+1$, every vertex in P_i has g neighbors in P_i $(i = 2, 3)$. Then $N(P_1)$ is an R^g-vertex-cut of G. □

Lemma 3.5. *Let $G = H_{n,d}$ be a Harary graph of the second type. Let g be an integer with $2 \le g \le \frac{d-1}{2}$. If $n \ge 3d+1$, and S be a minimal R^g-vertex-cut of $G-S$, then the smallest component of $G-S$ must be the subgraph induced by the vertex set $\{i, i+1, \cdots, i+g\}$ for some $i \in \{0, 1, 2, \cdots, n-1\}$.*

Proof. Let P be the smallest component of $G-S$. If P does not have consecutive vertex order $\{i, i+1, \cdots, i+g\}$, then P must have the following cases, in each case we will have a contradiction.

$P \cong G[\{n_1, n_2, \cdots, n_l\} \cup \{m_1, m_2, \cdots, m_j\}]$, where $l \ge g$, $1 < s \le l$, $n_s - n_{s-1} \le \frac{d-1}{2}$, $\{m_1, m_2, \cdots, m_j\} \subseteq \{n_1+\frac{n}{2}, n_1+1+\frac{n}{2}, n_1+2+\frac{n}{2} \cdots, n_l+$

$\frac{n}{2}\}$, $j = 0$ or $g \le j \le n_l - n_1 + 1$.

Case 1. $j > 0$. As $S = N(P) \supseteq \{n_1 - 1, n_1 - 2, \cdots, n_1 - \frac{d-1}{2}\} \cup \{n_l + 1, n_l + 2, \cdots, n_l + \frac{d-1}{2}\} \cup \{(m_1 - 1, m_1 - 2, \cdots, m_1 - \frac{d-1}{2}\} \cup \{m_j + 1, m_j + 2, \cdots, m_j + \frac{d-1}{2}\}$, we have $|S| = |N(P)| \ge 4 \times \frac{d-1}{2} = 2(d-1) > d + g$, a contradiction.

Case 2. $j = 0$.

Subcase 1. If there exits $s \in \mathbb{Z}_+$, $1 < s \le l$, such that $2 \le n_s - n_{s-1} \le \frac{d-1}{2}$, then $l \ge g + 1$. Clearly we have $N(P) \supseteq \{n_1 - 1, n_1 - 2, \cdots, n_1 - \frac{d-1}{2}\} \cup \{n_l + 1, n_l + 2, \cdots, n_l + \frac{d-1}{2}\} \cup \{n_1 + \frac{n}{2}, n_2 + \frac{n}{2}, \cdots, n_l + \frac{n}{2}\} \cup \{\{n_1, n_1 + 1, n_1 + 2, \cdots, n_l\} - \{n_1, n_2, \cdots, n_l\}\} \cup \{n_{s-1} + 1, n_{s-1} + 2, \cdots, n_s - 1\}$. Let$P' = G[V(P) \cup \{n_{s-1} + 1, n_{s-1} + 2, \cdots, n_s - 1\}]$, then $N(P') \supseteq \{n_1 - 1, n_1 - 2, \cdots, n_1 - \frac{d-1}{2}\} \cup \{n_l + 1, n_l + 2, \cdots, n_l + \frac{d-1}{2}\} \cup \{n_1 + \frac{n}{2}, n_2 + \frac{n}{2}, \cdots, n_l + \frac{n}{2}\} \cup \{\{n_1, n_1 + 1, n_1 + 2, \cdots, n_l\} - \{n_1, n_2, \cdots, n_l\} - \{n_{s-1} + 1, n_{s-1} + 2, \cdots, n_s - 1\}\}$;
Thus we have $|N(p)| \ge |N(P')| + 1$, and $N(P')$ is an R^g-vertex-cut, a contradiction.

Subcase 2. If there does not exist $s \in \mathbb{Z}_+$, $1 < s \le l$, such that $2 \le n_s - n_{s-1} \le \frac{d-1}{2}$, then $l \ge g+2$. As $S = N(P) = \{n_1 - 1, n_1 - 2, \cdots, n_1 - \frac{d-1}{2}\} \cup \{n_l + 1, n_l + 2, \cdots, n_l + \frac{d-1}{2}\} \cup \{n_1 + \frac{n}{2}, n_2 + \frac{n}{2}, \cdots, n_l + \frac{n}{2}\} \cup \{\{n_1, n_1 + 1, n_1 + 2, \cdots, n_l\} - \{n_1, n_2, \cdots, n_l\}\}$, then $|S| = |N(P)| = \frac{d-1}{2} + \frac{d-1}{2} + l = d - 1 + l = d - 1 + (g + 2) > d + g$, a contradiction. □

Theorem 3.6. *Let $G = H_{n,d}$ be a Harary graph of the second type. Let g be an integer with $2 \le g \le \frac{d-1}{2}$. If $n \ge 3d + 1$, then $\kappa^g(G) = d + g$ and each R^g-vertex-atom-part is isomorphic to a $(g+1)$-clique induced by the vertex set $\{i, i+1, \cdots, i+g\}$ for some $i \in \{0, 1, 2, \cdots, n-1\}$.*

Proof. By Lemma 3.4, if $n \ge 3d + 1$, then G has an R^g-vertex-cut. Let S be a minimal R^g-vertex-cut of G, and let P be the smallest component of $G - S$. By Lemma 3.5, $P \cong G[i, i+1, i+2, \cdots, i+g]$. Since $S = N(P)$, we have $|S| = |N(P)| = 3g + 2k + 1 = 2g + 2k + 1 + g = 2(g + k) + 1 + g = d + g$. Obviously, an R^g-vertex-atom-part has at least $g+1$ vertices and contiguous vertex order by Lemma 3.5. Since P is an R^g-vertex part relative with $N(P)$ with $|N(P)| = g + 1$, thus every R^g-vertex-atom-part has exactly $g + 1$ vertices and is isomorphic to $G[i, i + 1, i + 2, \cdots, i + g]$ for some $i \in \{0, 1, 2, \cdots, n - 1\}$.

□

3.4 $\kappa^g(G)$ of Harary Graph of the Second Type for $g = \frac{d+1}{2}$

Lemma 3.7. *Let $G = H_{n,d}$ be a Harary graph of the second type. Let g be an integer with $g = \frac{d+1}{2} \geq 2$. Then G has an R^g-vertex-cut if $n \geq 8g-4$.*

Proof. Let $P = G[\{i, i+1, \cdots, i+g-1\} \cup \{i+\frac{n}{2}, (i+1)+\frac{n}{2}, \cdots, i+g-1+\frac{n}{2}\}]$. As $N(P) = \{i-1, i-2, \cdots, i-(g-1)\} \cup \{(i+g-1)+1, (i+g-1)+2, \cdots, (i+2(g-1)\} \cup \{(i+\frac{n}{2})-1, (i+\frac{n}{2})-2, \cdots, (i+\frac{n}{2})-(g-1)\} \cup \{(i+g-1+\frac{n}{2})+1, (i+g-1+\frac{n}{2})+2, \cdots, (i+g-1+\frac{n}{2})+(g-1)\}$, we have $|N(P)| = (g-1)+(g-1)+(g-1)+(g-1) = 4(g-1)$. It is easy to see that every vertex in P has g neighbors. Since $n \geq 8g-4$, then $|V(G) - N(P) - V(P)| = n - 2g - 4(g-1) = n - 6g + 4 \geq 8g - 4 - 6g + 4 = 2g$. By the construction of the second type Harary graphs and $d = 2g-1$, there must be $\frac{|V(G)-V(P)-N(P)|}{2}$ vertices labeled consecutively between $i+2(g-1)$ and $(i+\frac{n}{2})-(g-1)$, and another $\frac{|V(G)-V(P)-N(P)|}{2}$ vertices labeled consecutively between $(i+g-1+\frac{n}{2})+(g-1)$ and $i-(g-1)$. Suppose P_1 be the induced subgraph by the vertices set which are labeled consecutively between $i+2(g-1)$ and $(i+\frac{n}{2})-(g-1)$, and P_2 be the induced subgraph by the vertices set which are labeled consecutively between $(i+g-1+\frac{n}{2})+(g-1)$ and $i-(g-1)$, clearly $|V(P_i)| = g$, every vertex in P_i has $g-1$ neighbors in P_i $(i = 1, 2)$ respectively. For any vertex in P, suppose it is labeled j, $i+2(j-1) \leq (i+\frac{n}{2})-(g-1)$, then $i+2(g-1)+\frac{n}{2} \leq j+\frac{n}{2} \leq i-(g-1)$, we have $(j, j+\frac{n}{2})$ is an edge between P_1 and P_2, then $P_1 \cup P_2$ is connected, and every vertex in $P_1 \cup P_2$ has g neighbors. We have proved $N(P)$ is an R^g-vertex-cut of G. □

Lemma 3.8. *Let $G = H_{n,d}$ be a Harary graph of the second type. Let g be an integer with $g \geq 2$ with $d = 2g-1$. If $n \geq 8g-4$, and S be a minimal R^g-vertex-cut, then every component of $G - S$ must be the subgraph induced by the vertex set $\{i, i+1, \cdots, i+j, i+\frac{n}{2}, i+1+\frac{n}{2}, i+j+\frac{n}{2}\}$ for some $i \in \{0, 1, 2, \cdots, n-1\}$, where $j \geq g-1$.*

Proof. The result is clear for $d = 3$. For $d \geq 5$, we prove our result by contradiction. If the Lemma is not true, then there exists a component P which is not induced by the vertex set $\{i, i+1, \cdots, i+j, i+\frac{n}{2}, i+1+\frac{n}{2}, i+j+\frac{n}{2}\}$ for some $i \in \{0, 1, \cdots, n-1\}$. Since every vertex in each component of $G - S$ has exactly g neighbors, then the vertex set of P must be the unit of several copies of $\{i, i+1, \cdots, i+j, i+\frac{n}{2}, i+1+\frac{n}{2}, i+j+\frac{n}{2}\}$. Without loss of generally, we only need to prove the following case: $P = G[\{i, i+1, i+2, \cdots, i+j_1; i+\frac{n}{2}, i+1+\frac{n}{2}, \cdots, i+j_1+\frac{n}{2}\} \cup \{(i+j_1)+m_1, (i+j_1)+m_1+1, \cdots, (i+j_1)+m_1+j_2; (i+j_1)+m_1+\frac{n}{2}, (i+j_1)+m_1+$

$1+\frac{n}{2}, \cdots, (i+j_1)+m_1+j_2+\frac{n}{2}\}]$, where $j_1, j_2 \geq g-1$, $2 \leq m_1 \leq g-1$. In this case, $S = N(P) = \{i-1, i-2, \cdots, i-(g-1)\} \cup \{(i+j_1)+m_1+j_2+1, \cdots, (i+j_1)+m_1+j_2+2, \cdots, (i+j_1)+m_1+j_2+(g-1)\} \cup \{i+\frac{n}{2}-1, i+\frac{n}{2}-2, \cdots, i+\frac{n}{2}-(g-1)\} \cup \{(i+j_1)+m_1+j_2+\frac{n}{2}+1, (i+j_1)+m_1+j_2+\frac{n}{2}+2, \cdots, (i+j_1)+m_1+j_2+\frac{n}{2}+(g-1)\} \cup \{(i+j_1)+1, (i+j_1)+2, \cdots, (i+j_1)+m_1-1\} \cup \{(i+j_1+\frac{n}{2})+1, (i+j_1+\frac{n}{2})+2, \cdots, (i+j_1+m_1\frac{n}{2})-1\}$, and $|S| = |N(P)| \geq 4 \times (g-1) + 2(m_1-1) \geq 4(g-1)+2 > 4(g-1)$, a contradiction. □

From the above two Lemmas, we can obtain a sufficient and necessary condition for the Harary graph $G = H_{n,d}$ having an R^g-vertex-cut when $g = \frac{d+1}{2} \geq 2$.

Corollary 3.9. *Let $G = H_{n,d}$ be a Harary graph of the second type. Let g be an integer with $g = \frac{d+1}{2} \geq 2$. Then there is an R^g-vertex-cut of G if and only if $n \geq 8g-4$.*

Theorem 3.10. *Let $G = H_{n,d}$ be a Harary graph of the second type. Let g be an integer with $g \geq 2$ and $d = 2g-1$. If $n \geq 8g-4$, then $\kappa^g(G) = 4(g-1)$, and the R^g-vertex-atom-part is isomorphic to the subgraph induced by the vertex set $\{i, i+1, \cdots, i+g-1, i+\frac{n}{2}, i+1+\frac{n}{2}, \cdots, i+g-1+\frac{n}{2}\}$ for some $i \in \{0, 1, 2, \cdots, n-1\}$.*

Proof. By Lemma 3.7, when $n \geq 8g-4$, G has an R^g-vertex-cut. Let S be a minimal R^g-vertex-cut of G, and let P is the smallest component of $G-S$. By Lemma 3.8, every component of $G-S$ must be the subgraph induced by the vertex set $\{i, i+1, \cdots, i+j, i+\frac{n}{2}, i+1+\frac{n}{2}, i+j+\frac{n}{2}\}$, where $j \geq g-1$, so does P. Thus we have $S = N(P) = \{i-1, i-2, \cdots, i-(g-1)\} \cup \{i+j+1, i+j+2, \cdots, i+j+(g-1)\} \cup \{(i+\frac{n}{2})-1, (i+\frac{n}{2})-2, \cdots, (i+\frac{n}{2})-(g-1)\} \cup \{i+j+\frac{n}{2}+1, i+j+\frac{n}{2}+2, \cdots, i+j+\frac{n}{2}+(g-1)\}$. It follows that $\kappa^g = |S| = |N(P)| = 4(g-1)$. By Lemma 3.8, the second part is obvious. □

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