Domination and Bondage number of $C_5 \times C_n$*

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Abstract

Let G be finite, simple graph, we denote by $\gamma(G)$ the *domination number* of G. The *bondage number* of G, denoted by $b(G)$, is the minimum number of the edges of G whose removal increase the domination number of G. C_n denote the cycle of n vertices. For $n \geq 5$ and $n \neq 5k+3$, the domination number of $C_5 \times C_n$ was determined in [6]. In this paper, we calculate the domination number of $C_5 \times C_n$ for $n = 5k+3 (k \geq 1)$, and also study the bondage number of this graph, where $C_5 \times C_n$ is the cartesian product of C_5 and C_n.

1 Introduction

The graphs considered here are finite, undirected, and simple (no loops or parallel edges). The set of vertices and edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively. $d_G(x), N_G(x)$ denote the degree, neighborhood of x in G, respectively. $N_G[x] := N_G(x) \cup \{x\}$. For $V' \subseteq V(G)$, $N_G[V'] := \cup_{x \in V'} N_G[x]$. We often omit the index G if it is clear from the context. For $x, y \in V(G)$, by xy we denote the edge joining x, y if they are adjacent. In this case, we also say x dominates y, or y dominates x. A set D of vertices of a graph G is called a dominating set if every vertex

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of $V(G) - D$ is adjacent to at least one element of D, i.e., $N[D] = V(G)$. The domination number of G, denoted by $\gamma(G)$, is the size of its smallest dominating set. The domination set with $\gamma(G)$ vertices is called minimum dominating set of G. Let D be a dominating set of G, if $y \in V(G) - D$ is adjacent to only $x \in D$, then y is called a private neighbor of x with respect to D. The bondage number, denoted by $b(G)$, is the minimum number of the edges of G whose removal increase the domination number, i.e., $b(G) = \min\{|E'| \mid E' \subseteq E(G) \text{ and } \gamma(G - E') > \gamma(G)\}$.

It has been proved that the decision problem corresponding to the domination number and bondage number for arbitrary graphs is NP-complete (see [3], Chapter 9). So, it is natural to turn to calculate these numbers for some special graphs. Let $G \times H$ be the graph, whose vertex set is $V(G) \times V(H)$, for $a, b \in V(G)$ and $x, y \in V(H)$, $(a, x), (b, y)$ are adjacent if and only if $x = y$ and $ab \in E(G)$, or $a = b$ and $xy \in E(H)$. It is called cartesian product of G and H.

In [6], the domination number of cartesian product of cycles were considered, the following result was proved.

Theorem 1 *[6] Let $n \geq 5$. Then,*

$$\gamma(C_5 \times C_n) = \begin{cases} n, & n = 5k; \\ n + 1, & n \in \{5k + 1, 5k + 2, 5k + 4\}, \end{cases}$$

For $k \geq 1$, $\gamma(C_5 \times C_{5k+3}) \leq 5(k + 1)$.

For bondage number, Fink et al. in [2] determined for complete graphs and complete t-partite graphs, the paths and the cycles. By [1] and [2], $b(T) \leq 2$ for any tree T. In [4], Hartnell and Rall gave an upper bound for general graph.

Theorem 2 *[4] If G is a graph, then for every pair u and v of adjacent vertices $b(G) \leq d(u) + d(v) - 1 - |N(u) \cap N(v)|$.*

For the cartesian product of cycles, the following results were obtained in [7] and [5], respectively.

Theorem 3 *[7] Let $n \geq 4$. Then,*

$$b(C_3 \times C_n) = \begin{cases} 2, & n = 4k; \\ 4, & n \in \{4k+1, 4k+2\}; \\ 5, & n = 4k+3 \end{cases}$$

Theorem 4 *[5] For $n \geq 4$, $b(C_4 \times C_n) = 4$.*

In this paper, we study the domination number and bondage number of $C_5 \times C_n$ for $n \geq 5$ (for smaller n the corresponding problem has been solved). We decide the exact value of $\gamma(C_5 \times C_{5k+3})$ for $k \geq 1$, and of $b(C_5 \times C_n)$ for $n \geq 5$ and $n \neq 5k+3$. For $n \geq 5$ and $n = 5k+3$ we present an upper bound for $b(C_5 \times C_n)$.

2 Main Results

In the below we always assume $n \geq 5$. We regard the vertex set of $C_5 \times C_n$ as an array of $5 \times n$ and denote by $V(C_5 \times C_n) = \{x_{ij} \mid i \in \{0, 1, \cdots 4\}, j \in \{0, 1, n-1\}\}$. For $0 \leq i \leq 4$, let H_i denote the vertex set of ith row, whose induced subgraph is a cycle C_n; For $0 \leq j \leq n-1$, let V_j denote the vertex set of jth column, whose induced subgraph is a cycle C_5. We distinguish the edges of $C_5 \times C_n$ to two kinds. An edge is called horizontal edge if it belongs to the cycle C_n induced by H_i $(0 \leq i \leq 4)$, an edge is called vertical edge if it belongs to the cycle C_5 induced by V_j $(0 \leq j \leq n-1)$. For $i = 0, 1, \cdots, 4$, we let

$$\begin{aligned} W_i &= \{x_{ij} \mid j \equiv 0 (mod\ 5)\} \cup \{x_{(i+2)j} \mid j \equiv 1 (mod\ 5)\} \\ &\cup \{x_{(i+4)j} \mid j \equiv 2 (mod\ 5)\} \cup \{x_{(i+1)j} \mid j \equiv 3 (mod\ 5)\} \\ &\cup \{x_{(i+3)j} \mid j \equiv 4 (mod\ 5)\}, \end{aligned}$$

for $i = 5, 6, \cdots, 9$, let

$$\begin{aligned} W_i &= \{x_{ij} \mid j \equiv 0 (mod\ 5)\} \cup \{x_{(i+3)j} \mid j \equiv 1 (mod\ 5)\} \\ &\cup \{x_{(i+1)j} \mid j \equiv 2 (mod\ 5)\} \cup \{x_{(i+4)j} \mid j \equiv 3 (mod\ 5)\} \\ &\cup \{x_{(i+2)j} \mid j \equiv 4 (mod\ 5)\}, \end{aligned}$$

where the first index of x takes modulo 5 and $0 \leq j \leq n-1$. We note that $W_0, W_1, \cdots, W_4$ are pairwise disjoint and their union is $V(C_5 \times C_n)$, and $W_5, \cdots, W_9$ have the same property.

Theorem 5 *For $k \geq 1$, $\gamma(C_5 \times C_{5k+3}) = 5(k+1)$.*

Proof. Let $k \geq 1$ and $n = 5k+3$. Clearly, $C_5 \times C_n$ is 4-regular and vertex-transitive. Here we will show $\gamma(C_5 \times C_{5k+3}) \geq 5(k+1) = n+2$, combining with Theorem 1 we have $\gamma(C_5 \times C_{5k+3}) = n+2$. By contradiction. Suppose, to the contrary, $\gamma(C_5 \times C_{5k+3}) \leq n+1$, we deduce a contradiction. For convenience, let $G := C_5 \times C_{5k+3}$. From the structure of G we have the following easy assertion.

Assertion 1 *Let D be a dominating set of G. If $D \cap V_j = \emptyset$, then $|D \cap (V_{j-1} \cup V_{j+1})| \geq 5$, where $0 \leq j \leq n-1$, and $j-1, j+1$ take modulo n.*

Now, let D be a dominating set of G with $n+1$ vertices (by our assumption, G has such a dominating set). We first check the distribution of D in the columns of G.

Case 1. G has one column, saying V_0, such that $|V_0 \cap D| = 5$. Let $D_1 := V_0 \cap D$ and $D_2 := D - D_1$. Then, $|N[D_1]| = 15$. As $k \geq 1$, $5n - 15 > 0$. So, there are at least $5n - 15 > 0$ vertices of G which are not dominated by any vertex in D_1. Since $|D_2| = (n+1) - 5$ and G is 4-regular, $|N[D_2]| \leq 5n - 20$. Thus, $|N[D]| \leq |N[D_1]| + |N[D_2]| \leq 5n - 5 < |V(G)|$, implying that D is not a dominating set of G, a contradiction.

Case 2. G has one column, saying V_0, such that $|V_0 \cap D| = 4$. Similarly, Let $D_1 := V_0 \cap D$, $D_2 := D - D_1$. Then, $|N[D_1]| \leq 13$ and $|N[D_2]| \leq 5(n+1-4) = 5n-15$, and hence, $|N[D]| \leq 5n-2 < |V(G)|$, still contradict the fact that D is not a dominating set of G.

Case 3. G has two columns V_{i_1}, V_{i_2} such that each of them contains 3 vertices of D. Let $D_1 := (V_{i_1} \cup V_{i_2}) \cap D$ and $D_2 := D - D_1$. Then, $|N[D_1]| \leq 2(5+6) = 22$. Since $|D_2| = (n+1) - 6$, $|N[D_2]| \leq 5n - 25$. Thus, $|N[D]| \leq 5n - 3 < |V(G)|$, also a contradiction.

Case 4. G has one column V_{i_1} which contains 3 vertices of D, and two columns V_{i_2}, V_{i_3} such that each of them contains 2 vertices of D. Let $D_1 := (\cup_{h=1}^{3} V_{i_h}) \cap D$ and $D_2 := D - D_1$. Then, $|N[D_1]| \leq (5+6) + 2(5+4) = 29$, and $|D_2| = (n+1) - (3+4)$, and $|N[D_2]| \leq 5(n-6) = 5n - 30$. Thus,

$|N[D]| \leq 5n-1 < |V(G)|$, also a contradiction.

Case 5. G has one column which contains 3 vertices of D and one column which contains 2 vertices of D. Then, there are at least two columns V_{i_1}, V_{i_2} of G such that $V_{i_1} \cap D = \emptyset = V_{i_2} \cap D$. Then, at least one of V_{i_1}, V_{i_2} does not satisfy the properties of Assertion 1, a contradiction.

Case 6. G has one column which contains 3 vertices of D and all of other column contain at most one vertices of D, then there is one column V_j such that $V_j \cap D = \emptyset$ which does not satisfy the properties of Assertion 1, a contradiction.

Summarizing the above cases, there is no column of G which contains more than two vertices of D. Then, by Assertion 1, each column of G contains at least one vertex of D. As $|D| = n+1$, without loss of the generality, we assume that $|V_j \cap D| = 1$ for $j = 0, 1, \cdots, n-2$ and $|V_{n-1} \cap D| = 2$. By symmetry, we may assume $D \cap V_0 = \{x_{00}\}$. As $|D \cap V_1| = |D \cap V_2| = 1$, we can easily verify that $D \cap V_1 = \{x_{21}\}$ or $\{x_{31}\}$. For the former case, we can deduce that $D' := D \cap \cup_{j=0}^{n-2} V_j = W_0 - \{x_{4(n-1)}\}$. Hence, $V(G) - N[D'] = (V_{n-1} - \{x_{0(n-1)}, x_{2(n-1)}\}) \cup \{x_{30}, x_{4(n-2)}\}$. Now we can see that the union of D' and any two vertices of V_{n-1} can not dominate G, a contradiction. Hence, $\gamma(G) \geq n+2$, and thus $\gamma(C_5 \times C_n) = n+2 = 5(k+1)$. This proves the theorem. ∎

Theorem 6 *For $n \geq 5$ and $n \neq 5k+3$,*

$$b(C_5 \times C_n) = \begin{cases} 3, & n \in \{5k, 5k+1\}; \\ 4, & n \in \{5k+2, 5k+4\}, \end{cases}$$

For $n = 5k+3$ $(k \geq 1)$, $b(C_5 \times C_n) \leq 7$.

Proof. For $n \geq 5$ and $n \neq 5k+3$, by Theorem 1, $\gamma(C_5 \times C_n) \leq n+1$. Still denote $G := C_5 \times C_n$, and let D be a minimum dominating set of G. By the same reason as the proof of Theorem 4, we have that, if $n = 5k$, then $|D \cap V_j| = 1$ for $0 \leq j \leq n-1$; if $n \in \{5k+1, 5k+2, 5k+4\}$, then, except for one column, $|D \cap V_j| = 1$ for $0 \leq j \leq n-1$, and the exception column contains two vertices of D. We distinguish the following cases.

1. $n = 5k$ and $G = C_5 \times C_{5k}$. By using the same reason as in the proof of Theorem 4, we can deduce that W_i $(0 \le i \le 9)$ are all minimum dominating sets of G. Let $e_1 = x_{00}x_{10}, e_2 = x_{10}x_{20}, e_3 = x_{30}x_{40}$ and $G' := G - \{e_1, e_2, e_3\}$. Then, we can easily verify that any of W_i $(0 \le i \le 9)$ can not dominate G'. Assume $\gamma(G') = n$, and let D' be a minimum dominating set of G'. Clearly, D' dominates G, and thus D' is also a minimum dominating set of G. So, D' must be one of W_i $(0 \le i \le 9)$, a contradiction. Hence, $\gamma(G') > n$, and thus, $b(C_5 \times C_{5k}) \le 3$.

Next we show $b(C_5 \times C_{5k}) \ge 3$. Let $e_1 = x_1y_1, e_2 = x_2y_2$ be any two edges of G. Clearly, there are at most four distinct vertices in x_1, y_1, x_2, y_2. Hence, there is a W_i $(0 \le i \le 4)$, saying W_4, which contains no these vertices. Then, we can see that in $C_5 \times C_{5k} - \{e_1, e_2\}$, each vertex v in W_4 is still adjacent to every element of $N_G(v)$, i.e., W_4 dominates $C_5 \times C_{5k} - \{e_1, e_2\}$. So, $b(C_5 \times C_{5k}) \ge 3$, and hence $b(C_5 \times C_{5k}) = 3$.

2. $n = 5k+1$ and $G = C_5 \times C_{5k+1}$. Recall that any minimum dominating set of G must intersect with each column of G. Let D be a minimum dominating set such that $|D \cap V_{n-1}| = 2$ and $D \cap V_0 = \{x_{00}\}, D \cap V_1 = \{x_{21}\}$, then we can verify that $D = W_0 \cup \{x_{3(n-1)}\}$. We note that the vertex in $D \cap V_j$ for $0 \le j \le n-2$ has two private neighbors in same column with respect to D, and that each vertex in $D \cap V_{n-1}$ has one private neighbor in V_{n-1}, and $x_{4(n-1)}$ is adjacent to both vertices in $D \cap V_{n-1}$. By symmetry, each minimum dominating set of G has similar properties as D.

Now let $e_1 = x_{0(n-1)}x_{1(n-1)}, e_2 = x_{2(n-1)}x_{3(n-1)}$ and $e_3 = x_{3(n-1)}x_{4(n-1)}$, and $G' := C_5 \times C_{5k+1} - \{e_1, e_2, e_3\}$. We come to prove that $\gamma(G') > n+1 = 5k+2$. By contradiction. Assume $\gamma(G') \le n+1$. As $\gamma(G') \ge \gamma(G) = n+1$, we have $\gamma(G') = n+1$. Let D' be a minimum dominating set of G'. As $|D'| = n+1$, D' is also a minimum dominating set of G. So, D' has similar properties as D. However, in G' each vertex of V_{n-1} has at most one neighbor in V_{n-1}, so, for any choices of the vertices of $V_{n-1} \cap D'$, they do not have the similar properties as described above, a contradiction. So, $\gamma(G') > n+1$, and thus $b(C_5 \times C_{5k+1}) \le 3$.

Next we show $b(C_5 \times C_{5k+1}) \ge 3$. Let $e_1 = x_1y_1, e_2 = x_2y_2$ be any two edges of G, and $G'' := G - \{e_1, e_2\}$. Similarly, we can assume that W_4

contains none of x_1, x_2, y_1, y_2. Then, it is clear to see that $W_4 \cup \{x_{10}\}$ or $W_4 \cup \{x_{2(n-1)}\}$ will dominate G'' unless $\{e_1, e_2\} \subseteq \{x_{10}x_{1(n-1)}, x_{20}x_{2(n-1)}, x_{10}x_{20}, x_{1(n-1)}x_{2(n-1)}\}$.

But, for the exceptions of $\{e_1, e_2\}$, $W_0 \cup \{x_{3(n-1)}\}$ dominates G''. Hence, $\gamma(G'') = n+1$, implying that $b(C_5 \times C_{5k+1}) \geq 3$, and hence $b(C_5 \times C_{5k+1}) = 3$.

3. $n = 5k+2$ and $G = C_5 \times C_{5k+2}$. Let e_1, e_2, e_3, e_4 are four edges incident to x_{00}, and $G' := G - \{e_1, e_2, e_3, e_4\}$. If G' has a dominating set of $n+1$ vertices, as x_{00} is an isolated vertex in G', then $G_1 := G - \{x_{00}\}$ has a dominating set of n vertices. Let D'' be a dominating set of n vertices of G_1. By the similar counting as in the proof of Theorem 4 (case 1–case 6), we can deduce that $|D'' \cap V_j| = 1$ for $j = 0, 1, \cdots n-1$. By symmetry, we may assume that $D'' \cap V_0 = \{v_{10}\}$ or $\{v_{20}\}$. Then, we can easily deduce that $D'' = W_1$ or W_6 for the former case, $D'' = W_2$ or W_7 for the latter case. By direct checking, none of W_1, W_2, W_6, W_7 dominate G_1, a contradiction. So, $\gamma(G_1) > n$, and thus $\gamma(G') > n+1$, implying $b(C_5 \times C_{5k+2}) \leq 4$.

Next we show $b(C_5 \times C_{5k+2}) \geq 4$. Let e_1, e_2, e_3 be any three edges of G, and $G'' := G - \{e_1, e_2, e_3\}$. We will prove $\gamma(G'') = n+1$. For $V' \subseteq V(G)$ we denote by $< V' >$ the subgraph induced of G by V'. We distinguish four cases.

Case 1. All of e_1, e_2, e_3 are vertical edges.

(1). Assume that e_1, e_2, e_3 are contained in $< V_{n-1} >$. By symmetry, we only need to consider two cases:

$$\begin{array}{rl} \{e_1, e_2, e_3\} & = \{x_{1(n-1)}x_{2(n-1)}, x_{2(n-1)}x_{3(n-1)}, x_{3(n-1)}x_{4(n-1)}\} \\ \text{or} & = \{x_{0(n-1)}x_{1(n-1)}, x_{2(n-1)}x_{3(n-1)}, x_{3(n-1)}x_{4(n-1)}\}. \end{array}$$

Then, $(W_2 - \{x_{4(n-1)}\}) \cup \{x_{3(n-2)}, x_{0(n-1)}\} = (W_2 \cap \cup_{j=0}^{n-2} V_j) \cup (W_3 \cap \cup_{j=n-2}^{n-1} V_j)$ dominates G'' for the former case; $W_3 \cup \{x_{1(n-1)}\}$ dominates G'' for the latter case.

(2). Assume that e_1, e_2 are contained in $< V_{n-1} >$, and that e_3 is not in $< V_{n-1} >$. By symmetry, we let $\{e_1, e_2\}$ be $\{x_{2(n-1)}x_{3(n-1)}, x_{3(n-1)}x_{4(n-1)}\}$, or $\{x_{1(n-1)}x_{2(n-1)}, x_{3(n-1)}x_{4(n-1)}\}$.

If $\{e_1, e_2\} = \{x_{2(n-1)}x_{3(n-1)}, x_{3(n-1)}x_{4(n-1)}\}$, then $W_3 \cup \{x_{1(n-1)}\}$ dominates $G - \{e_1, e_2\}$. By our choices, we assume that e_3 is incident to a vertex in $W_3 - \{x_{0(n-1)}\}$. Let $e_3 \in < V_m >$ $(0 \leq m < n-1)$. If $m = 5h$ $(0 \leq h \leq k)$, then, for $e_3 = x_{2m}x_{3m}$, $(W_2 \cap \cup_{j=0}^{m} V_j) \cup (W_3 \cap \cup_{j=m}^{n-1} V_j)$ dominates G''; for $e_3 = x_{3m}x_{4m}$, $(W_3 \cap \cup_{j=0}^{m} V_j) \cup (W_4 \cap \cup_{j=m}^{n-1} V_j)$ dominates G''. If $m = 5h+1$ $(0 \leq h < k)$, then, for $e_3 = x_{0(5h+1)}x_{4(5h+1)}$, $(W_2 \cap \cup_{j=0}^{m} V_j) \cup (W_3 \cap \cup_{j=m}^{n-1} V_j)$ dominates G''; for $e_3 = x_{0(5h+1)}x_{1(5h+1)}$, $(W_3 \cap \cup_{j=0}^{m} V_j) \cup (W_4 \cap \cup_{j=m}^{n-1} V_j)$ dominates G''. For other cases of m, by assuming e_3 is incident to a vertex $x_{im} \in W_3$, we can also deduce that, $(W_2 \cap \cup_{j=0}^{m} V_j) \cup (W_3 \cap \cup_{j=m}^{n-1} V_j)$ dominates G'' for $e_3 = x_{(i-1)m}x_{im}$; $(W_3 \cap \cup_{j=0}^{m} V_j) \cup (W_4 \cap \cup_{j=m}^{n-1} V_j)$ dominates G'' for $e_3 = x_{im}x_{(i+1)m}$, where $0 \leq i \leq 4$, $i-1, i+1$ take modulo 5.

If $\{e_1, e_2\} = \{x_{1(n-1)}x_{2(n-1)}, x_{3(n-1)}x_{4(n-1)}\}$, then $W_1 \cup \{x_{4(n-1)}\}$ dominates $G - \{e_1, e_2\}$. By our choices, we assume that e_3 is incident to a vertex in $W_1 - \{x_{3(n-1)}\}$. Then, $W_4 \cup \{x_{2(n-1)}\}$ dominates G''.

(3). By the symmetry, (1) and (2), we assume that each $< V_j >$ $(0 \leq j \leq n-1)$ contains at most one edge of e_1, e_2, e_3. Without loss of the generality, let $e_1 = x_{3(n-1)}x_{4(n-1)}$. Then, $D_1 := W_1 \cup \{x_{4(n-1)}\}$ dominates $G - \{e_1\}$. Clearly, if neither of e_2, e_3 is incident to any vertices of D_1, then D_1 dominates G''. By our assumption, we may let e_2 be incident to a vertex of $W_1 - \{x_{3(n-1)}\}$. Then, $W_4 \cup \{x_{2(n-1)}\}$ dominates $G - \{e_1, e_2\}$. Similarly, we let e_3 be incident to a vertex of $W_4 - \{x_{1(n-1)}\}$. Let $e_3 \in < V_m >, (0 \leq m < n-1)$. Then, we can similarly deduce as in the proof of (2) that, $(W_3 \cap \cup_{j=0}^{m} V_j) \cup (W_4 \cap \cup_{j=m}^{n-1} V_j)$ or $W_3 \cap \{x_{1(n-1)}\}$ dominates G''.

Case 2. One of e_1, e_2, e_3 is a horizontal edge and two of them are vertical edges. We assume that $e_1 = \{x_{00}x_{0(n-1)}\}$ is a horizontal edge. Then, $D_4 := W_4 \cup \{x_{2(n-1)}\}$ dominates $G - \{e_1\}$. Similarly, we assume that e_2 is incident to a vertices of D_4.

(1). $e_2 \neq x_{2(n-1)}x_{3(n-1)}$. Then, e_2 is incident to a vertex in W_4. Then, $D_1 := W_1 \cup \{x_{4(n-1)}\}$ dominates $G - \{e_1, e_2\}$. By the same reason, we assume that e_3 is incident to a vertex of D_1.

If $e_3 = x_{0(n-1)}x_{4(n-1)}$, then $D_0 := W_0 \cup \{x_{3(n-1)}\}$ dominates $G - \{e_1, e_3\}$. By the same reason, we assume e_2 is incident to a vertex of D_0. As e_2 is a vertical edge, it is impossible for e_2 to join a vertex of W_4 and $x_{3(n-1)}$. Then, e_2 joins a vertex of W_0 and a vertex of W_4. Let $e_2 \in < V_m >$ $(0 \leq m \leq n-1)$, then, $(W_4 \cap \cup_{j=0}^{m} V_j) \cup (W_0 \cap \cup_{j=m}^{n-1} V_j)$ dominates G''.

If $e_3 \neq x_{0(n-1)}x_{4(n-1)}$, then e_3 is incident to a vertex of W_1. When $e_3 \in < V_m >$ joins a vertex of W_1 and a vertex of W_2, then $(W_1 \cap \cup_{j=0}^{m}) \cup (W_2 \cap \cup_{j=m}^{n-1} V_j)$ dominates G'', otherwise e_3 joins a vertex of W_0 and a vertex of W_1. And then, if $e_3 \neq x_{00}x_{10}$, then $W_2 \cup \{x_{10}\}$ dominates G''. If $e_3 = x_{00}x_{10}$, by noting that $W_3 \cup \{x_{20}\}$ dominates $G - \{e_1, e_3\}$, then e_2 is also incident to a vertex of W_3. Then, e_2 joins a vertex of W_3 and a vertex of W_4. Let $e_2 \in < V_m >$ $(0 \leq m \leq n-1)$. Then, $(W_3 \cap \cup_{j=0}^{m}) \cup (W_4 \cap \cup_{j=m}^{n-1} V_j)$ dominates G''.

(2). $e_2 = x_{2(n-1)}x_{3(n-1)}$. Then, $D_0 := W_0 \cup \{x_{3(n-1)}\}$ dominates $G - \{e_1, e_2\}$. By the same reason, we assume that e_3 is incident to a vertex of D_0. If $e_3 = x_{3(n-1)}x_{4(n-1)}$, then $W_4 \cup \{x_{30}\}$ dominates G''. Otherwise, by our choice, we have that, either e_3 joins a vertex of W_0 and a vertex of W_4, or $e_3 \neq x_{2(n-1)}x_{3(n-1)}$ joins a vertex of W_0 and a vertex of W_1.

If e_3 joins a vertex of W_0 and a vertex of W_4, then $W_2 \cup \{x_{10}\}$ dominates G''. If $e_3 \neq x_{2(n-1)}x_{3(n-1)}$ joins a vertex of W_0 and a vertex of W_1, then $W_4 \cup \{x_{30}\}$ still dominates G''.

Case 3. $e_1 = x_{3(n-1)}x_{4(n-1)}$ is a vertical edge and e_2, e_3 are horizontal edges. Then, $D_1 := W_1 \cup \{x_{4(n-1)}\}$ dominates $G - \{e_1\}$. By the same reason, we assume e_2 is incident to a vertex of D_1.

(1). If e_2 is incident to $x_{4(n-1)}$, $D_3 := W_3 \cup \{x_{1(n-1)}\}$ dominates $G - \{e_1, e_2\}$. By the same reason, we assume e_3 is incident to a vertex of D_3. When e_3 is incident to $x_{1(n-1)}$, then $W_3 \cup \{x_{20}\}$ dominates G'', otherwise e_3 is incident to a vertex of W_3. Then, $W_4 \cup \{x_{2(n-1)}\}$ dominates G''.

(2). e_2 is incident to a vertex of W_1. Then, $W_0 \cup \{x_{40}\}$ dominates $G - \{e_1, e_2\}$. By the same reason, we assume that e_3 is incident to a vertex of W_0 or x_{40}.

If e_3 is incident to x_{40} and $e_3 = x_{40}x_{41}$, then $W_0 \cup \{x_{40}\}$ still dominates G''. If $e_3 = x_{40}x_{4(n-1)}$, then, for $e_2 \neq x_{3(n-2)}x_{3(n-1)}$ we have that $(W_2 - \{x_{4(n-1)}\}) \cup \{x_{3(n-2)}, x_{0(n-1)}\}$ dominates G''; for $e_2 = x_{3(n-2)}x_{3(n-1)}$ we have that $W_3 \cup \{x_{20}\}$ dominates G''.

Otherwise, e_3 is incident to a vertex of W_0. Let e_2 be incident to a vertex $x_{ij} \in W_1$. If $e_2 = x_{i(j-1)}x_{ij}$, then $W_4 \cup \{x_{30}\}$ dominates G'', where $j \leq n-1$ and $j-1, j+1$ take modulo n. If $e_2 = x_{ij}x_{i(j+1)}$, then $W_3 \cup \{x_{1(n-1)}\}$ dominates $G - \{e_1, e_2\}$, where $j \leq n-1$ and $j-1, j+1$ take modulo n. Hence, e_3 is also incident to a vertex of $W_3 \cup \{x_{1(n-1)}\}$. As e_3 is a horizontal edge, e_3 joins a vertex of W_0 and a vertex of W_3. Note that $e_2 \neq x_{3(n-2)}x_{3(n-1)}$, then we have $(W_2 \cap \bigcup_{j=0}^{n-2} V_j) \cup \{x_{3(n-2)}, x_{0(n-1)}\}$ dominates G''.

Case 4. All of e_1, e_2, e_3 are horizontal edges. Let $e_1 = x_{00}x_{0(n-1)}$. Then, $D_4 := W_4 \cup \{x_{2(n-1)}\}$ dominates $G - \{e_1\}$. By the same reason, we assume that e_2 is incident to a vertex of D_4.

(1). If e_2 is incident to $x_{2(n-1)}$, then $W_4 \cup \{x_{30}\}$ dominates $G - \{e_1, e_2\}$. By the same reason, we assume that e_3 is incident to a vertex of $W_4 \cup \{x_{30}\}$. First let e_3 be incident to x_{30}. Then, if $e_2 \neq x_{2(n-2)}x_{2(n-1)}$, then D_4 still dominates G''; otherwise, $e_2 = x_{20}x_{2(n-1)}$, then $W_0 \cup \{x_{40}\}$ dominates G''.

Next let e_3 be incident to a vertex of W_4. Then, if $e_2 = x_{2(n-2)}x_{2(n-1)}$, $W_3 \cup \{x_{20}\}$ dominates G''; if $e_2 = x_{20}x_{2(n-1)}$, $W_0 \cup \{x_{3(n-1)}\}$ dominates G''.

(2). If e_2 is incident to a vertex of W_4, then $D_0 := W_0 \cup \{x_{3(n-1)}\}$ dominates $G - \{e_1, e_2\}$. By the same reason, we assume that e_3 is incident to a vertex of D_0.

If e_3 is incident to $x_{3(n-1)}$, then $W_3 \cup \{x_{20}\}$ dominates G''.

So, let e_3 be incident to a vertex of W_0. Let e_2 be incident to a vertex $x_{ij} \in W_4$.

If $e_2 = x_{i(j-1)}x_{ij}$ (where $j-1$ takes modulo n), then $W_2 \cup \{x_{10}\}$ dominates $G - \{e_1, e_2\}$. From this, we may assume that e_3 joins a vertex of W_0 and a vertex of W_2. Let $x_{im} \in W_2$ and $e_3 = x_{im}x_{i(m+1)}$ (where $0 \leq m < n-1$) , then $W_3 \cup \{x_{1(n-1)}\}$ dominates G''.

If $e_2 = x_{ij}x_{i(j+1)}$ (where $j+1$ takes modulo n), then $W_3 \cup \{x_{20}\}$ dominates $G - \{e_1, e_2\}$. From this, we may assume that e_3 joins a vertex of W_0 and a vertex of W_3. Let $x_{im} \in W_3$ and $e_3 = x_{im}x_{i(m+1)}$ (where $0 \leq m < n-1$), then $W_1 \cup \{x_{4(n-1)}\}$ dominates G''.

Summarizing as above, we have that, $\gamma(G - \{e_1, e_2, e_3\}) \leq n + 1$ for any three edges of G, implying that $b(G) \geq 4$. Combining with the former inequality we have $b(G) = 4$. This proves that $b(C_5 \times C_{5k+2}) = 4$.

For the case of $n = 5k + 4$, we can similarly prove that $b(C_5 \times C_n) = 4$, we omit the details here.

For the case of $n = 5k + 3$, by Theorem 2 we have $b(G) \leq 7$. This proves Theorem 5. ■

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