On the Number of H-points in a Circle

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Abstract

Let an H-point be a vertex of a tiling of $\mathbb{R}^2$ by regular hexagons of side length 1, and $D(n)$ a circle of radius n ($n \in \mathbb{Z}^+$) centered at an H-point. In this paper we present an algorithm to calculate the number, $\mathcal{N}_H(D(n))$, of H-points that lie inside or on the boundary of $D(n)$. Furthermore, we show that the ratio $\mathcal{N}_H(D(n))/n^2$ tends to $2\pi/S$ as n tends to ∞, where $S = \frac{3\sqrt{3}}{2}$ is the area of the regular hexagonal tiles.

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1 Introduction

Let $\vec{u}$ and $\vec{v}$ be two linearly independent real vectors in $\mathbb{R}^2$. The set of all points $P = m\vec{u} + n\vec{v}$ ($m, n \in \mathbb{Z}$) is called a *general lattice* Λ generated by $\vec{u}$ and $\vec{v}$. A point of the lattice Λ is called a *lattice point*. Specially, if $\vec{u}$ and $\vec{v}$ are mutually orthogonal unit vectors, then the lattice Λ is called an *integral lattice*, which is denoted by $\mathbb{Z}^2$.

Let a planar tiling $\mathcal{T}$ be a countable family of closed sets $\mathcal{T} = \{T_1, T_2, \cdots\}$ such that the union of the sets $T_1, T_2, \cdots$ is to be the whole plane, and the interiors of the sets T_i are to be pairwise disjoint. Here $T_1, T_2, \cdots$ are known as the tiles of $\mathcal{T}$. The intersection of any finite set of tiles of $\mathcal{T}$ containing at least two distinct tiles may be empty or may consist of set of isolated points

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and straight line segments. The points of intersection are called vertices of the tiling and the segments of intersection are called edges of the tiling.

An edge-to-edge tiling is a type of tiling where each tile is a polygon and adjacent tiles only share full sides. A vertex around which, in cyclic order, we have an n_1-gon, an n_2-gon, etc., is said to be of type $[n_1.n_2.\cdots]$. Let an edge-to-edge tiling by regular polygons such that all vertices are of the same type be Archimedean Tilings. The Archimedean Tilings with of type $[n_1.n_2.\cdots.n_r]$ is called $[n_1.n_2.\cdots.n_r]$-tiling. As far as we know, a [3.3.3.3.3.3]-tiling, a [4.4.4.4]-tiling and a [6.6.6]-tiling are all Archimedean Tilings ([1]). In fact, the set of vertices of a [4.4.4.4]-tiling is an integral lattice $\mathbb{Z}^2$, and that of a [3.3.3.3.3.3]-tiling is a general lattice. Let $\mathcal{H}$ be a [6.6.6]-tiling by regular hexagons of side length 1, and H be the set of vertices of the tiling $\mathcal{H}$. A point of H is called an *H-point*. Some results on integral lattice points have been generalized to H-points, for example, Pick's theorem and some related properties on lattice polygons ([2], [3], [4], [5], [6]).

Let $D(n)$ be a circle of radius n centered at the origin of the integral lattice $\mathbb{Z}^2$, where $n \in \mathbb{Z}^+$. In 1837, C. F. Gauss [7] discussed the number $N(n)$ of lattice points that lie inside or on the boundary of $D(n)$. Furthermore, he showed that the ratio $N(n)/n^2$ tends to π as n tends to ∞. Recalling that the set of vertices of a [4.4.4.4]-tiling can be treated as an integral lattice, we are motivated to investigate the similar problem for H-points. Let $D(n)$ be a circle of radius n centered at an H-point, and let $\mathcal{N}_H(D(n))$ denote the number of H-points lying inside or on the boundary of $D(n)$. In this paper we present an algorithm to calculate the value of $\mathcal{N}_H(D(n))$. Furthermore, we show that the ratio $\mathcal{N}_H(D(n))/n^2$ tends to $2\pi/S$ as n tends to ∞, where $S = \frac{3\sqrt{3}}{2}$ is the area of the regular hexagonal tiles.

2 Main Results

In fact, H can be considered as the union of two disjoint triangular lattices denoted by H^+, H^- such that for any two points in H^+ (H^-) there exists a translation of the plane which maps one of the two points to the other and H to H. A point of H^+ (*resp.* H^-) is called an *H^+-point* (resp. *H^--point)*.

Let C denote the set of all centers of the hexagonal tiles which determine $\mathcal{H}$. A point of C is called a *C-point*. Then $H^+ \cup H^- \cup C$ forms a *triangular lattice* with the area of each triangular tile $\frac{\sqrt{3}}{4}$. We denote this triangular lattice by T and a point of T is called a *T-point*. If a segment $\overline{P_1P_2}$ contains some T-points, then the ordered sequence (from P_1 to P_2) of the T-points (to be precise, H^+-points, C-points, or H^--points) lying on it is called a

T-*sequence* of $\overline{P_1P_2}$, which is denoted by $(H^+, C, H^-, \cdots)$.

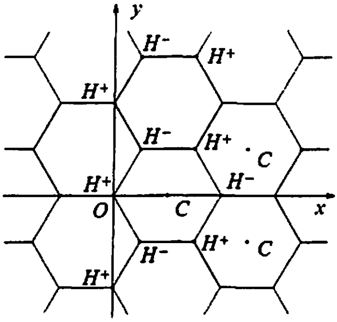

Figure 1: H^+-points, H^--points and C-points

Without loss of generality, we establish an $x-y$ coordinate system with an H^+-point as the origin, and the x-axis lying along one edge of a regular hexagonal tile, as shown in Figure 1. Let $D(n)$ be a circle of radius n centered at the origin O, where $n \in \mathbb{Z}^+$. Let the circumference of $D(n)$ intersect the x-axis (resp. y-axis) at points A_0, B_0 (resp. E, F). The line determined by A_0B_0 is denoted by L. Now we translate L in direction $\overrightarrow{OE}$. If L meets some T-points for the first time, then we stop translating and denote the chord obtained by l_1. Let A_1, B_1 denote the two endpoints of l_1, and O_1 the midpoint of l_1, as shown in Figure 2. We continue translating L in the same direction, and denote the second chord obtained in the same way by l_2. Similarly, we denote the two endpoints of l_2 by A_2 and B_2, and the midpoint of l_2 by O_2. We continue translating in this way, and finally we obtain k chords $l_1, l_2, \cdots, l_k$ with endpoints A_i, B_i and midpoints O_i, where $k = [\frac{2\sqrt{3}}{3}n]$ (here $[\cdot]$ denotes the greatest integer function). For the sake of convenience, let $l_0 = A_0B_0$ and $O_0 = O$. Clearly, all the chords l_i are parallel, and the distance between l_i and l_{i+1} is equal to $\frac{\sqrt{3}}{2}$, $i = 0, 1, 2, \cdots, k-1$.

Let $\mathscr{N}_H(D(n))$ denote the number of H-points that lie inside or on the boundary of $D(n)$. For the sake of brevity, we denote the number of H-points (resp. T-points, C-points) on a chord l_i by $\mathscr{N}_H(l_i)$ (resp. $\mathscr{N}_T(l_i)$, $\mathscr{N}_C(l_i)$). Suppose that the diameter l_0 cuts the circle $D(n)$ into two semicircles $D_1(n)$ and $D_2(n)$, and all the chords $l_0, l_1, l_2, \cdots, l_k$ defined above are chords of $D_1(n)$. Then we have $\mathscr{N}_H(D(n)) = \mathscr{N}_H(l_0) + 2 \cdot \sum_{i=1}^{k} \mathscr{N}_H(l_i)$, $\mathscr{N}_H(l_i) = \mathscr{N}_T(l_i) - \mathscr{N}_C(l_i)$. Let $x_i = \sqrt{n^2 - (\frac{\sqrt{3}}{2}i)^2}$; then the length of l_i is equal to $2x_i$, where $i = 0, 1, \cdots, k$. There are two cases to consider.

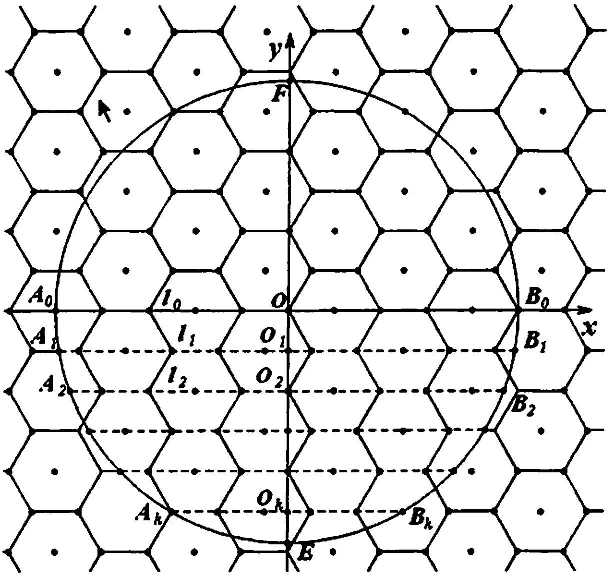

Figure 2: The chords of $D(n)$

Case 1. i is even.

In this case, the midpoint O_i of l_i is an H-point. Recall that the side length of each hexagonal tile is equal to 1, then both of the closed segments $\overline{A_iO_i}$ and $\overline{O_iB_i}$ contain $[x_i]+1$ T-points, sharing a common H-point O_i. Hence we have $\mathscr{N}_T(l_i) = 2[x_i]+1$. In order to calculate the value of $\mathscr{N}_C(l_i)$, there are three subcases to consider.

Subcase 1.1. $[x_i] \equiv 0 \pmod 3$.

Then the T-sequence of the segment $\overline{O_iB_i}$ is $(H^+, C, H^-, H^+, C, \cdots, C, H^-, H^+)$. Therefore, the number of C-points on the segment $\overline{O_iB_i}$ is $\frac{[x_i]}{3}$. Similarly, the T-sequence of the segment $\overline{A_iO_i}$ is $(H^+, C, H^-, H^+, C, \cdots, C, H^-, H^+)$. Thus the number of C-points on the segment $\overline{A_iO_i}$ is also $\frac{[x_i]}{3}$. Hence, $\mathscr{N}_C(l_i) = \frac{2[x_i]}{3}$ and therefore

$$\mathscr{N}_H(l_i) = \mathscr{N}_T(l_i) - \mathscr{N}_C(l_i) = \frac{4[x_i]}{3} + 1.$$

Subcase 1.2. $[x_i] \equiv 1 \pmod 3$.

Then the T-sequence of $\overline{O_iB_i}$ is $(H^+, C, H^-, H^+, C, \cdots, C, H^-, H^+, C)$. Thus, the number of C-points on the segments $\overline{O_iB_i}$ is $\left[\frac{[x_i]}{3}\right] + 1$. Similarly, the T-sequence of $\overline{A_iO_i}$ is $(H^-, H^+, C, H^-, H^+, C, \cdots, C, H^-, H^+)$, and hence the number of C-points on the segments $\overline{A_iO_i}$ is $\left[\frac{[x_i]}{3}\right]$. So,

$\mathscr{N}_C(l_i) = 2\left[\frac{[x_i]}{3}\right] + 1$ and then

$$\mathscr{N}_H(l_i) = \mathscr{N}_T(l_i) - \mathscr{N}_C(l_i) = 2[x_i] + 1 - \left(2\left[\frac{[x_i]}{3}\right] + 1\right)$$
$$= 2\left([x_i] - \left[\frac{[x_i]}{3}\right]\right) = 4\left[\frac{[x_i]}{3}\right] + 2.$$

Subcase 1.3. $[x_i] \equiv 2 \pmod 3$.

By a method similar to that for case Subcase 1.2, we know that the numbers of C-points on the segments $\overline{O_iB_i}$ and $\overline{A_iO_i}$ are both $\left[\frac{[x_i]}{3}\right] + 1$. Hence, we have

$$\mathscr{N}_H(l_i) = \mathscr{N}_T(l_i) - \mathscr{N}_C(l_i) = 2[x_i] + 1 - \left(2\left[\frac{[x_i]}{3}\right] + 2\right)$$
$$= 2\left([x_i] - \left[\frac{[x_i]}{3}\right]\right) - 1 = 4\left[\frac{[x_i]}{3}\right] + 3.$$

Case 2. i is odd.

In this case, the midpoint O_i of l_i is not an H-point anymore. Since the smallest distance between O_i and T-points on l_i is 0.5, both the segments $\overline{A_iO_i}$ and $\overline{O_iB_i}$ contain $[x_i - 0.5] + 1$ T-points. Thus $\mathscr{N}_T(l_i) = 2[x_i - 0.5] + 2$. There are also three subcases to consider.

Subcase 2.1. $[x_i - 0.5] \equiv 0 \pmod 3$.

Then the T-sequence of $\overline{O_iB_i}$ is $(H^-, H^+, C, H^-, H^+, C, \cdots, H^-, H^+, C, H^-)$. So the number of C-points on the segment $\overline{O_iB_i}$ is $\frac{[x_i-0.5]}{3}$, and the number of C-points on the segment $\overline{A_iO_i}$ is $\frac{[x_i-0.5]}{3} + 1$. Thus we have

$$\mathscr{N}_{l_i}(H) = \mathscr{N}_{l_i}(T) - \mathscr{N}_{l_i}(C) = 2(\lfloor x_i - 0.5\rfloor + 1) - \left(\frac{2\lfloor x_i - 0.5\rfloor}{3} + 1\right)$$
$$= \frac{4\lfloor x_i - 0.5\rfloor}{3} + 1.$$

Subcase 2.2. $[x_i - 0.5] \equiv 1 \pmod 3$.

We omit the details here and have

$$\mathscr{N}_H(l_i) = 4\left[\frac{[x_i - 0.5]}{3}\right] + 3.$$

Subcase 2.3. $[x_i - 0.5] \equiv 2 \pmod 3$.

By calculating we have

$$\mathscr{N}_H(l_i) = 4\left[\frac{[x_i - 0.5]}{3}\right] + 4.$$

Now we present an algorithm to calculate $\mathcal{N}_H(D(n))$:

1. *Start with an arbitrary nonnegative integer* n*, where* n *is the radius of the circle* $D(n)$*. Set* $k = [\frac{2\sqrt{3}}{3}n]$*,* $i = 0$ *and* $N = 0$*.*

2. *Set* $y_i = \sqrt{n^2 - (\frac{\sqrt{3}}{2}i)^2}$*. If* i *is even, then go to step 3. Otherwise, go to step 4.*

3. *Compute the number of H-points* a_i *on the chord* l_i*. If* $[y_i] \equiv 0 \pmod 3$*, then set* $a_i = \frac{4[y_i]}{3} + 1$*. If* $[y_i] \equiv 1 \pmod 3$*, then set* $a_i = 4\left[\frac{[y_i]}{3}\right] + 2$*. Otherwise, set* $a_i = 4\left[\frac{[y_i]}{3}\right] + 3$*. Replace* N *by* $N + 2a_i$ *and go to step 5.*

4. *Compute the number of H-points* a_i *on the chord* l_i*. If* $[y_i - 0.5] \equiv 0 \pmod 3$*, then set* $a_i = \frac{4[y_i - 0.5]}{3} + 1$*. If* $[y_i - 0.5] \equiv 1 \pmod 3$*, then set* $a_i = 4\left[\frac{[y_i - 0.5]}{3}\right] + 3$*. Otherwise, set* $a_i = 4\left[\frac{[y_i - 0.5]}{3}\right] + 4$*. Replace* N *by* $N + 2a_i$ *and go to step 5.*

5. *If* $i \leqslant k$*, then replace* i *by* $i + 1$ *and go to step 2. Otherwise, stop and output the value of* $N - a_0$*.*

By this algorithm, we can compute $\mathcal{N}_H(D(n))$ for all $n \in \mathbb{Z}^+$. Some values of $\mathcal{N}_H(D(n))$ are listed in Table 1.

$r = n$	$\mathcal{N}_H(D(n))$	$\frac{\mathcal{N}_H(D(n))}{n^2}$	$r = n$	$\mathcal{N}_H(D(n))$	$\frac{\mathcal{N}_H(D(n))}{n^2}$
10	244	2.44	10000	241839646	2.41839646
20	979	2.4475	20000	967359343	2.4183983575
50	6049	2.4196	50000	6045997801	2.4183991204
100	24202	2.4202	100000	24183991576	2.4183991576
200	96715	2.417875	200000	96735966373	2.418399159325
500	644597	2.418388	500000	604599788545	2.41839915418
1000	2418358	2.418358	1000000	2418399151576	2.418399151576
2000	9673627	2.41840675	2000000	9673596608725	2.41839915218125
5000	60460099	2.41840396	5000000	60459978806305	2.4183991522522

Table 1: Some values of $\mathcal{N}_H(D(n))$

From the table we find that as n increases, the ratio of $\mathcal{N}_H(D(n))/n^2$ tends to $2.418399152\cdots$. In fact, we have the following theorem.

Theorem 2.1. *Let* $D(n)$ *be a circle of radius* n $(n \in \mathbb{Z}^+)$ *centered at the origin* O*. Then* $\lim\limits_{n\to\infty} \frac{\mathcal{N}_H(D(n))}{n^2} = \frac{2\pi}{S}$*, where* $S = \frac{3\sqrt{3}}{2}$ *is the area of the regular hexagonal tiles.*

Proof. Let the family $\mathcal{A} = \{l \mid l$ is a line with an equation $x = 1 + \frac{3}{2}m,\ m \in \mathbb{Z}\}$, and $\mathcal{B} = \{l \mid l$ is a line with an equation $y = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2}m,\ m \in \mathbb{Z}\}$. Then all the lines in $\mathcal{A}$ and $\mathcal{B}$ divide the whole plane into small rectangles. Furthermore, each H-point is related to such a rectangle (it is unique), as shown in Figure 3. If an H-point lies inside or on the boundary of the circle $D(n)$, then we shade the rectangle related to it, also see Figure 3. Now denote the area of the regular hexagonal tile by S. Then the area of the rectangles in Figure 3 is equal to $\frac{3\sqrt{3}}{4} = \frac{S}{2}$. Therefore, the area of the shaded region is equal to $\mathscr{N}_H(D(n)) \cdot \frac{S}{2}$.

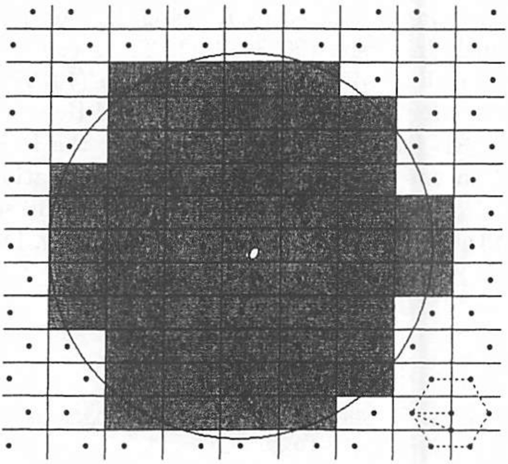

Figure 3: Rectangles determined by H-points

Observe, however, that some parts of the shaded region are outside of the disk $x^2 + y^2 \leqslant n$, and that the disk is not entirely shaded. This observation lets us bound the shaded region from below and above. In fact, we only need find the largest disk whose interior is completely shaded, and the smallest disk whose exterior is completely unshaded. Because the diagonal of the rectangle is $\sqrt{3}$, all the shaded rectangles must be contained in a circle of radius $r = n + \sqrt{3}$. Similarly, the circle of radius $r = n - \sqrt{3}$ is contained entirely within the shaded rectangles. It follows that

$$\begin{aligned}\pi(n^2 - 2\sqrt{3}n - 3) \leqslant& \pi(n^2 - 2\sqrt{3}n + 3) = \pi(n - \sqrt{3})^2 \leqslant \mathscr{N}_H(D(n)) \cdot \frac{S}{2} \\ \leqslant& \pi(n + \sqrt{3})^2 = \pi(n^2 + 2\sqrt{3}n + 3),\end{aligned}$$

which implies that

$$\left| \frac{\mathscr{N}_H(D(n))}{n^2} \cdot \frac{S}{2} - \pi \right| \leqslant \pi \left(\frac{2\sqrt{3}}{n} + \frac{3}{n^2} \right).$$

Therefore we have

$$\lim_{n\to\infty} \frac{\mathscr{N}_H(D(n))}{n^2} \cdot \frac{S}{2} = \pi,$$

that is, $\lim_{n\to\infty} \frac{\mathscr{N}_H(D(n))}{n^2} = \frac{2\pi}{S}$, as desired. □

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