Toughness of Infinite Graphs

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Abstract

We initiate a study of the toughness of infinite graphs by considering a natural generalization of that for finite graphs. After providing general calculation tools, computations are completed for several examples. Avenues for future study are presented, including existence problems for tough-sets and calculations of maximum possible toughness. Several open problems are posed.

1 Introduction

Throughout this article we consider only locally finite graphs G. For each graph G, we use $V(G)$, $E(G)$, $\kappa(G)$ and $\Delta(G)$ to denote, respectively, the vertex set, edge set, connectivity, and maximum degree for G. Given a set of vertices U in a graph G, the number of components in the subgraph of G induced by U is denoted by $\omega(U)$. A separating set for G is a set $S \subseteq V(G)$ such that $\omega(G \setminus S) > 1$. Note that, if a locally finite graph G is connected and S is a finite subset of $V(G)$, then $\omega(G \setminus S)$ is finite. We extend Chvátal's definition [1] of the toughness of a non-complete graph G and allow G to be infinite by

$$\tau(G) = \min\{\frac{|S|}{\omega(G \setminus S)} : S \text{ is a finite separating set for } G\}.$$

We further adopt the convention of Pippert [7] and define $\tau(K_n) = n - 1$. A graph G is said to be t-tough if $\tau(G) \geq t$. A tough-set for G is a separating set S for which $\tau(G) = |S|/\omega(G \setminus S)$. All standard notation and terminology not presented here can be found in [8].

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2 General Bounds

The following two theorems extend directly from finite graphs to infinite graphs.

Theorem 2.1 ([1]). *For a graph G, $\tau(G) \leq \kappa(G)/2$.*

Theorem 2.2 ([6]). *For a graph G, if m is the largest integer such that the complete bipartite graph $K_{1,m}$ is an induced subgraph, then $\tau(G) \geq \kappa(G)/m$.*

The proofs of Theorems 2.1 and 2.2 for infinite graphs are the same as those given for finite graphs. However, note that although the proof of Theorem 2.2 given in [6] refers to a tough-set, it suffices in that proof to merely consider a separating set. We shall see in Example 3.3 that an infinite graph need not have a tough-set and consider this issue further in Section 4.

Theorem 2.3. *Let $t \in \mathbb{R}$, and suppose that there is a sequence of subgraphs $G_1 \subseteq G_2 \subseteq \cdots \subseteq G$ such that $\bigcup V(G_k) = V(G)$ and G_k is t-tough for k sufficiently large. Then, G is t-tough.*

Proof. Suppose to the contrary that $\tau(G) < t$. So there is some separating set $S \subseteq V(G)$ with $|S|/\omega(G \setminus S) < t$. Since $|S|$ and $\omega(G \setminus S)$ are finite, we have some N such that G_N is t-tough, $S \subseteq V(G_N)$, and each component of $G \setminus S$ has at least one vertex in $V(G_N)$. Hence, $\omega(G_N \setminus S) \geq \omega(G \setminus S)$. We thus have

$$\tau(G_N) \leq \frac{|S|}{\omega(G_N \setminus S)} \leq \frac{|S|}{\omega(G \setminus S)} < t,$$

a contradiction. □

Since Hamiltonian graphs are 1-tough [1], we have an immediate corollary.

Corollary 2.4. *If there is a sequence of subgraphs $G_1 \subseteq G_2 \subseteq \cdots \subseteq G$ such that $\bigcup V(G_n) = V(G)$ and G_n is Hamiltonian for n sufficiently large, then $\tau(G) \geq 1$.*

Theorem 2.5. *Let G be a connected locally finite graph with $\tau(G) = 0$. Then, G is infinite and $\Delta(G)$ is infinite.*

Proof. Let $n \in \mathbb{Z}^+$. Since $\tau(G) = 0$, we have a separating set $S \subseteq V(G)$ such that $|S|/\omega(G \setminus S) < 1/n$. Since G is connected, each component of $G \setminus S$ is adjacent to at least one element of S. We claim that some vertex in S is adjacent to at least $n+1$ vertices. If not, then each vertex in S is adjacent to at most n components of $G \setminus S$ and we have $\omega(G \setminus S) \leq n|S|$. This gives

$$\frac{1}{n} \leq \frac{|S|}{\omega(G \setminus S)} < \frac{1}{n},$$

a contradiction, and establishes our claim. Since n is arbitrary, $\Delta(G)$ must be infinite, and hence G must be infinite. □

Corollary 2.6. *If G is a connected locally finite graph with $\Delta(G)$ finite, then $\tau(G) > 0$.*

3 Examples

The converse of Corollary 2.6 does not hold.

Example 3.1. Consider the graph G such that $V(G)$ is the set of positive integers $\mathbb{Z}^+$ and $E(G) = \{\{i, i+j\} : 1 \leq j \leq i\}$. So $\Delta(G)$ is infinite, and it follows from Theorems 2.1 and 2.2 that $\tau(G) = 1/2 > 0$. Of course, $\{2\}$ is a tough-set.

Theorem 3.2. *Let T be a tree. Then $\tau(T) = 1/\Delta(T)$ if $\Delta(T)$ is finite, and $\tau(T) = 0$ otherwise.*

Proof. If $\Delta(T)$ is finite, then it follows from Theorem 2.2 that a vertex of maximum degree forms a tough-set for T. So assume that $\Delta(T)$ is infinite. Let $n \in \mathbb{Z}^+$. So there must be some vertex v in T of degree at least n. Hence,

$$\tau(T) \leq \frac{1}{\omega(T \setminus \{v\})} \leq \frac{1}{n}.$$

Since n is arbitrary, $\tau(T) = 0$. □

The following is an example of a tree with toughness 0.

Example 3.3. Let $V = \{(1,0)\} \cup \{(x,y) : x, y \in \mathbb{Z}, 0 \leq y \leq x-2\}$. Form a tree T with vertex set V by connecting two vertices (x_1, y_1) and (x_2, y_2) by an edge if and only if either $y_1 = y_2 = 0$ and $|x_2 - x_1| = 1$ or $x_1 = x_2$, $y_1 \neq y_2$, and $y_1 y_2 = 0$. Note that $\deg((n,0)) = n$. So $\Delta(T)$ is infinite and thus $\tau(T) = 0$. Moreover, T has no tough-set.

We shall use $\{1, \ldots, n\}$ as the vertex set for the path P_n, the cycle C_n, and the complete graph K_n. Let P_∞ denote the two-way infinite path on $\mathbb{Z}$, and let P_∞^+ denote the one-way infinite path on $\mathbb{Z}^+$. It follows from Theorem 3.2 that $\tau(P_\infty) = \tau(P_\infty^+) = 1/2$.

Theorem 3.4. *The following graphs have toughness* 1.

(a) $P_\infty^+ \times P_n$ *for* $n \geq 2$,

(b) $P_\infty^+ \times C_n$ *for even* $n \geq 4$,

(c) $P_\infty^+ \times P_\infty^+$,

(d) $P_\infty \times P_n$ *for* $n \geq 2$*,*

(e) $P_\infty \times C_n$ *for even* $n \geq 4$*,*

(f) $P_\infty \times P_\infty^+$*, and*

(g) $P_\infty \times P_\infty$*.*

Proof. To establish 1 as a lower bound, we apply Theorem 2.3, with appropriate choices of finite subgraphs G_k for the graph G in each case. (a) $G_k = P_{2k} \times P_n$. (b) $G_k = P_{2k} \times C_n$. (c) $G_k = P_{2k} \times P_k$. (d) $G_k = (P_\infty \cap [-k, k-1]) \times P_n$. (e) $G_k = (P_\infty \cap [-k, k-1]) \times C_n$. (f) $G_k = (P_\infty \cap [-k, k-1]) \times P_k$. (g) $G_k = (P_\infty \cap [-k, k-1]) \times (P_\infty \cap [-k, k-1])$. That these graphs G_k are 1-tough is shown in [6].

To establish 1 as an upper bound in each case, let S_k be the set of vertices (i, j) in G_k with $i \not\equiv j \pmod 2$, and observe that

$$\lim_{k \to \infty} \frac{|S_k|}{\omega(G \setminus S_k)} = 1. \qquad \square$$

Note that the lower bound for parts (e) and (g) of Theorem 3.4 also follows directly from Theorem 2.2.

Theorem 3.5. *For odd* $n \geq 5$, $1 \leq \tau(P_\infty^+ \times C_n) \leq \tau(P_\infty \times C_n) \leq n/(n-1)$.

Proof. We apply Theorem 2.3. To establish $n/(n-1)$ as an upper bound, let S_k be the set of vertices (i, j) in $P_{2k} \times C_n$ with $i \not\equiv j \pmod 2$, and note that

$$\lim_{k \to \infty} \frac{|S_k|}{\omega((P_\infty \times C_n) \setminus S_k)} = \frac{n}{n-1}.$$

To establish the asserted middle inequality, let S be a finite separating set for $P_\infty \times C_n$, and let l the the smallest first coordinate of a vertex in S. By adding $|l| + 2$ to the first coordinate of each vertex in S, we obtain a finite separating set S' for $P_\infty^+ \times C_n$. Since

$$\tau(P_\infty^+ \times C_n) \leq \frac{|S'|}{\omega((P_\infty^+ \times C_n) \setminus S')} = \frac{|S|}{\omega((P_\infty \times C_n) \setminus S)},$$

and S is arbitrary, we see that $\tau(P_\infty^+ \times C_n) \leq \tau(P_\infty \times C_n)$.

The lower bound of 1 follows from Corollary 2.4 and the observation that $P_k \times C_n$ is Hamiltonian. $\square$

It follows from Theorem 2.3 that the following infinite analog of Conjecture 5.7 from [6] is equivalent to it.

Conjecture 3.6. *For odd* $n \geq 5$, $\tau(P_\infty^+ \times C_n) = n/(n-1)$.

Theorem 3.7. *For $n \geq 2$, $\tau(P_\infty^+ \times K_n) = \tau(P_\infty \times K_n) = (n+1)/3$.*

Proof. In both cases, the separating set S given by the neighborhood of the vertex $(2,1)$ establishes $(n+1)/3$ as an upper bound. For $G = P_\infty^+ \times K_n$, let $G_k = P_k \times K_n$, and, for $G = P_\infty \times K_n$, let $G_k = (P_\infty \cap [-k,k]) \times K_n$. By Theorem 5.5 of [6], $\tau(G_k) = (n+1)/3$, and thus our desired lower bound follows from Theorem 2.3. □

4 Tough-Sets

Note that $P_\infty^+ \times P_3$ has tough-set $\{(1,2),(2,1)\}$. Despite the fact that the related graph $P_\infty \times P_3$ has the same toughness, it has no tough-set. In our proof of this, we make repeated use of the following easily proven lemma restricting the local structure of a tough-set.

Lemma 4.1 ([4, 3]). ***Separation Rule.*** *Let S be a tough-set for a graph G. If $v \in S$, then v is adjacent to at least two components of $G \setminus S$.*

Example 4.2. $P_\infty \times P_3$ has no tough-set.

Proof. Suppose to the contrary that $P_\infty \times P_3$ has a tough-set, and let S be a tough-set of smallest possible size. By Theorem 3.4(d), $\omega((P_\infty \times P_3) \setminus S) = |S|$. Without loss of generality, assume that 1 is the smallest first coordinate for a vertex in S, and consider a subgraph $P_k \times P_3$ of $P_\infty \times P_3$ such that k is even and $S \subseteq P_k \times P_3$. By Theorem 5.2 of [6], $\tau(P_k \times P_3) = 1$.

Note that we cannot have all of $(1,1),(1,2),(1,3) \in S$, since the Separation Rule would then force $(2,1),(2,2),(2,3) \notin S$, which would contradict the assumption that S is a tough-set. Moreover, there must be a finite component of $(P_\infty \times P_3) \setminus S$, and S is a separating set for $P_k \times P_3$ with $\omega((P_k \times P_3) \setminus S) \geq \omega((P_\infty \times P_3) \setminus S)$. Since

$$1 = \tau(P_\infty \times P_3) = \frac{|S|}{\omega((P_\infty \times P_3) \setminus S)} \geq \frac{|S|}{\omega((P_k \times P_3) \setminus S)} \geq 1,$$

it follows that we have equality throughout and S is a tough-set for $P_k \times P_3$. We consider cases, based upon the portion of $\{(1,1),(1,2),(1,3)\}$ in S.

Case 1: just $(1,2) \in S$. It follows from the Separation Rule for $(1,2)$ in $P_k \times P_3$ that $(1,1)$ and $(1,3)$ must be in distinct components of $(P_k \times P_3) \setminus S$. Since $(1,1)$ and $(1,3)$ collapse to the same component of $(P_\infty \times P_3) \setminus S$, this contradicts the fact that S is a tough-set for $P_\infty \times P_3$, also having toughness 1.

Case 2: just $(1,1) \in S$. By the Separation Rule for $(1,1)$, we must have $(1,2)$ and $(2,1)$ in distinct components of $(P_\infty \times P_3) \setminus S$. Letting

$S' = S \setminus \{(1,1)\}$, we see that $\omega((P_\infty \times P_3) \setminus S') = \omega((P_\infty \times P_3) \setminus S) - 1$. However,

$$\frac{|S'|}{\omega((P_\infty \times P_3) \setminus S')} = \frac{|S|}{\omega((P_\infty \times P_3) \setminus S)} = 1,$$

and S' is seen to be a tough-set for $P_\infty \times P_3$ of smaller size that S, a contradiction.

Case 3: $(1,1),(1,2) \in S$. This contradicts the Separation Rule for $(1,1)$ in $P_k \times P_3$.

Case 4: just $(1,1),(1,3) \in S$. It follows from the Separation Rule that $(2,2) \in S$. Letting $S' = S \setminus \{(1,1),(1,3)\}$, we see that $\omega((P_\infty \times P_3) \setminus S') \geq \omega((P_\infty \times P_3) \setminus S) - 2$. However,

$$\frac{|S'|}{\omega((P_\infty \times P_3) \setminus S')} \leq \frac{|S|}{\omega((P_\infty \times P_3) \setminus S)} = 1,$$

which contradicts the assumption that S is a tough-set of smallest possible size.

Since any other case is symmetric to those considered, we see that S cannot be a tough-set. □

In light of the arguments used to establish upper bounds in Theorems 3.4 and 3.5, we extend the definition of a tough-set to allow infinite tough-sets.

Definition 4.3. Given a graph G, call an infinite separating set $S \subseteq V(G)$ an *infinite tough-set* if there is a sequence of finite subsets $S_1 \subseteq S_2 \subseteq \cdots \subseteq S$ such that S_k is a separating set for k sufficiently large and

$$\tau(G) = \lim_{k \to \infty} \frac{|S_k|}{\omega(G \setminus S_k)}.$$

Based on Definition 4.3, $P_\infty \times P_3$ has the infinite tough-set

$$S = \{(i,j) : i,j \in \mathbb{Z},\ 1 \leq j \leq 3,\ \text{and } i \not\equiv j \pmod 2\}.$$

Moreover, we see from the proofs of Theorems 3.4 and 3.5 that each of the graphs therein has a tough-set or an infinite tough-set.

Question 4.4. *What characterizes those graphs which have a finite tough-set, those graphs which only have an infinite tough-set, and those graphs with no tough-sets of any kind?*

5 Possible Toughness Values

The toughness of a finite connected graph is obviously a positive rational number. In fact, any such number is achievable by both a finite graph and an infinite graph.

Theorem 5.1. *For any positive rational number q, there is a non-complete finite graph G such that $\tau(G) = q$.*

Proof. Write $q = a/b$ for integers a and b with $b > 1$. From Theorem 3 of [7], we see that the join $K_a + E_b$ of the complete graph on a vertices and the empty graph on b vertices has $\tau(K_a + E_b) = a/b = q$. □

Theorem 5.2. *For any nonnegative rational number q, there is a connected infinite graph G such that $\tau(G) = q$.*

Proof. In light of Example 3.3, it remains to consider rational $q > 0$. Write $q = a/b$ for integers a and b with $b > 1$. Note that the join $K_a + (E_b \times K_a)$ provides an alternative to the construction used in the finite case. We extend this construction to build an infinite graph. First, let H be the quotient graph of $P_\infty^+ \times E_b$ obtained by identifying all vertices of the form $(1, j)$ to a single vertex v. To form G, we replace each vertex of H by a copy of K_a. Two vertices in G are adjacent if and only if they come from the same vertex of H or come from adjacent vertices in H. The set S of vertices in G corresponding to the vertex v from H forms a separating set that establishes $\tau(G) \leq a/b = q$. Notice that G is a-connected and that the largest integer m such that the complete bipartite graph $K_{1,m}$ is an induced subgraph of G is b. Hence, Theorem 2.2 gives that $\tau(G) \geq a/b = q$. □

Chvátal [1] conjectures that there is a t such that any t-tough graph must be Hamiltonian. It follows from Theorem 5.2 that there exist infinite, and hence obviously non-Hamiltonian, graphs with arbitrarily high toughness.

Question 5.3. *Does there exist an infinite graph with irrational toughness?*

If the answer to Question 5.3 is yes, then what irrational numbers are achievable? Is there some restriction to algebraic numbers?

6 Maximum Toughness

The determination of the maximum possible toughness among finite graphs with a fixed number of vertices and edges has been considered extensively [2, 3, 5]. In that work, the possible values for r-regular graphs are the most significant.

Question 6.1. *What is the maximum possible toughness among r-regular infinite graphs?*

It is easy to see that the answer to Question 6.1 in the case that $r = 2$ is 1/2, the toughness value achieved by P_∞. Note that this is less than the known value 1 when $r = 2$ in the analog of Question 6.1 for finite graphs [5]. Since $P_\infty \times K_{r-1}$ is r-regular, we know from Theorem 3.7 that the answer to Question 6.1 is at least $r/3$. However, in the finite case [2], the value $r/2$ is generally achievable.

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