On Cordial Labelings of the Second Power of Paths with Other Graphs

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Abstract. A graph is said to be cordial if it has a 0-1 labeling that satisfies certain properties. The second of paths P_n^2, is the graph obtained from the path P_n by adding edges that join all vertices u and v with $d(u, v) = 2$. In this paper, we show that certain combinations of second power of paths, paths, cycles and stars are cordials. Specifically, we investigate the cordiality of the join and the union of pairs of second power of paths and graphs consisting of one second power of path and one path and one cycle.
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1. Introduction.

It is well known that graph theory has applications in many other fields of study, including physics, chemistry, biology, communication, psychology, sociology, economics, engineering, operator research, and especially computer science.

One area of graph theory of considerable recent research is that of graph labeling. Labeled graphs serve as useful models for a broad range of applications such as : coding theory, X- ray crystallography, radar, circuit design, communication network addressing and data base management.

In a labeling of a particular type, the vertices are assigned values from a given set, the edges have a prescribed induced labeling, and the labeling must satisfy certain properties. An excellent reference on this subject in the survey by Gallian [4].

Two of the most important types of labelings are called graceful and harmonious. Graceful labelings were introduced independently by Rosa [7] in 1966 and Golomb [5] in 1972, while harmonious labelings were first studied by Graham and Sloane [6] in1980. A third important type of labeling, which contains aspects of both of the other two, is called cordial and was introduced by Cahit [1] in 1990.Whereas the label of an edge u w for graceful and harmonious labelings is given respectively by $|f(v) - f(w)|$ and $f(v) + f(w)$ (modulo the number of edges), cordial labelings use only labels 0 and 1 and the induced label $(f(v) + f(w))(\text{mod } 2)$, which of course equals $|f(v) - f(w)|$.

Because arithmetic modulo 2 is an integral part of computer science, cordial labelings have close connections with that field.

More precisely, cordial graphs are defined as follows.

Let $G = (V, E)$ be a graph, let $f : V \to \{0,1\}$ be a labeling of its vertices, and let $f^* : E \to \{0,1\}$ be the extension of f to the edges of G by the formula $f^*(vw) = f(v) + f(w)(\mathrm{mod} 2)$. Thus, for any edge e, $f^*(e) = 0$ if its two vertices have the same label and $f^*(e) = 1$ if they have different labels). Let v_0 and v_1 be the numbers of vertices labeled 0 and 1 respectively, and let e_0 and e_1 be the corresponding numbers of edges. Such a labeling is called cordial if both $|v_0 - v_1| \leq 1$ and $|e_0 - e_1| \leq 1$. A graph is called cordial if it has a cordial labeling.

Given two disjoint graphs G and H, their union $G \cup H$ is simply the unions of their sets of vertices and edges, while their join $G + H$ is obtained from $G \cup H$ by adding all edges that join a vertex of G to a vertex of H.

The second power of paths P_n^2, is the graph obtained from the path P_n by adding edges that join all vertices u and v with $d(u,v) = 2$. So, the order of the second power of paths P_n^2 is n and its size is $2n-3$, in particular $P_1^2 = P_1$, $P_2^2 = P_2$ and $P_3^2 = C_3$.

The main object of this paper is to extend some important results on paths P_n to the second power of paths P_n^2. Specifically, in [2,3,8], we determined that the join of two paths P_n and P_m is cordial for all n and all m except for $P_2 + P_2$, that the join of the path P_n and the cycle C_m is cordial for all n and all m except for $(m,n) = (3,1), (3,2)$ and $(3,3)$, that the union of two paths P_n and P_m is cordial for all n and all m except $P_2 \cup P_2$, that the union of the path P_n and the cycle C_m is cordial for all n and all m if and only if it is not isomorphic to $P_1 \cup C_m$ with $m \equiv 2 \pmod 4$. In section 3, we show that the join of pairs of second power of paths P_n^2 and P_m^2 is cordial for all n and all m except for $(n,m) = (1,3), (3,1), (2,3), (3,2), (2,4), (4,2), (3,4), (4,3), (2,2), (3,3)$ and $(4,4)$. Also, we show that the union of pairs of second power of paths P_n^2 and P_m^2 is cordial for all n and all m except for $(n,m) = (2,2)$ and $(3,3)$. In section 4,we show that the join $P_n^2 + P_m$ of the second power of the path P_n^2 and the path P_m is cordial for all n and all m except for $(n,m) = (3,1), (3,2), (2,2), (3,3)$ and $(4,2)$, and we also show that the union $P_n^2 \cup P_m$ of the second power of the path P_n^2 and the path P_m is cordial for all n and all m except for $(n,m) = (2,2)$. In section 5, we show that show that the join $P_n^2 + C_m$ of the second power of the path P_n^2 and a cycle C_m is cordial for all n and all m if and only if $(n,m) \neq (1,3), (2,3)$ and $(3,3)$. Also, we prove that if $(n,m) \neq (3,3)$ and $P_n^2 \cup C_m$ is not isomorphic to $P_1 \cup C_m$ with $m \equiv 2 \pmod 4$, then the union $P_n^2 \cup C_m$ of the second power of the path P_n^2 and a cycle C_m is cordial for all n and all m.

2.Terminology and notations. We introduce some notation and terminology for a graph with $4r$ vertices, we let L_{4r} denote the labeling 00110011...0011, S_{4r} denote the labeling 11001100...1100, ,M_r denotes the labelling 0101...r-times (zero- one repeated r- times), M_r' denotes the labelling 10101...r-times (one-zero repeated r- times), O_r denotes the labelling

0000...0000 (zero repeated r- times) and I_r denotes the labelling 111...1111 (one repeated r- times) ,i.e., O_5 is 00000 , I_5 is 11111 , M_5 is 01010 , M_6 is 010101(This means that if r is odd number ,then M_r is 010...01010 and if r is even number, then M_r is 010...010101) ,M_5' is 10101 and M_6' is 101010. In most cases, we then modify this by adding symbols at one end or the other (or both). Thus $01L_{4r}$ denotes the labeling 0100110011...0011 of either P_{4r+2}^2 or C_{4r+2} or P_{4r+2} or .... For specific labeling L and M of $G \cup H$ and $G+H$, where G and H are second power of paths or paths and cycles, we let $[L; M]$ denote the joint labeling.

Throughout of this paper all graphs are finite and simple, and we use the additional notation that is the following. For a given labeling of the join $G+H$ or the union $G \cup H$, we let v_i and e_i (for $i = 0, 1$) be the numbers of labels that are i as before, we let x_i and a_i be the corresponding quantities for G, and we let y_i and b_i be those for H. It follows that $v_0 = x_0 + y_0, v_1 = x_1 + y_1$, $e_0 = a_0 + b_0 + x_0 y_0 + x_1 y_1$ and $e_1 = a_1 + b_1 + x_0 y_1 + x_1 y_0$, thus $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1)$. When it comes to the proof, we only need to show that, for each specified combination of labeling, $|v_0 - v_1| \leq 1$ and $|e_0 - e_1| \leq 1$.

3. Joins and Unions of Pairs of Second Power of Paths.

In [8, 4], we determined that the second power of paths P_n^2 is cordial for all n, that the join of two paths P_n and P_m is cordial for all n and all m except for $P_2 + P_2$ and that the union of two paths P_n and P_m is cordial for all n and all m except for $(n, m) = (2, 2)$. In this section, we extend the above result to show that the join of pairs of second power of paths P_n and P_m is cordial for all n and all m except for $(n, m) = (1, 3), (3, 1), (2, 3), (3, 2), (2, 4), (4, 2), (3, 4), (4, 3), (2, 2), (3, 3)$ and $(4, 4)$. Moreover, we show that the union of pairs of second power of paths P_n^2 and P_m^2 is cordial for all n and all m except for $(n, m) = (2, 2)$ and $(3, 3)$.

Lemma 3.1. If $1 \leq n \leq 6$ and $1 \leq m \leq 6$, then the join $P_n^2 + P_m^2$ of second power of paths P_n^2 and P_m^2 is cordial for all n and all m except for $(n, m) = (1, 3), (3, 1), (2, 3), (3, 2), (2, 4), (4, 2), (3, 4), (4, 3), (2, 2), (3, 3)$ and $(4, 4)$.

Proof. From the fact that the graph $P_n^2 + P_m^2$ is isomorphic to the graph $P_m^2 + P_n^2$, we can consider the cases of either n or m separately.

Case1. $n = 1$. The following labelings is suffice.

$P_1^2 + P_1^2 = P_1 + P_1 : [0; 1], P_1^2 + P_2^2 = P_1 + P_2 : [1; 00], P_1^2 + P_4^2 = P_1 + P_4^2 : [0; 0011]$,

$P_1^2 + P_5^2 = P_1 + P_5^2 : [0; 00111]$ and $P_1^2 + P_6^2 = P_1 + P_6^2 : [0; 010101]$.

Case 2. $n = 2$. The following labelings is suffice.

$P_2^2 + P_5^2 = P_2 + P_5^2 : [01; 00111]$ and $P_2^2 + P_6^2 = P_2 + P_6^2 : [00; 001111]$.

Case 3. $n = 3$. The following labelings is suffice.

$P_3^2 + P_5^2 = C_3 + P_5^2 : [001; 00111]$ and $P_3^2 + P_6^2 = C_3 + P_6^2 : [001; 001111]$.

Case 4. $n = 4$. The following labelings is suffice.
$P_5^2 + P_5^2$: [0101; 00111] and $P_5^2 + P_6^2$: [0001; 001111].
Case 5. $n = 5$. The following labelings is suffice.
$P_5^2 + P_5^2$: [10101; 00011] and $P_5^2 + P_6^2$: [00011; 010101].
Case 6. $n = 6$. The following labelings is suffice.
$P_6^2 + P_6^2$: [001111; 101000], the lemma follows.

Lemma 3.2. If $1 \leq n \leq 6$ and $m \geq 7$, then the join $P_n^2 + P_m^2$ of second power of paths P_n^2 and P_m^2 is cordial for all n and all m.

Proof. Without loss of generality, we assume that $1 \leq n \leq 6$ and $m = 4s+j$ (for $j = 0, 1, 2, 3$), then for given values of n and j with $1 \leq n \leq 6$ and $0 \leq j \leq 3$, we use the labeling A_n for P_n^2 and B_j for P_m^2 as given in Table3.1. Using Table 3.1 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1)$, we can compute the values shown in the last two columns of Table 3.2. Since these are all 0,1, or -1, the lemma follows.

$1 \leq n \leq 6$	Labeling of P_n^2	x_0	x_1	a_0	a_1
$n = 1$	$A_1 = 1$	0	1	0	0
$n = 2$	$A_2 = 01$	1	1	0	1
$n = 3$	$A_3 = 011$	1	2	1	2
$n = 4$	$A_4 = 0011$	2	2	2	3
$n = 5$	$A_5 = 10101$	2	3	3	4
$n = 6$	$A_6 = 010101$	3	3	4	5

$m = 4s + j$, $j = 0, 1, 2, 3$	Labeling of P_m^2	y_0	y_1	b_0	b_1
$j = 0$	$B_0 = O_3I_3M_{4s-6}, s > 1$	$2s$	$2s$	$4s - 1$	$4s - 2$
$j = 1$	$B_1 = O_3I_3M_{4s-6}, 0, s > 1$	$2s + 1$	$2s$	$4s$	$4s - 1$
$j = 2$	$B_2 = O_3I_3M_{4s-4}, s > 1$	$2s + 1$	$2s + 1$	$4s + 1$	$4s$
$j = 3$	$B_3 = O_3I_3M_{4s-4}, 0, s \geq 1$	$2s + 2$	$2s + 1$	$4s + 2$	$4s + 1$

Table 3.1. Labelings of P_n^2 and P_m^2

$1 \leq n \leq 6$	$m = 4s + j$, $j = 0, 1, 2, 3$	P_n^2	P_m^2	$v_0 - v_1$	$e_0 - e_1$
1	0	A_1	B_0	-1	1
1	1	A_1	B_1	0	0
1	2	A_1	B_2	-1	1
1	3	A_1	B_3	0	0
2	0	A_2	B_0	0	0
2	1	A_2	B_1	1	0
2	2	A_2	B_2	0	0
2	3	A_2	B_3	1	0
3	0	A_3	B_0	-1	0
3	1	A_3	B_1	0	-1
3	2	A_3	B_2	-1	0
3	3	A_3	B_3	0	-1
4	0	A_4	B_0	0	0
4	1	A_4	B_1	1	0
4	2	A_4	B_2	0	0
4	3	A_4	B_3	1	0
5	0	A_5	B_0	-1	0
5	1	A_5	B_1	0	-1
5	2	A_5	B_2	-1	0
5	3	A_5	B_3	0	-1
6	0	A_6	B_0	0	0
6	1	A_6	B_1	1	0
6	2	A_6	B_2	0	0
6	3	A_6	B_3	1	0

Table 3.2.Combinations of labelings.

Lemma 3.3. If $n \geq 7$ and $m \geq 7$, then the join $P_n^2 + P_m^2$ of second power of paths P_n^2 and P_m^2 is cordial for all n and all m.

Proof. We let $n = 4r+i$ (for $i = 0, 1, 2, 3$) and $m = 4s+j$ (for $j = 0, 1, 2, 3$), then for given values of i and j with $0 \leq i \leq 3$ and $0 \leq j \leq 3$, we use the labeling A_i or A'$_i$ for P_n^2 and B_j for P_m^2 as given in Table 3.3. Using Table 3.3 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1)$, we can compute the values shown in the last two columns of Table 3.4. Since these are all 0,1, or -1, the lemma follows.

$n = 2r+i$, $i = 0, 1$	Labeling of P_n^2	x_0	x_1	a_0	a_1
$i = 1$	$A_0 = M_{2r}$	r	r	$2r-2$	$2r-1$
$i = 2$	$A_1 = M_{2r}0$,	$r+1$	r	$2r-1$	$2r$
	or$A_1' = M_{2r}1$	r	$r+1$	$2r-1$	$2r$

$m = 4s+j$, $j = 0, 1, 2, 3$	Labeling of P_m^2	y_0	y_1	b_0	b_1
$j = 0$	$B_0 = O_3I_3M_{4s-6}, s > 1$	$2s$	$2s$	$4s-1$	$4s-2$
$j = 1$	$B_1 = O_3I_3M_{4s-6}, 0, s > 1$	$2s+1$	$2s$	$4s$	$4s-1$
$j = 2$	$B_2 = O_3I_3M_{4s-4}, s > 1$	$2s+1$	$2s+1$	$4s+1$	$4s$
$j = 3$	$B_3 = O_3I_3M_{4s-4}, 0, s \geq 1$	$2s+2$	$2s+1$	$4s+2$	$4s+1$

Table 3.3. Labelings of P_n^2 and P_m^2.

$n = 2r+i$, $i = 0, 1$,	$m = 4s+j$, $j = 0, 1, 2, 3$	P_n^2	P_m^2	$v_0 - v_1$	$e_0 - e_1$
0	0	A_0	B_0	0	0
0	1	A_0	B_1	1	0
0	2	A_0	B_3	0	0
0	3	A_0	B_3	1	0
1	0	A_1	B_0	1	0
1	1	A_1'	B_1	0	-1
1	2	A_1'	B_2	-1	0
1	3	A_1'	B_3	0	-1

Table 3.4. Combinations of labelings.

It should be remarked that the complete graph K_n [4] is cordial if and only if $n \leq 3$. Therefore $P_2^2+P_2^2 = P_2+P_2 \equiv K_4, P_1^2+P_3^2 = P_1+C_3 \equiv K_4$, $P_2^2+P_3^2 = P_2+C_3 \equiv K_5$ and $P_3^2+P_3^2 = C_3+C_3 \equiv K_6$ are not cordials.

Theorem 3.1. The join $P_n^2 + P_m^2$ of second power of paths P_n^2 and P_m^2 is cordial for all n and all m except for $(n,m) = (1,3), (3,1), (2,3), (3,2), (2,4), (4,2), (3,4), (4,3), (2,2), (3,3)$ and $(4,4)$.

Proof. The proof follows directly from lemma 3.1, lemma 3.2 and lemma 3.3, the theorem follows.

Lemma 3.4. If $1 \leq n \leq 6$ and $1 \leq m \leq 6$, then the join $P_n^2 \cup P_m^2$ of second power of paths P_n^2 and P_m^2 is cordial except for $(n,m) = (2,2)$ and $(3,3)$.

Proof. From the fact that the graph $P_n^2 \cup P_m^2$ is isomorphic to the graph $P_m^2 \cup P_n^2$, we can consider the cases of either n or m separately.

<u>Case 1.</u> $n = 1$. The following labelings is suffice.

$P_1^2 \cup P_1^2 = P_1 \cup P_1 : [0;1], P_1^2 \cup P_2^2 = P_1 \cup P_2 : [1;01], P_1^2 \cup P_3^2 = P_1 \cup C_3 :$ $[1;001]$,

$P_1^2 \cup P_4^2 = P_1 \cup P_4^2 : [0;0101], P_1^2 \cup P_5^2 = P_1 \cup P_5^2 : [0;00111]$ and $P_1^2 \cup P_6^2 = P_1 \cup P_6^2 : [0;010101]$.

Case 2. $n = 2$. The following labelings is suffice.

$P_2^2 \cup P_3^2 = P_2 \cup C_3 : [00;011], P_2^2 \cup P_4^2 = P_2 \cup P_4^2 : [11;0010]$, $P_2^2 \cup P_5^2 = P_2 \cup P_5^2 : [01;00111]$ and $P_2^2 \cup P_6^2 = P_2 \cup P_6^2 : [00;011101]$.

Case 3. $n = 3$. The following labelings is suffice.

$P_3^2 \cup P_4^2 = C_3 \cup P_4^2 : [011;0001], P_3^2 \cup P_5^2 = C_3 \cup P_5^2 : [011;00011]$ and $P_3^2 \cup P_6^2 : [111;010110]$.

Case 4. $n = 4$. The following labelings is suffice.

$P_4^2 \cup P_4^2 : [0001;1011], P_4^2 \cup P_5^2 : [0100;00111]$ and $P_4^2 \cup P_6^2 : [1011;101000]$.

Case 5. $n = 5$. The following labelings is suffice.

$P_5^2 \cup P_5^2 : [10101;00011]$ and $P_5^2 \cup P_6^2 : [00011;010101]$.

Case 6. $n = 6$. The following labelings is suffice.

$P_6^2 \cup P_6^2 : [011101;101000]$, the lemma follows.

Lemma 3.5. If $1 \le n \le 6$ and $m \ge 7$, then the union $P_n^2 \cup P_m^2$ of second power of paths P_n^2 and P_m^2 is cordial.

Proof. By using the label of vertices of P_n^2 (for $1 \le n \le 6$) and P_m^2 as given in Table 3.1 of lemma 3.2 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1)$, we can compute the values shown in the last two columns of Table 3.5. Since these are all 0,1, or -1, the lemma follows.

$1 \leq n \leq 6$	$m = 4s + j$, $j = 0, 1, 2, 3$	P_n^2	P_m^2	$v_0 - v_1$	$e_0 - e_1$
1	0	A_1	B_0	-1	1
1	1	A_1	B_1	0	1
1	2	A_1	B_2	-1	1
1	3	A_1	B_3	0	1
2	0	A_2	B_0	0	0
2	1	A_2	B_1	1	0
2	2	A_2	B_2	0	0
2	3	A_2	B_3	1	0
3	0	A_3	B_0	-1	0
3	1	A_3	B_1	0	0
3	2	A_3	B_2	-1	0
3	3	A_3	B_3	0	0
4	0	A_4	B_0	0	0
4	1	A_4	B_1	1	0
4	2	A_4	B_2	0	0
4	3	A_4	B_3	1	0
5	0	A_5	B_0	-1	0
5	1	A_5	B_1	0	0
5	2	A_5	B_2	-1	0
5	3	A_5	B_3	0	0
6	0	A_6	B_0	0	0
6	1	A_6	B_1	1	0
6	2	A_6	B_2	0	0
6	3	A_6	B_3	1	0

Table 3.5. Combinations of labelings.

Lemma 3.6. If $n \geq 7$ and $m \geq 7$, then the union $P_n^2 \cup P_m^2$ of second power of paths P_n^2 and P_m^2 is cordial for all n and all m.
Proof. By using the label of vertices of P_n^2 and P_m^2as given in Table 3.3 of lemma 3.3 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1)$, we can compute the values shown in the last two columns of Table 3.6. Since these are all 0,1, or -1,the lemma follows.

$n=2r+i$, $i=0,1$,	$m=4s+j$, $j=0,1,2,3$	P_n^2	P_m^2	v_0-v_1	e_0-e_1
0	0	A_0	B_0	0	0
0	1	A_0	B_1	1	0
0	2	A_0	B_2	0	0
0	3	A_0	B_3	1	0
1	0	A_1	B_0	1	0
1	1	A_1'	B_1	0	0
1	2	A_1'	B_2	-1	0
1	3	A_1'	B_3	0	0

Table 3.6. Combinations of labelings.

Theorem 3.2. The union $P_n^2 \cup P_m^2$ of second power of paths P_n^2 and P_m^2 is cordial for all n and all m except for $(n,m)=(2,2)$ and $(3,3)$.
Proof. The proof follows directly from lemma 3.4, lemma 3.5 and lemma 3.6, the theorem follows.
4. Joins and Unions of Second Power of Paths and Paths. Diab [2], has proved that the join $C_m + P_n$ of the cycle C_m and the path P_n is cordial for all n and all m if and only if $(n,m) \neq (3,1), (3,2)$ and $(3,3)$, Seoud, Diab and Elsakhawi [8], have proved $P_n + P_m$ is cordial for all n and all m except for $(n,m)=(2,2)$, and Diab and Elsakhawi [3], have shown that $P_n \cup P_m$ is cordial unless $n=m=2$ and $C_n \cup P_m$ is cordial if and only if it is not isomorphic to $P_1 \cup C_n$ with $n \equiv 2 \pmod 4$. In this section, we extend the above results to show that $P_n^2 + P_m$ is cordial for all n and all m except for $(n,m)=$ (3,1), (3,2), (2,2), (3,3) and (4,2). Moreover, we show that $P_n^2 \cup P_m$ is cordial for all n and all m except $(n,m)=$ (2,2).
Lemma 4.1. If $n \geq 4$ and $m \geq 4$, then the join $P_n^2 + P_m$ of the second power of the path P_n^2 and the path P_m is cordial.
Proof. The labelings that we use are given in Table 4.1, along with the corresponding values of x_i and a_i or y_i and b_i (for $i=$ 0,1). We let $n=2r+i$ (for $i=$ 0,1) and $m=4s+j$ (for $j=$ 0,1,2,3), then for given values of i and j with $0 \leq i \leq 1$ and $0 \leq j \leq 3$, we use the labeling A_i or A_i' for P_n^2 and B_j or B_j' for the path P_m as given in Table 4.1. Using Table 4.1 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1)$, we can compute the values shown in the last two columns of Table 3.2. Since these are all 0,1, or -1, the lemma follows.

$n=2r+i$, $i=0,1,$	Labeling of P_n^2	x_0	x_1	a_0	a_1
$i=1$	$A_0=M_{2r}$	r	r	$2r-2$	$2r-1$
$i=2$	$A_1=M_{2r}0$,	$r+1$	r	$2r-1$	$2r$
	or $A_1'=M_{2r}1$,	r	$r+1$	$2r-1$	$2r$

$m=4s+j$, $j=0,1,2,3,$	Labeling of P_m	y_0	y_1	b_0	b_1
$j=0$	$B_0=L_{4s}$	$2s$	$2s$	$2s$	$2s-1$
$j=1$	$B_1=L_{4s}0$,	$2s+1$	$2s$	$2s$	$2s$
	or $B_1'=L_{4s}1$,	$2s$	$2s+1$	$2s+1$	$2s-1$
$j=2$	$B_2=10L_{4s}$	$2s+1$	$2s+1$	$2s+1$	$2s$
$j=3$	$B_3=L_{4s}001$,	$2s+2$	$2s+1$	$2s+1$	$2s+1$
	or $B_3'=S_{4s}011$,	$2s+1$	$2s+2$	$2s+2$	$2s$

Table 4.1. labelings of P_n^2 and P_m.

$n=2r+i$, $i=0,1,$	$m=4s+j$, $j=0,1,2,3$	P_n^2	P_m	v_0-v_1	e_0-e_1
0	0	A_0	B_0	0	0
0	1	A_0	B_1	1	-1
0	2	A_0	B_2	0	0
0	3	A_0	B_3	1	-1
1	0	A_1	B_0	1	0
1	1	A_1	B_1'	0	0
1	2	A_1'	B_2	-1	0
1	3	A_1	B_3'	0	0

Table 4.2. Combinations of labelings.

Lemma 4.2. If $1 \le n \le 3$ and $1 \le m \le 3$, then $P_n^2 + P_m$ is cordial for all n and all m except for (n,m) = (3,1), (3,2), (2,2) and (3,3).
Proof. The proof follows directly from the fact that $P_1^2 = P_{1,P_2^2} = P_2$ and the following theorem [8], which state that $P_n + P_m$ is cordial for all n and all m except (n,m) = (2,2), the lemma follows.
Lemma 4.3. If $1 \le n \le 3$ and $m \ge 4$, then $P_n^2 + P_m$ is cordial for all m.
Proof. We let $m4s+j$ (for j = 0,1,2,3), then for given values of n and j with $1 \le n \le 3$ and $0 \le j \le 3$, we use the labeling A_n or A_n' for P_n^2 and B_j or B_j' for P_m as given in Table 4.3. Using Table 4.3 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1)$, we can compute the values shown in the last two columns of Table 4.4. Since these are all 0,1, or -1, the lemma follows.

$1 \le n \le 3$	Labeling P_n^2	x_0	x_1	a_0	a_1
$n=1$	$A_1 = 1$	0	1	0	0
$n=2$	$A_2 = 01$	1	1	0	1
$n=3$	$A_3 = 011$,	1	2	1	2
	or $A_3' = 001$	2	1	1	2

$n = 4s+j, s > 1$ $j = 0,1,2,3,$	Labeling of P_m	x_0	x_1	a_0	a_1
$j=0$	$B_0 = L_{4s}$	$2s$	$2s$	$2s$	$2s-1$
$j=1$	$B_1 = L_{4s}0$,	$2s+1$	$2s$	$2s$	$2s$
	or $B_1' = L_{4s}1$,	$2s$	$2s+1$	$2s+1$	$2s-1$
$j=2$	$B_2 = 10L_{4s}$	$2s+1$	$2s+1$	$2s+1$	$2s$
$j=3$	$B_3 = L_{4s}001$,	$2s+2$	$2s+1$	$2s+1$	$2s+1$
	or $B_3' = S_{4s}011$,	$2s+1$	$2s+2$	$2s+2$	$2s$

Table 4.3. labelings of P_n^2 and P_m.

$1 \le n \le 3$	$m = 4s+j$ $j = 0,1,2,3$	P_n^2	P_m	$v_0 - v_1$	$e_0 - e_1$
1	0	A_1	B_0	-1	1
1	1	A_1	B_1	0	-1
1	2	A_1	B_2	-1	1
1	3	A_1	B_3	0	-1
2	0	A_2	B_0	0	0
2	1	A_2	B_1	1	-1
2	2	A_2	B_2	0	0
2	3	A_2	B_3	1	-1
3	0	A_3	B_0	-1	0
3	1	A_3'	B_1'	0	0
3	2	A_3	B_2	-1	0
3	3	A_3'	B_3'	0	0

Table 4.4. Combination of labelings.

Lemma 4.3. If $n \ge 4$ and $1 \le m \le 3$, then $P_n^2 + P_m$ is cordial except for $(n,m) = (4,2)$.

__Proof.__ For labelings of P_n^2, where $n \ge 4$, we have two cases :

__Case 1.__ $4 \le n \le 6$ and $1 \le m \le 3$. The following labelings is suffice. $P_4^2 + P_1$: [0011;0], $P_4^2 + P_3$: [0011;001], $P_5^2 + P_1$: [00111;0], $P_5^2 + P_2$:[00111;01], $P_5^4 + P_3$: [00111;001], $P_6^2 + P_1$: [010101;0], $P_6^2 + P_2$: [001111;00] and $P_6^2 + P_3$: [001111;001].

__Case 2.__ $n \ge 7$ and $1 \le m \le 3$.

We let $n = 4r + j$ (for j = 0,1,2,3) and $1 \le m \le 3$, then for given values of j and m with $1 \le m \le 3$ and $0 \le j \le 3$, we use the labeling A_n for P_n^2 and B_m for P_m as given in Table 4.5. Using Table 4.5 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1)$, we can compute the values shown in the last two columns of Table 4.6. Since these are all 0,1, or -1, the lemma follows.

$n = 4r + j$, $j = 0,1,2,3$	Labeling of P_n^2	x_0	x_1	a_0	a_1
$j = 0$	$A_0 = O_3 I_3 M_{4r-6}, r > 1$	$2r$	$2r$	$4r - 1$	$4r - 2$
$j = 1$	$A_1 = O_3 I_3 M_{4r-6} 0, r > 1$	$2r + 1$	$2r$	$4r$	$4r - 1$
$j = 2$	$A_2 = O_3 I_3 M_{4r-4}, r > 1$	$2r + 1$	$2r + 1$	$4r + 1$	$4r$
$j = 3$	$A_3 = O_3 I_3 M_{4r-4} 0, r \ge 1$	$2r + 2$	$2r + 1$	$4r + 2$	$4r + 1$

$1 \le m \le 3$,	Labeling of P_m	y_0	y_1	b_0	b_1
$m = 1$	$B_1 = 0$,	1	0	0	0
	or $B_1' = 1$	0	1	0	0
$m = 2$	$B_2 = 01$	1	1	0	1
$m = 3$	$B_3 = 011$	1	2	1	1

Table 4.5. labelings of P_n^2 and P_m.

$n = 4r + j$, $j = 0,1,2,3$	$1 \le m \le 3$	P_n^2	P_m	$v_0 - v_1$	$e_0 - e_1$
0	1	A_0	B_1	1	1
1	1	A_1	B_1'	0	0
2	1	A_2	B_1	1	1
3	1	A_3	B_1'	0	0
0		A_0	B_2	0	0
1	2	A_1	B_2	1	0
2	2	A_2	B_2	0	0
3	2	A_3	B_2	1	0
0	3	A_0	B_3	-1	1
1	3	A_1	B_3	0	0
2	3	A_2	B_3	-1	1
3	3	A_3	B_3	0	0

Table 4.6 Combination of labelings.

Theorem 4.1 The join $P_n^2 + P_m$ of the second power of the path P_n^2 and the path P_m is cordial for all n and all m except for $(n, m) =$ (3,1), (3,2), (2,2), (3,3) and (4,2).

Proof. The proof follows directly from lemma 4.1, lemma 4.2, lemma 4.3

and lemma 4.4, the theorem follows.

Lemma 4.5. If $n \geq 4$ and $m \geq 4$, then the union $P_n^2 \cup P_m$ of the second power of the path P_n^2 and the path P_m is cordial.

Proof. By using the labelings of P_n^2 and P_m as given in Table 4.1 of lemma 4.1.Using Table 4.1 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1)$, we can compute the values shown in the last two columns of Table 4.7. Since these are all 0,1, or -1, the lemma follows.

$n = 2r + i$, $i = 0, 1$	$m = 4s + j$, $j = 0, 1, 2, 3$	P_n^2	P_m	$v_0 - v_1$	$e_0 - e_1$
0	0	A_0	B_0	0	0
0	1	A_0	B_1	1	-1
0	2	A_0	B_2	0	0
0	3	A_0	B_3	1	-1
1	0	A_1	B_0	1	0
1	1	A_1	B_1'	0	1
1	2	A_1'	B_2	-1	0
1	3	A_1	B_3'	0	1

Table 4.7. Combinations of labelings.

Lemma 4.6. $P_3^2 \cup P_m = C_3 \cup P_m$ is cordial for all m.

Proof. The proof follows directly from the fact that $P_3^2 = C_3$ and the following theorem [3], which state that $C_n \cup P_m$ is cordial if and only if it is not isomorphic to $P_1 \cup C_n$ with $n \equiv 2 (\mathrm{mod} 4)$, the lemma follows.

Lemma 4.7. If n < 3, then the union $P_n^2 \cup P_m$ is cordial for all n and all m except for (n, m) =(2,2).

Proof. The proof follows directly from the fact that $P_1^2 = P_1$, $P_2^2 = P_2$ and the following theorem [3], which state that except $2P_2$, the union $P_n \cup P_m$ of two paths is cordial, the lemma follows.

Theorem 4.2. The union $P_n^2 \cup P_m$ of the second power of the path P_n^2 and the path P_m is cordial for all n and all m except for $(n, m) =$ (2,2).

Proof. The proof follows directly from lemma 4.5, lemma 4.6 and lemma 4.7, the theorem follows.

5. Joins and Unions of Second Power of Paths and Cycles.

Seoud, Diab and Elsakhawi [8], have proved that if n is not congruent to 2(mod 4) and m is not congruent to 0(mod 4), then the join $C_n + C_m$ of two cycles is cordial for all n and all m. Diab [2], has proved that the join $C_n + P_m$ of the cycle C_n and the path P_m is cordial for all n and all m if and only if $(n, m) \neq$ (1,3), (2,3) and (3,3). Also, Diab and Elsakhawi [3], have proved that the union $C_n \cup C_m$ of two cycles C_n and C_m is cordial if and only if $n + m$ is not congruent to 2 (mod 4), and the union $P_n \cup C_m$ is cordial if and only if it is not isomorphic to $P_1 \cup C_m$ with $m \equiv 2$ (mod 4).

In this section, we extend the above results to show that the join $P_n^2 + C_m$ of the second power of the path P_n^2 and a cycle C_m is cordial for all n and all m if and only if $(n, m) \neq$ (1,3), (2,3) and (3,3). Moreover, we prove that if $(n, m) \neq$ (3,3) and $P_n^2 \cup C_m$ is not isomorphic to $P_1 \cup C_m$ with $m \equiv$ 2(mod 4), then the union $P_n^2 \cup C_m$ of the second power of the path P_n^2 and a cycle C_m is cordial for all n and all m.

Lemma 5.1. If $n \geq 3$, then the join $P_n^2 + C_m$ of the second power of the path P_n^2 and the cycle C_m is cordial for all m except for $(n, m) =$ (3,3).

Proof. The labelings that we use are given in Table 5.1, along with the corresponding values of x_i and a_i or y_i and b_i (for $i =$ 0,1). We let $n = 2r+i$ (for $i =$ 0,1) and $m = 4s + j$ (for $j =$ 0,1,2,3), then for given values of i and j with $0 \leq i \leq 1$ and $0 \leq j \leq 3$, we use the labeling A_i or A'_i for P_n^2 and B_j for the cycle C_m as given in Table 5.1. Using Table 5.1 and the fact that $v_0 - v_1 = (x_0 - x_1) + +(y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1)$, we can compute the values shown in the last two columns of Table 5.2. Since these are all 0,1, or -1, the lemma follows.

$n = 2r + i$, $i = 0, 1$	Labeling of P_n^2	x_0	x_1	a_0	a_1
$i = 1$	$A_0 = M_{2r}, r > 1$	r	r	$2r - 2$	$2r - 1$
$i = 2$	$A_1 = M_{2r}0, r \geq 1$,	$r + 1$	r	$2r - 1$	$2r$
	or $A'_1 = M_{2r}1, r \geq 1$,	r	$r + 1$	$2r - 1$	$2r$

$m = 4s + j$, $j = 0, 1, 2, 3$,	Labeling of C_m	y_0	y_1	b_0	b_1
$j = 0$	$B_0 = L_{4s}$	$2s$	$2s$	$2s$	$2s$
$j = 1$	$B_1 = L_{4s}0$,	$2s + 1$	$2s$	$2s + 1$	$2s$
$j = 2$	$B_2 = 0L_{4s}1$	$2s + 1$	$2s + 1$	$2s + 2$	$2s$
$j = 3$	$B_3 = S_{4s}001$,	$2s + 2$	$2s + 1$	$2s + 2$	$2s + 1$

Table 5.1. labelings of P_n^2 and C_m.

$n = 2r + i$, $i = 0, 1$,	$m = 4s + j$, $j = 0, 1, 2, 3$	P_n^2	C_m	$v_0 - v_1$	$e_0 - e_1$
0	0	A_0	B_0	0	-1
0	1	A_0	B_1	1	0
0	2	A_0	B_2	0	1
0	3	A_0	B_3	1	0
1	0	A_1	B_0	1	-1
1	1	A_1'	B_1	0	-1
1	2	A_1	B_2	1	1
1	3	A_1'	B_3	0	-1

Table 5.2. Combinations of labelings.

Lemma 5.2. If n < 3, then the join $P_n^2 + C_m$ of the second power of the path P_n^2 and the cycle C_m is cordial for all m except for $(n, m) =$ (1,3) and (2,3).

Proof. The proof follows directly from the fact that $P_1^2 = P_1$, $P_2^2 = P_2$ and the following theorem [2], which state that the join $P_n + C_m$ of the path P_n and the cycle C_m is cordial for all n and all m if and only if $(n, m) \neq$ (1,3), (2,3) and (3,3), the lemma follows.

Example 5.1. The following graphs $P_1^2 + C_3, P_2^2 + C_3$ and $P_3^2 + C_3$ are not cordials.

Solution. It is well known that $P_1^2 = P_1, P_2^2 = P_2, P_3^2 = C_3$ and the complete graph K_n is cordial if and only if $n \leq 3$. Therefore $P_1^2 + C_3 = P_1 + C_3 K_4$, $P_2^2 + C_3 = P_2 + C_3 \equiv K_5$ and $P_3^3 + C_3 = C_3 + C_3 \equiv K_6$ are not cordials.

Theorem 5.1. The join $P_n^2 + C_m$ of the second power of the path P_n^2 and the cycle C_m is cordial for all n and m if and only if $(n, m) \neq$ (1,3), (2,3) and (3,3).

Proof. The proof follows directly from lemma 5.1 and lemma 5.2 and example 5.1, the theorem follows.

Lemma 5.3. If $n \geq 3$, then the union $P_n^2 \cup C_m$ of the second power of the path P_n^2 and the cycle C_m is cordial for all m except for $(n, m) =$ (3,3).

Proof. By using the labelings of the second power of the path P_n^2 and the cycle C_m as given in Table 5.1 of lemma 5.1, and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1)$, we can compute the values shown in the last two columns of Table 5.3. Since these are all 0,1, or -1, the lemma follows.

$n=2r+i$, $i=0,1,$	$m=4s+j$, $j=0,1,2,3$	P_n^2	C_m	v_0-v_1	e_0-e_1
0	0	A_0	B_0	0	-1
0	1	A_0	B_1	1	0
0	2	A_0	B_2	0	1
0	3	A_0	B_3	1	0
1	0	A_1	B_0	1	-1
1	1	A_1'	B_1	0	0
1	2	A_1	B_2	1	1
1	3	A_1'	B_3	0	0

Table 5.3. Combinations of labelings.

Lemma 5.4. If $n < 3$, then the union $P_n^2 \cup C_m$ of the second power of the path P_n^2 and the cycle C_m is cordial for all m if is not isomorphic to $P_1 \cup C_m$ with $m \equiv 2 (\bmod\ 4)$.
Proof. The proof follows directly from the fact that $P_1^2 = P_1$, $P_2^2 = P_2$ and the following theorem [3], which state that the union $P_n \cup C_m$ is cordial if and only if it is not isomorphic to $P_1 \cup C_m$ with $m \equiv 2 \pmod 4$, the lemma follows.
Theorem 5.2. If $(n, m) \neq (3,3)$ and $P_n^2 \cup C_m$ is not isomorphic to $P_1 \cup C_m$ with $m \equiv 2 (\bmod\ 4)$, then the union $P_n^2 \cup C_m$ of the second power of the path P_n^2 and a cycle C_m is cordial for all n and all m.
Proof. The proof follows directly from lemma 5.3 and lemma 5.4, the theorem follows.

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