The dimension of the kernel for intersections of certain starshaped sets in $\mathbb{R}^3$

Marilyn Breen

Abstract

Let $\mathcal{S}$ be a finite family of sets in $\mathbb{R}^d$, each a finite union of polyhedral sets at the origin and each having the origin as an extreme point. Fix d and $k, 0 \leq k \leq d \leq 3$. If every $d+1$ (not necessarily distinct) members of $\mathcal{S}$ intersect in a starshaped set whose kernel is at least k-dimensional, then $\cap\{S_i : S_i \text{ in } \mathcal{S}\}$ also is a starshaped set whose kernel is at least k-dimensional. For $k \neq 0$, the number $d+1$ is best possible.

Mathematics Subject Classification (2000):Primary 52.A30, 52.A35.
Keywords: Starshaped sets, intersections of starshaped sets, kernels.

1 Introduction.

We begin with some familiar definitions: Let S be a set in $\mathbb{R}^d$. A point s of S is called an *extreme point* of S if and only if s is not on any segment (a, b) contained in set S. For points x, y in S, we say *s sees y* (*x* is *visible* from y) via S if and only if the corresponding segment $[x, y]$ lies in S. A set S is called *starshaped* if and only if for some point p in S, p sees each point of S via S, and the set of all such points p is the (convex) *kernel* of S, denoted *ker* S. A set S is called a *cone* if and only if for some point v in S and for every point s in $S, s \neq v$, the associated ray $R(v, s)$ emanating from v through s lies in S. The point v is called a *vertex* of S. Clearly v belongs to *ker* S, so every cone will be starshaped.

Recent work by Bobylev [1] provides a starshaped set analogue of Helly's familiar theorem on intersections of convex sets. Since there are some well known Helly-type theorems that concern the dimension of an intersection of convex sets (see Grünbaum [6] and Katchalski [8]), it is reasonable to expect analogous results to hold for the dimension of the kernel in an intersection of starshaped sets. Some theorems in [2] and [3] offer these kinds of analogues for the planar case but establish few answers for higher dimensions. In this paper, we present a small result concerning the dimension of the kernel for intersections of certain starshaped sets in $\mathbb{R}^3$ and a Helly-type result for intersections of cones in $\mathbb{R}^d$.

Throughout the paper, *bdry* S and *conv* S will denote the boundary and convex hull, respectively, for set S. The reader may refer to Valentine [10] to Lay [9], to Danzer, Grünbaum, Klee [4], and to Eckhoff [5] for discussions on Helly-type theorem and starshaped sets.

2 Preliminary comments.

In this section we present some terminology and results from [2] and [3] that will be useful in subsequent arguments.

Fix d and $k, 0 \leq k \leq d$, and let $\mathcal{S}$ be a finite family of closed sets in $\mathbb{R}^d$. Assume that every $d+1$ (not necessarily distinct) members of $\mathcal{S}$ intersect in a starshaped set whose kernel is at least k-dimensional. Define set $S = \cap \{S_i : S_i \text{ in } \mathcal{S}\}$. By Bobylev's results [1], set S is nonempty and starshaped. Adapting Bobylev's proof, for each set S_i we define an associated set $M_i = \{x : x \text{ in } S_i, x \text{ sees each point of S via } S_i\}$. Every set M_i is closed, and we let $\mathcal{M}$ denote the family of all the sets M_i. Observe that $ker\, S_i \subseteq ker\, M_i$ for each i. Similarly, for any $d+1$ (not necessarily distinct) members $S_1, \ldots, S_{d+1}$ of $\mathcal{S}, ker(S_1 \cap \ldots \cap S_{d+1}) \subseteq ker(M_1 \cap \ldots \cap M_{d+1})$. Therefore, every $d+1$ members of $\mathcal{M}$ have a starshaped intersection, and by a version of Helly's topological theorem [7], $\cap\{M_i : M_i \text{ in } \mathcal{M}\} \neq \phi$.

Select any point w in $\cap\{M_i : M_i \text{ in } \mathcal{M}\}$. It is not hard to show that $w \in ker\, M_i$ for every i. Similarly, w is in the kernel of every intersection of sets M_i. The point w belongs to $ker\, S$ as well, and it is easy to verify that $\cap\{M_i : M_i \text{ in } \mathcal{M}\} = ker\, S$. For each i, since $ker\, S_i \subseteq ker\, M_i$ and $w \in ker\, M_i$, $ker\, M_i$ contains a k-dimensional set at w. Likewise, every $d+1$ (not necessarily distinct) members of $\mathcal{S}$ meet in a starshaped set having at least a k-dimensional kernel, so every $d+1$ members of $\mathcal{M}$ meet in a starshaped set having at least a k-dimensional kernel at w.

3 The results.

The first lemma and theorem concern the dimension of the kernel for intersections of certain starshaped sets.

Lemma 1. Let $\mathcal{S} = \{S_i : 1 \leq i \leq n\}$ be a finite family of sets in $\mathbb{R}^d$, each a finite union of polyhedral sets at the origin and each having the origin as an extreme point. If every $d+1$ (not necessarily distinct) members of $\mathcal{S}$ meet in a starshaped set whose kernel is at least one-dimensional, then $S \equiv \cap\{S_i : S_i \text{ in } \mathcal{S}\}$ also is a starshaped set whose kernel is at least one-dimensional. The number $d+1$ is best possible.

Proof. As in our preliminary remarks, for each set S_i in $\mathcal{S}$, define the associated set $M_i = \{x : x \text{ in } S_i, x \text{ sees each point of } S \text{ via } S_i\}$. By our preliminary comments, every $d+1$ of the M_i sets meet in a starshaped set having at least a one-dimensional kernel. Clearly this kernel contains the origin θ.

Let B denote a d-dimensional ball centered at θ. For each i, define $B_i = \cup\{R(\theta, m) \cap bdry\, B : m \in M_i, m \neq \theta\}$. Since S_i is a finite union of polyhedral sets, so is M_i, and B_i is closed. We assert that B_i is starshaped relative to $bdry\, B$ in the following sense: There is a point p in B_i such that, for each b in B_i, B_i contains the smaller great circle arc from p to b. Also, b cannot be antipodal to p. To verify this, let $[\theta, s_p]$ be a nondegenerate segment in $ker\, S_i \subseteq ker\, M_i$, and let $\{p\} = R(\theta, s_p) \cap bdry\, B$. For b in B_i, say $\{b\} = R(\theta, m_b) \cap bdry\, B$ for m_b in M_i. Clearly b and p cannot be antipodal since θ is an extreme point of S_i. Segment $[\theta, s_p]$ sees m_b via M_i, and the family of rays from θ through $[s_p, m_b]$ meets $bdry\, B$ in the required arc.

Similarly, every $d+1$ of the B_i sets have an intersection which is starshaped relative to $bdry\, B$. By the topological version of Helly's theorem, $\cap\{B_i : 1 \leq i \leq n\} \neq \phi$. Select z in this intersection. For every $i, 1 \leq i \leq n$, there is a corresponding nondegenerate segment $[\theta, z_i]$ in $R(\theta, z) \cap M_i$. The shortest of these segments lies in $\cap\{M_i : 1 \leq i \leq n\} = ker\, S$ and satisfies the lemma.

Example 1 will show that the number $d+1$ in the lemma is best possible.

Theorem 1. Let $\mathcal{S} = \{S_i : 1 \leq i \leq n\}$ be a finite family of sets in $\mathbb{R}^d$, each a finite union of polyhedral sets at the origin and each having the origin as an extreme point. Fix d and $k, 0 \leq k \leq d \leq 3$. If every $d+1$ (not necessarily distinct) members of $\mathcal{S}$ intersect in a starshaped set whose kernel is at least k-dimensional, then $\cap\{S_i : S_i \text{ in } \mathcal{S}\}$ also is a starshaped set whose kernel is at least k-dimensional. For $k \neq 0$, the number $d+1$ is best possible.

Proof. If $k=0$, the result is trivial, and if $k=d$, the result follows from [2, Corollary 1.2]. When $k=1$, Lemma 1 above establishes the result. Thus we may restrict our attention to the case for $k=2, d=3$. We adapt an argument by Katchalski [8], also employed in [3]. Using the notation from the proof of Lemma 1, for each $i, 1 \leq i \leq n$, define set M_i, and select a nondegenerate segment $[\theta, z]$ in $\cap\{M_i : 1 \leq i \leq n\} = ker\, S$. Let H be a plane orthogonal to $[\theta, z]$ at a point z' on (θ, z). Using our preliminary remarks, since every $d+1 = 4$ of the S_i sets intersect in a starshaped set whose kernel is at least two-dimensional, every four of the M_i sets intersect in such a set as well. Furthermore, $[\theta, z]$ is a subset of these kernels. That is, for any four of the M_i sets, say $M_1, M_2, M_3, M_4, ker(M_1 \cap M_2 \cap M_3 \cap M_4)$ is at least two-dimensional and contains $[\theta, z]$. Since θ and z are in opposite open halfplanes determined by $H, H \cap ker(M_1 \cap M_2 \cap M_3 \cap M_4)$ is at least one-dimensional. Furthermore, since $H \cap ker(M_1 \cap M_2 \cap M_3 \cap M_4) \subseteq ker((M_1 \cap H) \cap (M_2 \cap H) \cap (M_3 \cap H) \cap (M_4 \cap H))$, every four of the sets $M_i \cap H$ intersect in a starshaped set in H whose kernel is at least one-dimensional. By [2, Theorem 2], $\cap\{M_i \cap H : 1 \leq i \leq n\}$ is a starshaped set whose kernel is at least one-dimensional. Hence $\cap\{M_i : 1 \leq i \leq n\} = ker\, S$ contains a one-dimensional convex set C in H as well as the segment $[\theta, z]$, and $conv(C \cup [\theta, z])$ is a two-dimensional subset of $ker\, S$, the desired result.

Example 1 will demonstrate that the number $d + 1$ is best for $k \neq 0$.

The second theorem is a Helly-type result for intersections of cones.

Theorem 2. Let $\mathcal{S} = \{S_i : 1 \leq i \leq n\}$ be a finite family of closed cones in $\mathbb{R}^d$. If every $d + 1$ (not necessarily distinct) members of $\mathcal{S}$ intersect in a cone, then $S \equiv \cap\{S_i : S_i \text{ in } \mathcal{S}\}$ also is a cone. The number $d + 1$ is best possible.

Proof. Again we use a technique from Bobylev [1]. By the topological version of Helly's theorem [7], observe that $S \equiv \cap\{S_i : 1 \leq i \leq n\} \neq \phi$. If S is degenerate, there is nothing to prove, so assume that this is not the case. For each $i, 1 \leq i \leq n$, define an associated set $A_i = \{v : \text{for all } s \text{ in } S, s \neq v, R(v, s) \subseteq S_i\}$. Clearly A_i contains every vertex of S_i, so $A_i \neq \phi$. Similarly, for any $d+1$ (not necessarily distinct) sets S_i and corresponding sets $A_i, 1 \leq i \leq d + 1$, if v is a vertex of $\cap\{S_i : 1 \leq i \leq d + 1\}$, then $v \in \cap \{A_i : 1 \leq i \leq d + 1\}$, so every $d + 1$ of the A_i sets have a nonempty intersection as well.

We assert that each A_i set is starshaped, as are intersections of $d + 1$ of these sets: For $A_1, \ldots, A_{d+1}$ (not necessarily distinct), let v be a vertex of $\cap\{S_i : 1 \leq i \leq d + 1\}$, to prove that $v \in ker \cap \{A_i : 1 \leq i \leq d + 1\}$. That is, for $a \in \cap \{A_i : 1 \leq i \leq d + 1\} \subseteq \cap\{S_i : 1 \leq i \leq d + 1\}$, we will show that $[v, a] \subseteq \cap\{A_i : 1 \leq i \leq d + 1\}$. If $v = a$, there is nothing to show, so assume that $v \neq a$. For s in $S, s \neq a, R(a, s) \subseteq \cap\{S_i : 1 \leq i \leq d + 1\}$, so for each point t of $R(a, s)\backslash\{v\}, R(v, t) \subseteq \cap\{S_i : 1 \leq i \leq d + 1\}$. This implies that for each point u of $[v, a]\backslash\{s\}, R(u, s) \subseteq \cap\{S_i : 1 \leq i \leq d + 1\}$. If $s = a$, since $R(v, a) \subseteq \cap\{S_i : 1 \leq i \leq d + 1\}$, again for each point u of $[v, a]\backslash\{s\}, R(u, s) \subseteq \cap\{S_i : 1 \leq i \leq d + 1\}$. Because this is true for every s in $S, [v, a] \subseteq \cap\{A_i : 1 \leq i \leq d + 1\}$. Hence $v \in ker \cap \{A_i : 1 \leq i \leq d + 1\}$, and $\cap\{A_i : 1 \leq i \leq d + 1\}$ is starshaped.

Finally, since the S_i sets are closed, so are the A_i sets. Again by the topological version of Helly's theorem [7], $\cap\{A_i : 1 \leq i \leq n\} \neq \phi$. For w

in $\cap\{A_i : 1 \le i \le n\} \subseteq \cap\{S_i : 1 \le i \le n\}, R(w,s) \subseteq S_i$ for all s in $S\backslash\{w\}$ and for all $i, 1 \le i \le n$. We conclude that w is a vertex for $\cap\{S_i : 1 \le i \le n\} = S$, and S is a cone, finishing the proof.

The following easy examples will show that the number $d+1$ is best in Lemma 1, Theorem 1, and Theorem 2.

Example 1. Let S be a d-dimensional simplex in $\mathbb{R}^d$, with the origin θ interior to S. Let $F_i, 1 \le i \le d+1$, denote the facets of S, and consider the convex (hence starshaped) cones $C_i \equiv \cup\{R(\theta, x) : x \in F_i\}, 1 \le i \le d+1$. Clearly every d of the cones C_i meet in a ray, yet $\cap\{C_i : 1 \le i \le d+1\} = \{\theta\}$. If we enlarge each cone slightly but keep the origin as the vertex, every d of these new cones will meet in a full d-dimensional convex cone while all $d+1$ cones will meet in the origin. Thus the number $d+1$ in Lemma 1 and Theorem 1 is best possible (for $k \ne 0$).

Similarly, let H_i denote the hyperplane determined by facet F_i above, with S in the closed halfspace clH_{i1} determined by H_i. Every d of the convex cones $clH_{i1}, 1 \le i \le d+1$, meet in a cone at a vertex of S, but $\cap\{clH_{i1} : 1 \le i \le d+1\} = S$ is not a cone. Thus the number $d+1$ in Theorem 2 is best, also.

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The University of Oklahoma
Department of Mathematics
Norman, Oklahoma 73019
U.S.A.
Email: mbreen@ou.edu