Monochromatic paths and monochromatic cycles in edge-coloured k-partite tournaments

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2000 Mathematic Subject Classification:05C20
Keywords kernel, kernel by monochromatic paths, k-partite tournament.

Abstract

We call the digraph D an *m-coloured digraph* if the arcs of D are coloured with m colours. A subdigraph H of D is called *monochromatic* if all of its arcs are coloured alike.

A set $N \subseteq V(D)$ is said to be a *kernel by monochromatic paths* if it satisfies the following two conditions:

(i) For every pair of different vertices $u, v \in N$ there is no monochromatic directed path between them.

(ii) For every vertex $x \in V(D) - N$, there is a vertex $y \in N$ such that there is an xy-monochromatic directed path.

In this paper it is proved that if D is an m-coloured k-partite tournament such that every directed cycle of length 3 and every directed cycle of length 4 is monochromatic, then D has a kernel by monochromatic paths.

Some previous results are generalized.

1 Introduction

In 1982 Sands, Sauer and Woodrow, have proved that any 2-coloured digraph has a kernel by monochromatic paths [24]. In particular any 2-coloured tournament has a kernel by monochromatic paths.

The known sufficient conditions for the existence of a kernel by monochromatic paths in m-coloured digraphs ($m \geq 3$) tournaments (or nearly tournaments), ask for the monochromaticity or quasi-monochromaticity of small subdigraphs as directed cycles of length at most 4 or transitive tournaments of order 3 (a subdigraph H of an m-coloured digraph D is *quasi-monochromatic* if with at most one exception all of its arcs are coloured alike).

In 1988 [23] Shen Minggang proved that if T is an m-coloured tournament such that every directed cycle of length 3 and every transitive tournament of order 3 is quasi-monochromatic, then T has a kernel by monochromatic paths; he also proved that the result is best possible for $m \geq 5$ (In [13] it was proved that the result is best possible for $m \geq 4$). In [9] it was proved that if T is an m-coloured tournament such that every directed cycle of length at most 4 is quasi-monochromatic then T has a kernel by monochromatic paths. Results similar to those of [23] and [9] were proved for the digraph obtained from a tournament by the deletion of a single arc in [11] and [12] respectively.

In [10] it was proved that if T is an m-coloured tournament such that every directed cycle of length 3 is monochromatic, then T has a kernel by monochromatic paths. And in [14] it was proved that if T is an m-coloured bipartite tournament such that every directed cycle of length 4 is monochromatic, then T has a kernel by monochromatic paths. Another interesting result in kernels by monochromatic paths can be found in [15].

In this paper is proved that if D is an m-coloured k-partite tournament such that every directed cycle of length 3 and every directed cycle of length 4 is monochromatic, then D has a kernel by monochromatic paths. Clearly this result generalizes the named result of [14].

2 Preliminaries

For general concepts we refer the reader to [1] and [2]. The topic of domination in graphs has been widely studied by several authors, a complete study can be found in [16] and [17].

A special class of domination is the domination in digraphs. Let D be a digraph, a set of vertices $S \subseteq V(D)$ is *dominating* whenever for every

$w \in V(D) - S$ there exists a wS-arc in D. A *kernel* of a digraph D is a dominating independent set of vertices. Interesting surveys of kernels in digraphs can be found in [6], [7] and [8].

The concept of kernel has found many applications, see for example [4],[5],[22].

Clearly the concept of kernel by monochromatic paths is a generalization of that of kernel, another interesting generalization is the concept of (k, l)-kernel introduced by Kwäsnik in [20]. Other results about (k, l)-kernels can be found in [18],[19], and [21].

The topic of k-partite tournaments also has been widely studied, a survey in this topic can be found in [25].

Let D be a digraph, $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D respectively. All the paths, cycles and walks considered in this paper will be directed paths, cycles or walks. Let $S_1, S_2 \subseteq V(D)$, the path $(x_0, x_1, \ldots, x_n)$ will be called an S_1S_2-*path* whenever $x_0 \in S_1$ and $x_n \in S_2$. An arc (x_1, x_2) is *asymmetrical* (resp. *symmetrical*) whenever $(x_2, x_1) \notin A(D)$ (resp. $(x_2, x_1) \in A(D)$).

Let D be an m-coloured digraph, the *closure* of D denoted by $\mathcal{C}(D)$ is the digraph defined as follows: $V(\mathcal{C}(D)) = V(D)$ and $(u, v) \in A(\mathcal{C}(D))$ if and only if there exists an uv-monochromatic path in D. Clearly N is a kernel by monochromatic paths of D if and only if N is a kernel of $\mathcal{C}(D)$.

A digraph D is said to be *kernel-perfect digraph* whenever each one of its induced subdigraphs has kernel.

Definition 2.1. A digraph D is called a k-partite tournament whenever there exists a partition of $V(D)$ into k subsets $\{V_1, V_2, \ldots, V_k\}$ such that:

1. For each $i \in \{1, 2, \ldots, k\}$, $A(D[V_i]) = \emptyset$ (where $D[V_i]$ is the subdigraph of D induced by V_i).

2. If $\{i, j\} \subseteq \{1, 2, \ldots, k\}$, then for each $u \in V_i$ and each $w \in V_j$ there exists exactly one arc between u and w in D.

In what follows we will denote by $\Gamma^-(u) = \{z \in V(D) \mid (z, u) \in A(D)\}$ and $\Gamma^+(u) = \{z \in V(D) \mid (u, z) \in A(D)\}$.

The following Lemma will be used along the paper without more explanations. The proof is very easy and it is let to the reader.

Lemma 2.1. *Let D be a k-partite tournament, $k \geq 2$, and $x, y \in V(D)$. If $(x, y) \notin A(D)$ and $(y, x) \notin A(D)$ then $(x, u) \in A(D)$ or $(u, x) \in A(D)$ for each $u \in \Gamma^-(y) \cup \Gamma^+(y)$.*

The following result will be very useful to prove the main result of this paper:

Theorem 2.2 ([3]). *Let D be a digraph. If every cycle of D has a symmetrical arc then D is a kernel-perfect digraph.*

3 The main result

Along the proof we will need the following two Lemmas:

Lemma 3.1. *Let D be an m-coloured k-partite tournament such that every $\vec{C}_3$ (cycle of length 3) and every $\vec{C}_4$ (cycle of length 4) contained in D is monochromatic; $u, v \in V(D)$, $T = (u_0, u_1, \ldots, u_n)$ a uv-monochromatic path with $l(T) \geq 3$. If for each $i \in \{0, 1, \ldots, n-2\}$ and each $j \in \{i+2, \ldots, n\}$, $(u_i, u_j) \notin A(D)$, then for each $i \in \{1, \ldots, n\}$ there exists a $u_i u$-monochromatic path coloured as T.*

Proof: Let $u, v \in V(D)$ and $T = (u = u_0, u_1, \ldots, u_n = v)$ a monochromatic path (coloured say 1) as in the hypothesis. We proceed by induction to prove that for each $i \in \{1, \ldots, n\}$ there exists a $u_i u$-monochromatic path coloured 1. For $i = 1$ we have: If $(u_2, u_0) \in A(D)$ then $\vec{C}_3 = (u_0, u_1, u_2, u_0)$ is a cycle of length 3 and from the hypothesis it is monochromatic. Since (u_0, u_1) is coloured 1 it follows that $(u_1, u_2, u_0 = u)$ is a monochromatic path coloured 1. If $(u_2, u_0) \notin A(D)$ then from Lemma 2.1 there exists an arc between u_0 and u_3 (notice that from the hypothesis we have $(u_0, u_2) \notin A(D)$). So $(u_3, u_0) \in A(D)$ (from the hypothesis $(u_0, u_3) \notin A(D)$). Thus $\vec{C}_4 = (u_0, u_1, u_2, u_3, u_0)$ is a monochromatic cycle; moreover since (u_0, u_1) is coloured 1 we have that $(u_1, u_2, u_3, u_0 = u)$ is a monochromatic path coloured 1.

Assume that for $i \in \{1, \ldots, n-1\}$ there exists a $u_i u$-monochromatic path coloured 1 in D, let T_i' be such a path.

Now we will prove that there exists a $u_{i+1}u$-m.p. (monochromatic path) coloured 1 in D.

If $(u_{i+1}, u_{i-1}) \in A(D)$ then $\vec{C}_3' = (u_{i-1}, u_i, u_{i+1}, u_{i-1})$ is monochromatic (by the hypothesis). Since (u_{i-1}, u_i) is coloured 1, we conclude that $\vec{C}_3'$ is coloured 1. Thus $(u_{i+1}, u_{i-1}, u_i) \cup T_i'$ contains a $u_{i+1}u$-m.p. coloured 1.

If $(u_{i+1}, u_{i-1}) \notin A(D)$ then we consider the two following cases:

Case (a) $i \leq n-2$. Since $(u_{i+1}, u_{i-1}) \notin A(D)$ and from the hypothesis $(u_{i-1}, u_{i+1}) \notin A(D)$ and $(u_{i-1}, u_{i+2}) \notin A(D)$; it follows from Lemma 2.1 that $(u_{i+2}, u_{i-1}) \in A(D)$. Thus $\vec{C}_4' = (u_{i+1}, u_{i+2}, u_{i-1}, u_i, u_{i+1})$ is monochromatic (by hypothesis) and since (u_{i+1}, u_{i+2}) is coloured 1, it follows that this cycle is coloured 1 and then $(u_{i+1}, u_{i+2}, u_{i-1}, u_i) \cup T_i'$ contains a $u_{i+1}u$-m.p. coloured 1.

Case (b) $i = n - 1$. Since $\ell(T) \geq 3$ we have $i \geq 2$. Now $(u_{i+1}, u_{i-2}) \in A(D)$ (Notice that $(u_{i+1}, u_{i-1}) \notin A(D)$, by hypothesis $(u_{i-1}, u_{i+1}) \notin A(D)$ and $(u_{i-2}, u_{i+1}) \notin A(D)$, so the assertion follows from Lemma 2.1). Thus $\vec{C}_4' = (u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i-2})$ is coloured 1 (it is monochromatic by hypothesis and the arc (u_{i-2}, u_{i-1}) is coloured 1). Hence $(u_{i+1}, u_{i-2}, u_{i-1}, u_i) \cup T_i'$ contains a $u_{i+1}u$-m.p. coloured 1. ■

Lemma 3.2. *Let D be an m-coloured k-partite tournament such that each cycle of length 3 and each cycle of length 4 is monochromatic. If $u, v \in V(D)$ are such that there is a uv-m.p. and there is no vu-m.p., then $(u, v) \in A(D)$ or there exists a uv-m.p. of length 2.*

Proof: Let $u, v \in V(D)$ as in the hypothesis, and T a uv-m.p. of minimum length; assume it is coloured 1 and $T = (u = u_0, u_1, \ldots, u_n = v)$. When $\ell(T) \in \{1, 2\}$, the assertion in the Lemma holds. So we will suppose $\ell(T) \geq 3$.

Case (a) There exists $i \in \{0, 1, \ldots, n-2\}$ such that $(u_i, u_j) \in A(D)$ for some $j \geq i + 2$. Let $i_0 = \min\{i \in \{0, 1, \ldots, n-2\} \mid (u_i, u_j) \in A(D)$ for some $j \geq i + 2\}$, and $j_0 = \max\{j \in \{i_0 + 2, \ldots, n\} \mid (u_{i_0}, u_j) \in A(D)\}$.

Subcase (a.1) $j_0 \leq n - 2$.

- $(u_{i_0}, u_{j_0+1}) \notin A(D)$. This follows directly from the choice of j_0.
- $(u_{j_0+1}, u_{i_0}) \notin A(D)$. When $(u_{j_0+1}, u_{i_0}) \in A(D)$ we have that $C = (u_{i_0}, u_{j_0}, u_{j_0+1}, u_{i_0})$ is a monochromatic cycle coloured 1 (from the hypothesis and the fact that (u_{j_0}, u_{j_0+1}) is coloured 1). Thus $T' = (u = u_0, T, u_{i_0}) \cup (u_{i_0}, u_{j_0}) \cup (u_{j_0}, T, u_n = v)$ (where (x, T, y) means the xy-path contained in T, for $x, y \in V(T)$) is a uv-m.p. coloured 1 with $\ell(T') < \ell(T)$, a contradiction.
- $(u_{j_0+2}, u_{i_0}) \in A(D)$. It follows the two previous assertions and from Lemma 2.1 that there exists an arc between u_{i_0} and u_{j_0+2}; and from the definition of j_0, we have $(u_{i_0}, u_{j_0+2}) \notin A(D)$.

Thus $C' = (u_{i_0}, u_{j_0}, u_{j_0+1}, u_{j_0+2}, u_{i_0})$ is a monochromatic cycle coloured 1 (this follows from the hypothesis and the fact (u_{j_0}, u_{j_0+1}) is coloured 1). Hence $T' = (u = u_0, T, u_{i_0}) \cup (u_{i_0}, u_{j_0}) \cup (u_{j_0}, T, u_n = v)$ is a uv-m.p. coloured 1, with $\ell(T') < \ell(T)$, a contradiction. So, this subcase is impossible.

Subcase (a.2) $j_0 = n - 1$.

Clearly when $i_0 = 0$ we have $(u = u_0, u_{n-1}, u_n = v)$ a uv-path of length 2 and the assertion of Lemma 3.2 holds. So, we will assume $i_0 \geq 1$.

- $(u_{i_0}, u_n) \notin A(D)$. This follows directly from the choice of j_0.

- $(u_n, u_{i_0}) \notin A(D)$. Assume, for a contradiction that $(u_n, u_{i_0}) \in A(D)$, then $C = (u_{i_0}, u_{n-1}, u_n, u_{i_0})$ is coloured 1 (it is monochromatic by the hypothesis, and (u_{n-1}, u_n) is coloured 1). Thus $T' = (u = u_0, T, u_{i_0}) \cup (u_{i_0}, u_{n-1}, u_n = v)$ is a uv-m.p. coloured 1, with $\ell(T') < \ell(T)$, a contradiction.

- $(u_n, u_{i_0-1}) \in A(D)$. It follows the two previous assertions and Lemma 2.1 that there exists an arc between u_{i_0-1} and u_n. Now from the definition of i_0 we have $(u_{i_0-1}, u_n) \notin A(D)$.

Thus $C' = (u_{i_0-1}, u_{i_0}, u_{n-1}, u_n, u_{i_0-1})$ is a monochromatic cycle coloured 1 (As C' is monochromatic from the hypothesis and (u_{n-1}, u_n) is coloured 1).

Hence $T' = (u, T, u_{i_0}) \cup (u_{i_0}, u_{n-1}, u_n = v)$ is a uv-m.p. coloured 1, with $\ell(T') < \ell(T)$, a contradiction. So, when $j_0 = n - 1$ we have $i_0 = 0$ and we are done.

Subcase (a.3) $j_0 = n$.

- $i_0 \neq 0$. Because we are assuming $\ell(T) \geq 3$ and T has minimum length. Clearly when $i_0 = 1$, we have $(u = u_0, T, u_{i_0}) \cup (u_{i_0}, u_n = v)$ is a uv-path of length 1 or 2 and in this case the assertion of Lemma 3.2 holds. Thus we will assume $i_0 \geq 2$.

- $(u_{i_0-1}, u_n) \notin A(D)$. From the definition of i_0.

- $(u_n, u_{i_0-1}) \notin A(D)$. Assume for a contradiction that $(u_n, u_{i_0-1}) \in A(D)$. Hence $C = (u_{i_0-1}, u_{i_0}, u_n, u_{i_0-1})$ is a monochromatic cycle coloured 1 (As it is monochromatic by hypothesis and (u_{i_0-1}, u_{i_0}) is coloured 1). Thus $(u_0, T, u_{i_0}) \cup (u_{i_0}, u_n = v)$ is a uv-m.p. coloured 1 whose length is lesser than $\ell(T)$, a contradiction.

- $(u_n, u_{i_0-2}) \in A(D)$. From the two previous assertions and Lemma 2.1 we have that there exists an arc between u_n and u_{i_0-2}, and from the definition of i_0 we have $(u_{i_0-2}, v = u_n) \notin A(D)$.

Hence $C' = (u_{i_0-2}, u_{i_0-1}, u_{i_0}, u_n, u_{i_0-2})$ is a monochromatic cycle coloured 1 (the hypothesis and the fact (u_{i_0-1}, u_{i_0}) is coloured 1). Thus $T' = (u = u_0, T, u_{i_0}) \cup (u_{i_0}, u_n = v)$ is a uv-m.p. coloured 1, with $\ell(T') < \ell(T)$, a contradiction. We conclude that when $j_0 = n$ we have $i_0 = 1$, and we are done.

Case (b) For each $i \in \{0, 1, \ldots, n-2\}$ and for each $j \in \{i+2, \ldots, n\}$, $(u_i, u_j) \notin A(D)$. From Lemma 3.1 we have that for each $i \in \{1, \ldots, n\}$ there exists a $u_i u$-m.p. coloured 1. Particularly there exists a vu-m.p. coloured 1, a contradiction. Thus this case is impossible. ■

The following remark is a directed consequence of the Definition of k-partite tournament and will be useful to prove the main result.

Remark 3.1. Let D be an m-coloured k-partite tournament. If C is a closed walk of length at most 5, then C is a cycle.

The next theorem is the main result of this paper.

Theorem 3.3. *Let D be an m-coloured k-partite tournament. If each cycle of length 3 and each cycle of length 4 is monochromatic, then the closure of D, $\mathcal{C}(D)$, is a kernel-perfect digraph.*

Proof: We will prove that each cycle in the closure of D, $\mathcal{C}(D)$, possesses at least one symmetrical arc. Thus the assertion in Theorem 3.3 will follow directly from Theorem 2.2.

We proceed by contradiction. Assume (for a contradiction) that there exists a cycle $C = (u_0, u_1, \ldots, u_n = u_0)$ in $\mathcal{C}(D)$ which has no symmetrical arc. Thus, for each $i \in \{0, 1, \ldots, n-1\}$ we have that the following assertions hold:

Claim 1. There exists a $u_i u_{i+1}$-m.p. in D. Since $(u_i, u_{i+1}) \in A(\mathcal{C}(D))$, the definition of $\mathcal{C}(D)$ implies that there exists a $u_i u_{i+1}$-m.p. in D.

Claim 2. There is no $u_{i+1} u_i$-m.p. in D. Since C has no symmetrical arc, then $(u_{i+1}, u_i) \notin A(\mathcal{C}(D))$ and from the definition of $\mathcal{C}(D)$ we conclude that there is no $u_{i+1} u_i$-m.p. in D.

Claim 3. $(u_i, u_{i+1}) \in A(D)$ or there exists a $u_i u_{i+1}$-path of length 2 in D. It follows from the Claims 1 and 2 and Lemma 3.2.

Claim 4. If P is a closed monochromatic walk in D then for each $j \in \{0, 1, \ldots, n-1\}$ $\{u_j, u_j + 1\} \not\subseteq V(P)$. This follows directly from Claim 2.

For each $i \in \{0, 1, \ldots, n-1\}$ let

$$T_i = \begin{cases} (u_i, u_{i+1}) \text{ whenever } (u_i, u_{i+1}) \in A(D), \\ \text{a } u_i u_{i+1}\text{-path of length 2 whenever } (u_i, u_{i+1}) \notin A(D). \end{cases}$$

and $C^1 = \bigcup_{i=0}^{n-1} T_i$, C^1 is a closed walk in D. Let $C^1 = (z_0, z_1, \ldots, z_{k-1}, z_k = z_0)$ and define the function φ: $\{0, 1, \ldots, k-1\} \to V(C^1)$ as follows: If

$T_i = (u_i = z_{i_0}, z_{i_0+1}, \ldots, z_{i_0+r} = u_{i+1})$ with $r \in \{1,2\}$ (as $1 \le \ell(T_i) \le 2$), then $\varphi(j) = z_{i_0}$ for each $j \in \{i_0, i_0+1, \ldots, i_0+r-1\}$. We will say that the index i of the vertex $z_i \in V(C^1)$ is a principal index whenever $z_i = \varphi(i)$; and we will denote by I_p the set of principal indices. The sums of indices will be taken modulo k. Now we have the following assertions:

Claim 5. There is no monochromatic closed walk C'' such that for some $i \in \{0,1,\ldots,k-1\}$, $\{z_i, z_{i+1}, z_{i+2}, z_{i+3}\} \subseteq V(C'')$. We proceed by contradiction. Assume (for a contradiction) that there exists a monochromatic closed walk C'' as in Claim 5. From the definition of principal index we have that there exists $r \in \{i, i+1\}$ such that r is a principal index. Thus $z_r = u_j$ for some $j \in \{0,1,\ldots,n\}$; again, the definition of principal index implies $u_{j+1} \in \{z_{i+1}, z_{i+2}, z_{i+3}\}$. Since $\{z_i, z_{i+1}, z_{i+2}, z_{i+3}\} \subseteq V(C'')$ it follows that C'' contains a $u_{j+1}u_j$-m.p. which contradicts Claim 2.

Claim 6. For each $i \in \{0,1,\ldots,k-1\}$ $(z_{i+3}, z_i) \notin A(D)$. We proceed by contradiction. Suppose that for some $i \in \{0,1,\ldots,k-1\}$ we have $(z_{i+3}, z_i) \in A(D)$. Then $(z_i, z_{i+1}, z_{i+2}, z_{i+3}, z_i)$ is a cycle of length 4 and hencefort it is monochromatic, contradicting Claim 5.

Let $i \in \{0,1,\ldots,k-1\}$ be such that $i \in I_p$, say $z_i = u_j$ for some $j \in \{0,1,\ldots,n\}$, then we have the Claims 7-10:

Claim 7. $(z_{i+2}, z_i) \notin A(D)$. We proceed by contradiction. Suppose that $(z_{i+2}, z_i) \in A(D)$. Then $(z_i, z_{i+1}, z_{i+2}, z_i)$ is a monochromatic cycle (Remark 3.1 and the hypothesis). From the construction of C^1 we have $u_{j+1} \in \{z_{i+1}, z_{i+2}\}$. Hence there exists a $u_{j+1}u_j$-m.p. in D, which contradicts Claim 2.

Claim 8. $(z_i, z_{i-2}) \notin A(D)$. Assume, for a contradiction that $(z_i, z_{i-2}) \in A(D)$. Then $(z_i, z_{i-2}, z_{i-1}, z_i)$ is a monochromatic cycle (Remark 3.1 and the hypothesis). Now, it follows from the definition of C^1 that $u_{j-1} \in \{z_{i-1}, z_{i-2}\}$. Clearly this implies that there exists a u_ju_{j-1}-m.p. in D, which contradicts Claim 2.

Claim 9. $(z_i, z_{i+2}) \in A(D)$ or $(z_i, z_{i+3}) \in A(D)$. If $(z_i, z_{i+2}) \in A(D)$ then Claim 9 holds. So, Assume $(z_i, z_{i+2}) \notin A(D)$; from Claim 7 we have $(z_{i+2}, z_i) \notin A(D)$. Thus from Lemma 2.1 we have $(z_i, z_{i+3}) \in A(D)$ or $(z_{i+3}, z_i) \in A(D)$. Now, from Claim 6 $(z_{i+3}, z_i) \notin A(D)$. We conclude that $(z_i, z_{i+3}) \in A(D)$.

Claim 10. $(z_{i-2}, z_i) \in A(D)$ or $(z_{i-3}, z_i) \in A(D)$. When $(z_{i-2}, z_i) \in A(D)$ Claim 10 holds and we are done. Suppose $(z_{i-2}, z_i) \notin A(D)$. From Claim 8 we have $(z_i, z_{i-2}) \notin A(D)$. Now from Lemma 2.1 there exists an arc between z_{i-3} and z_i. From Claim 6 we have $(z_i, z_{i-3}) \notin A(D)$. So, $(z_{i-3}, z_i) \in A(D)$.

Claim 11. $(z_2, z_0) \notin A(D)$, $(z_3, z_0) \notin A(D)$, $(z_0, z_{k-2}) \notin A(D)$ and

$(z_0, z_{k-3}) \notin A(D)$. Since $0 \in I_p$, Claim 11 follows directly from Claims 6, 7 and 8.

Claim 12. $(z_0, z_2) \in A(D)$ or $(z_0, z_3) \in A(D)$. Since $0 \in I_p$, Claim 12 follows directly from Claim 9.

Claim 13. $(z_{k-2}, z_0) \in A(D)$ or $(z_{k-3}, z_0) \in A(D)$. Since $0 \in I_p$, this assertion follows from Claim 10.

Now let $i_0 = \min\{i \in \{3, 4, \ldots, k-2\} \mid (z_{i+1}, z_0) \in A(D)\}$. Thus $(z_{i_0+1}, z_0) \in A(D)$ and we have the following assertion:

Claim 14 $(z_0, z_{i_0}) \in A(D)$ or $(z_0, z_{i_0-1}) \in A(D)$. When $(z_0, z_{i_0}) \in A(D)$ the assertion in Claim 14 holds. Thus, suppose $(z_0, z_{i_0}) \notin A(D)$ (we will prove that $(z_0, z_{i_0-1}) \in A(D)$). It follows from the definition of i_0 that $(z_{i_0}, z_0) \notin A(D)$. Since $z_{i_0-1} \in \Gamma^-(z_{i_0})$ then from Lemma 2.1 we have $(z_0, z_{i_0-1}) \in A(D)$ or $(z_{i_0-1}, z_0) \in A(D)$. Again, it follows from the definition of i_0 that $(z_{i_0-1}, z_0) \notin A(D)$. We conclude $(z_0, z_{i_0-1}) \in A(D)$.

Thus $(z_0, z_{i_0}) \in A(D)$ or $(z_0, z_{i_0-1}) \in A(D)$. We continue the proof of Theorem 3.3 by considering this two possible cases.

Case (a) $(z_0, z_{i_0}) \in A(D)$. Since $(z_{i_0+1}, z_0) \in A(D)$ then $C'' = (z_0, z_{i_0}, z_{i_0+1}, z_0)$ is a cycle and from the hypothesis it is monochromatic coloured say 1.

Subcase (a.1) $i_0 \notin I_p$. In this subcase we have the Claims 15-18:

Claim 15 $i_0 - 1 \in I_p$ and $i_0 + 1 \in I_p$ .This follows from the definition of I_p.

Let $j \in \{1, 2, \ldots, n\}$ be such that $z_{i_0-1} = u_j$ and $z_{i_0+1} = u_{j+1}$.

Claim 16 $(z_0, z_{i_0-1}) \notin A(D)$. We proceed by contradiction. Suppose that
$(z_0, z_{i_0-1}) \in A(D)$. Then $\vec{C} = (z_0, z_{i_0-1} = u_j, z_{i_0}, z_{i_0+1} = u_{j+1}, z_0)$ is a cycle of length 4 and from the hypothesis it is monochromatic. Thus there exists a $u_{j+1}u_j$-m.p., a contradiction to Claim 2.

Claim 17 $(z_0, z_{i_0-2}) \in A(D)$. It follows from Claim 16 that $(z_0, z_{i_0-1}) \notin A(D)$ and from the definition of i_0 we have $(z_{i_0-1}, z_0) \notin A(D)$. Since $z_{i_0-2} \in \Gamma^-(z_{i_0-1})$, we have from Lemma 2.1 that $(z_0, z_{i_0-2}) \in A(D)$ or $(z_{i_0-2}, z_0) \in A(D)$. Now, the definition of i_0 implies $(z_{i_0-2}, z_0) \notin A(D)$. Thus $(z_0, z_{i_0-2}) \in A(D)$.

Claim 18 $(z_{i_0-1}, z_{i_0+1}) \in A(D)$. In the proof of Claim 17 we have observed that $(z_0, z_{i_0-1}) \notin A(D)$ and $(z_{i_0-1}, z_0) \notin A(D)$. Since $z_{i_0+1} \in \Gamma^-(z_0)$ it follows from Lemma 2.1 that $(z_{i_0-1}, z_{i_0+1}) \in A(D)$ or $(z_{i_0+1}, z_{i_0-1}) \in A(D)$. From Claim 7 we have $(z_{i_0+1}, z_{i_0-1}) \notin A(D)$. Thus $(z_{i_0-1}, z_{i_0+1}) \in A(D)$.

Now $(z_{i_0-1}, z_{i_0+1}, z_0, z_{i_0-2}, z_{i_0-1})$ is a cycle of length 4 and so, from the hypothesis it is monochromatic coloured 1 (because C'' is coloured 1 and

$(z_{i_0+1}, z_0) \in C''$); and this contradicts Claim 4. We conclude that Subcase (a.1) is impossible.

Subcase (a.2) $i_0 \in I_p$. Let $j \in \{1, 2, \ldots, n\}$ be such that $z_{i_0} = u_j$.

In this subcase we will consider the two possibilities: $(z_0, z_{i_0-1}) \in A(D)$ or $(z_0, z_{i_0-1}) \notin A(D)$.

Subcase (a.2.1) $(z_0, z_{i_0-1}) \in A(D)$.

Then $(z_0, z_{i_0-1}, z_{i_0}, z_{i_0+1}, z_0)$ is a cycle of length 4 and from the hypothesis it is monochromatic, moreover it is coloured 1 (as $(z_{i_0+1}, z_0) \in A(C'')$ and C'' is coloured 1). Now, we have Claims 19-22.

Claim 19 $i_0 - 2 \in I_p$ and $z_{i_0-2} = u_{j-1}$. Since $i_0 \in I_p$, it follows from the definition of C^1 that $i_0 - 2 \in I_p$ or $i_0 - 1 \in I_p$. If $i_0 - 1 \in I_p$ then $z_{i_0-1} = u_{j-1}$ and $(z_{i_0} = u_j, z_{i_0+1}, z_0, z_{i_0-1} = u_{j-1})$ is a $u_j u_{j-1}$-m.p. which contradicts Claim 2. Hence, $i_0 - 2 \in I_p$, moreover $z_{i_0-2} = u_{j-1}$.

Claim 20 $(z_{i_0-2}, z_{i_0+1}) \notin A(D)$. We proceed by contradiction. Suppose that $(z_{i_0-2}, z_{i_0+1}) \in A(D)$. If $(z_0, z_{i_0-2}) \in A(D)$ then $(z_0, z_{i_0-2}, z_{i_0+1}, z_0)$ is a cycle of length 3 and from the hypothesis it is monochromatic, moreover it is coloured 1 (as $(z_{i_0+1}, z_0) \in A(C'')$ and C'' is coloured 1). So, $(z_{i_0} = u_j, z_{i_0+1}, z_0, z_{i_0-2} = u_{j-1})$ is a $u_j u_{j-1}$-m.p. which contradicts Claim 2. Then we may assume that $(z_0, z_{i_0-2}) \notin A(D)$, this implies $i_0 \geq 2$, so $i_0 \geq 4$. From the definition of i_0 we have $(z_{i_0-2}, z_0) \notin A(D)$. Since $z_{i_0-3} \in \Gamma^-(z_{i_0-2})$, it follows from Lemma 2.1 that $(z_0, z_{i_0-3}) \in A(D)$ or $(z_{i_0-3}, z_0) \in A(D)$. Again the definition of i_0 implies that $(z_{i_0-3}, z_0) \notin A(D)$ so $(z_0, z_{i_0-3}) \in A(D)$. Now, $(z_0, z_{i_0-3}, z_{i_0-2}, z_{i_0+1}, z_0)$ is a cycle of length 4 which is monochromatic (from the hypothesis) and coloured 1 (as $(z_{i_0+1}, z_0) \in A(C'')$ and C'' is coloured 1). Hence, $(z_{i_0} = u_j, z_{i_0+1}, z_0, z_{i_0-3}, z_{i_0-2} = u_{j-1})$ is a $u_j u_{j-1}$-m.p. contradicting Claim 2. We conclude $(z_{i_0-2}, z_{i_0+1}) \notin A(D)$.

Claim 21 $(z_{i_0-2}, z_{i_0}) \in A(D)$. Claim 6 implies that $(z_{i_0+1}, z_{i_0-2}) \notin A(D)$ and from Claim 20 we have $(z_{i_0-2}, z_{i_0+1}) \notin A(D)$. Since $z_{i_0} \in \Gamma^-(z_{i_0+1})$, it follows from the Lemma 2.1 that $(z_{i_0-2}, z_{i_0}) \in A(D)$ or $(z_{i_0}, z_{i_0-2}) \in A(D)$. From Claim 8 we have $(z_{i_0}, z_{i_0-2}) \notin A(D)$ then $(z_{i_0-2}, z_{i_0}) \in A(D)$.

Claim 22 $(z_{i_0}, z_{i_0-2}) \in A(D)$. We have proved (see proof of Claim 21) that $(z_{i_0+1}, z_{i_0-2}) \notin A(D)$ and $(z_{i_0-2}, z_{i_0+1}) \notin A(D)$. Since $z_{i_0} \in \Gamma^+(z_{i_0+1})$, it follows from the Lemma 2.1 that $(z_{i_0-2}, z_0) \in A(D)$ or $(z_0, z_{i_0-2}) \in A(D)$. From the definition of i_0 we have $(z_{i_0-2}, z_0) \notin A(D)$ hence $(z_0, z_{i_0-2}) \in A(D)$.

Now $(z_0, z_{i_0-2} = u_{j-1}, z_{i_0} = u_j, z_{i_0+1}, z_0)$ is a cycle of length 4 and so, from the hypothesis it is monochromatic coloured 1 (because C'' is coloured 1 and $(z_{i_0+1}, z_0) \in C''$); and this contradicts Claim 4. We conclude that

Subcase (a.2.1) is impossible.

Subcase (a.2.2) $(z_0, z_{i_0-1}) \notin A(D)$.

In this subcase we have Claims 23-30.

Claim 23 $(z_0, z_{i_0-2}) \in A(D)$. From the definition of i_0 we have $(z_{i_0-1}, z_0) \notin A(D)$. We are assuming $(z_0, z_{i_0-1}) \notin A(D)$. Thus from Lemma 2.1 we have $(z_0, z_{i_0-2}) \in A(D)$ or $(z_{i_0-2}, z_0) \in A(D)$. The definition of i_0 implies $(z_{i_0-2}, z_0) \notin A(D)$. We conclude $(z_0, z_{i_0-2}) \in A(D)$.

Claim 24 $(z_{i_0-1}, z_{i_0+1}) \notin A(D)$. We proceed by contradiction. Suppose that $(z_{i_0-1}, z_{i_0+1}) \in A(D)$. Then $(z_0, z_{i_0-2}, z_{i_0-1}, z_{i_0+1}, z_0)$ is a cycle of length 4 and from the hypothesis it is monochromatic, moreover it is coloured 1 (as $(z_{i_0+1}, z_0) \in A(C'')$ and C'' is coloured 1). Since $i_0 \in I_p$, we have from the definition of C^1 that $i_0 - 1 \in I_p$ or $i_0 - 2 \in I_p$, this implies $u_{j-1} \in \{z_{i_0-2}, z_{i_0-1}\}$. In any case we obtain a $u_j u_{j-1}$-m.p. coloured 1 which contradicts Claim 2. We conclude $(z_{i_0-1}, z_{i_0+1}) \notin A(D)$.

Claim 25 $(z_{i_0+1}, z_{i_0-1}) \in A(D)$. We have $(z_{i_0-1}, z_0) \notin A(D)$ (by the definition of i_0), $(z_0, z_{i_0-1}) \notin A(D)$ (assumption in this case) and $z_{i_0+1} \in \Gamma^-(z_0)$ (by the definition of i_0). Thus from Lemma 2.1 we have $(z_{i_0+1}, z_{i_0-1}) \in A(D)$ or $(z_{i_0-1}, z_{i_0+1}) \in A(D)$. Now from Claim 24 we have $(z_{i_0-1}, z_{i_0+1}) \notin A(D)$. We conclude that $(z_{i_0+1}, z_{i_0-1}) \in A(D)$.

Claim 26 $(z_{i_0-1}, z_{i_0} = u_j)$ and (z_{i_0+1}, z_{i_0-1}) are coloured 1. We have that $(z_{i_0-1}, z_{i_0}, z_{i_0+1}, z_{i_0-1})$ is a monochromatic cycle coloured 1 (by hypothesis and the fact that $(z_{i_0}, z_{i_0+1}) \in A(C'')$ and C'' is coloured 1). Thus $(z_{i_0-1}, z_{i_0} = u_j)$ and (z_{i_0+1}, z_{i_0-1}) are coloured 1.

Claim 27 $i_0 - 2 \in I_p$. Since $i_0 \in I_p$, it follows from the construction of C^1 that $i_0 - 1 \in I_p$ or $i_0 - 2 \in I_p$ which implies $u_{j-1} \in \{z_{i_0-2}, z_{i_0-1}\}$. We have proved (see proof of Claim 26) that $C = (z_{i_0-1}, z_{i_0} = u_j, z_{i_0+1}, z_{i_0-1})$ is a monochromatic cycle coloured 1. From Claim 4 $u_{j-1} \notin V(C)$. So, we conclude $u_{j-1} = z_{i_0-1}$, and then $i_0 - 2 \in I_p$.

Claim 28 $(z_{i_0-2}, z_{i_0+1}) \notin A(D)$. We proceed by contradiction. Suppose that $(z_{i_0-2}, z_{i_0+1}) \in A(D)$. Then $(z_0, z_{i_0-2}, z_{i_0+1}, z_0)$ is a monochromatic cycle coloured 1 (by hypothesis and the fact $(z_{i_0+1}, z_0) \in A(C'')$ and C'' is coloured 1). Hence $(z_{i_0} = u_j, z_{i_0+1}, z_0, z_{i_0-2} = u_{j-1})$ is a $u_j u_{j-1}$-m.p. coloured 1 which contradicts Claim 2. We conclude $(z_{i_0-2}, z_{i_0+1}) \notin A(D)$.

Claim 29 $(z_{i_0+1}, z_{i_0-2}) \notin A(D)$. We proceed by contradiction. Suppose that $(z_{i_0+1}, z_{i_0-2}) \in A(D)$. Then $(z_{i_0+1}, z_{i_0-2} = u_{j-1}, z_{i_0-1}, z_{i_0} = u_j, z_{i_0+1})$ is a monochromatic cycle (by the hypothesis) which contains u_{j-1} and u_j, contradicting Claim 4. Hence $(z_{i_0+1}, z_{i_0-2}) \notin A(D)$.

Claim 30 $(z_{i_0-2}, z_{i_0}) \in A(D)$. We have $(z_{i_0-2}, z_{i_0+1}) \notin A(D)$ (Claim

28) and $(z_{i_0+1}, z_{i_0-2}) \notin A(D)$ (Claim 29), clearly $(z_{i_0}, z_{i_0+1}) \in A(D)$. Thus from Lemma 2.1 we have that $(z_{i_0-2}, z_{i_0}) \in A(D)$ or $(z_{i_0}, z_{i_0-2}) \in A(D)$. Since $i_0 - 2 \in I_p$ (see Claim 27), it follows from Claim 7 that $(z_{i_0}, z_{i_0-2}) \notin A(D)$. Hence $(z_{i_0-2}, z_{i_0}) \in A(D)$.

Finally, we have that $(z_0, z_{i_0-2} = u_{j-1}, z_{i_0} = u_j, z_{i_0+1}, z_0)$ is a monochromatic cycle (by hypothesis), but this contradicts Claim 4. We conclude that this subcase is impossible.

Case (b) $(z_0, z_{i_0}) \notin A(D)$.

We have Claims 31-38.

Claim 31 $(z_0, z_{i_0-1}) \in A(D)$. From the definition of i_0 we have $(z_{i_0}, z_0) \notin A(D)$ and from the assumption on this case we have $(z_0, z_{i_0}) \notin A(D)$. Since $z_{i_0} \in \Gamma^-(z_{i_0})$, it follows from Lemma 2.1 that $(z_0, z_{i_0-1}) \in A(D)$ or $(z_{i_0-1}, z_0) \in A(D)$. Now from the definition of i_0 we have $z_{i_0-1}, z_0)$ $\notin A(D)$. Thus $(z_0, z_{i_0-1}) \in A(D)$.

Claim 32 $C^2 = (z_0, z_{i_0-1}, z_{i_0}, z_{i_0+1}, z_0)$ is a monochromatic cycle. This follows directly from the previous assertion and the hypothesis. Say C^2 is coloured 1.

Claim 33 $i_0 - 1 \notin I_p$. We proceed by contradiction. Suppose $i_0 - 1 \in I_p$, from the construction of C^1 we have $i_0 \in I_p$ or $i_0 + 1 \in I_p$, so there are two elements of I_p in the monochromatic cycle C^2 (Claim 32) and this contradicts Claim 4. Thus $i_0 - 1 \notin I_p$.

Claim 34 $i_0 \in I_p$ and $i_0 - 2 \in I_p$. This follows from Claim 33 and the definition of C^1.

Let $j \in \{0, 1, \ldots, n-1\}$ be such that $z_{i_0-1} = u_{j-1}$ and $z_{i_0} = u_j$.

Claim 35 $i_0 + 1 \notin I_p$. Proceeding by contradiction suppose that $i_0 + 1 \in I_p$, then $z_{i_0+1} = u_{j+1}$. Thus $\{u_j, u_{j+1}\} \subseteq V(C^2)$, and C^2 is a monochromatic cycle (see Claim 32). This is a contradiction to Claim 4. Hence $i_0 + 1 \notin I_p$.

Claim 36 $(z_0, z_{i_0-2}) \notin A(D)$. Proceeding by contradiction, suppose that $(z_0, z_{i_0-2}) \in A(D)$. From the definition of i_0 we have $(z_{i_0}, z_0) \notin A(D)$ and from our assumption on this case $(z_0, z_{i_0}) \notin (AD)$. Since $z_{i_0-2} \in \Gamma^+(z_0)$, it follows from Lemma 2.1 that $(z_{i_0}, z_{i_0-2}) \in A(D)$ or $(z_{i_0-2}, z_{i_0}) \in A(D)$. When $(z_{i_0}, z_{i_0-2}) \in A(D)$ we have that $(u_j = z_{i_0}, z_{i_0-2} = u_{j-1})$ is a $u_j u_{j-1}$-m.p. which contradicts Claim 2. When $z_{i_0-2}, z_{i_0}) \in A(D)$ we have that $C^3 = (z_0, z_{i_0-2} = u_{j-1}, z_{i_0} = u_j, z_{i_0+1}, z_0)$ is a cycle of length 4, and then from the hypothesis it is monochromatic. Clearly this contradicts Claim 4. Thus $(z_0, z_{i_0-2}) \notin A(D)$.

Claim 37 $(z_0, z_{i_0-3}) \in A(D)$. From Claim 36 we have that $(z_0, z_{i_0-2}) \notin A(D)$. From the definition of i_0 we have that $(z_{i_0-2}, z_0) \notin A(D)$. Thus $i_0 - 2 \geq 2$ and $i_0 \geq 4$. Since $z_{i_0-3} \in \Gamma^-(z_{i_0-2})$, it follows from Lemma 2.1

that $(z_0, z_{i_0-3}) \in A(D)$ or $(z_{i_0-3}, z_0) \in A(D)$. Again from the definition of i_0 we have $(z_{i_0-3}, z_0) \notin A(D)$. We conclude $(z_0, z_{i_0-3}) \in A(D)$.

Claim 38 $(z_{i_0-2}, z_{i_0+1}) \in A(D)$. Notice that: $(z_0, z_{i_0-2}) \notin A(D)$ (Claim 36) and $(z_{i_0-2}, z_0) \notin A(D)$ (definition of i_0). Since $z_{i_0+1} \in \Gamma^-(z_0)$, it follows from Lemma 2.1 that $(z_{i_0-2}, z_{i_0+1}) \in A(D)$ or $(z_{i_0+1}, z_{i_0-2}) \in A(D)$. When $(z_{i_0+1}, z_{i_0-2}) \in A(D)$ we have that $(z_{i_0-2}, z_{i_0-1}, z_{i_0}, z_{i_0+1}, z_{i_0-2})$ is a monochromatic cycle which contradicts Claim 5. Thus $(z_{i_0-2}, z_{i_0+1}) \in A(D)$.

Now $(z_0, z_{i_0-3}, z_{i_0-2}, z_{i_0+1}, z_0)$ is a monochromatic cycle coloured 1 (from the hypothesis and the fact that (z_{i_0+1}, z_0) is coloured 1). Thus $(u_j = z_{i_0}, z_{i_0+1}, z_0, z_{i_0-3}, z_{i_0-2} = u_{j-1})$ is a $u_j u_{j-1}$-m.p. coloured 1 (recall Claim 32) which contradicts Claim 2. Thus Case (b) is impossible.

We conclude that each cycle in $\mathcal{C}(D)$ possesses at least one symmetrical arc; and from Theorem 2.2 $\mathcal{C}(D)$, is a kernel-perfect digraph. ■

Remark 3.2. The bipartite tournament D_1 in figure 1 shows that we can not remove the condition about C_4 from the Theorem 3.3. D_1 is a 3-coloured bipartite tournament which contains (u, x, w, z, u), a C_4 that is not monochromatic, and D_1 has no kernel by monochromatic paths.

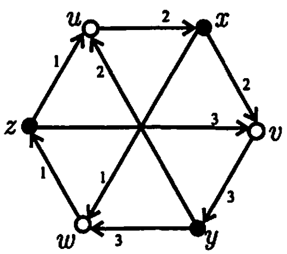

Figure 1: D_1

Remark 3.3. The 3-coloured C_3 is an example of a 3-partite tournament which shows that the condition over C_3 can not be removed from the Theorem 3.3.

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