The Linear Arboricity of Planar Graphs without 5-cycles and 6-cycles*

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Abstract

Let G be a planar graph with maximum degree Δ. It's proved that if $\Delta \geq 5$ and G does not contain 5-cycles and 6-cycles, then $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$.

Key words: planar graph; linear arboricity; cycle

1 Introduction

In this paper, all graphs are finite, simple and undirected. For a real number x, $\lceil x \rceil$ is the least integer not less than x and $\lfloor x \rfloor$ is the largest integer not larger than x. Let G be a graph. We use $\Delta(G)$ and $\delta(G)$ to denote the maximum(vertex) degree and the minimum (vertex) degree, respectively. A k-, k^+- or k^-- vertex is a vertex of degree k, at least k, or at most k, respectively.

A *linear forest* is a graph in which each component is a path. A map φ from $E(G)$ to $\{1, 2, \ldots, t\}$ is called a t-linear coloring if the induced subgraph of edges having the same color α is a linear forest for $1 \leq \alpha \leq t$. The linear arboricity $la(G)$ of a graph G defined by Harary [5] is the minimum number t for which G has a t-linear coloring.

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Akiyama, Exoo, and Harary [1] conjectured that $la(G) = \lceil\frac{\Delta(G)+1}{2}\rceil$ for any regular graph G. It is obvious that $la(G) \geq \lceil\frac{\Delta(G)}{2}\rceil$. So the conjecture is equivalent to the following conjecture.

Conjecture A. For any graph G, $\lceil\frac{\Delta(G)}{2}\rceil \leq la(G) \leq \lceil\frac{\Delta(G)+1}{2}\rceil$.

The linear arboricity has been determined for complete bipartite graphs [1] and regular graphs with $\Delta = 3, 4$ [1] and [2], 5, 6, 8 [3], and 10 [4].

Conjecture A has already been proved to be true for all planar graphs, see [7] and [9]. Wu also proved in [7] that for a planar graph G with girth g and maximum degree Δ, $la(G) = \lceil\frac{\Delta(G)}{2}\rceil$ if $\Delta(G) \geq 13$, or $\Delta(G) \geq 7$ and $g \geq 4$, or $\Delta(G) \geq 5$ and $g \geq 5$, or $\Delta(G) \geq 3$ and $g \geq 6$. In this paper, we obtain that if G is a planar graph with $\Delta(G) \geq 5$ and without 5-cycles and 6-cycles, then $la(G) = \lceil\frac{\Delta(G)}{2}\rceil$.

2 Main results and their proofs

In this section, all graphs are planar graphs which have been embedded in the plane. For a planar graph G, the degree of a face f, denoted by $d(f)$, is the number of edges incident with it, where each cut-edge is counted twice. A k-, k^+- or k^-- face is a face of degree k, at least k, or at most k, respectively. $F(v) = \{f \in F(G)$: the face is incident with $v\}$. For $v \in G$, we use $n_i(v)$ to denote the number of i-vertices which are adjacent to v, $f_i(v)$ to denote the number of i-faces incident with v. A k-face with consecutive vertices v_1, v_2, ..., v_k along its boundary in some direction is often said to be a $(d(v_1), d(v_2), \ldots, d(v_k))$-face.

Given a t-linear coloring φ and a vertex v of G, we denote $C^i_\varphi(v)$ the set of colors that appear i times at v, where $i = 0, 1, 2$. Let $C_\varphi(u, v) = C^2_\varphi(u) \cup C^2_\varphi(v) \cup (C^1_\varphi(u) \cap C^1_\varphi(v))$, that is , $C_\varphi(u, v)$ is the set of colors that appear at least two times at u and v. A monochromatic path is a path whose edges receive the same color. For two different edges e_1 and e_2 of G, they are said to be in the same color component, denoted by $e_1 \leftrightarrow e_2$ if there is a monochromatic path of G connecting them. Furthermore, if the ends of e_1 and e_2 are known, say that, $e_i = x_iy_i$ $(i = 1, 2)$, then $x_1y_1 \leftrightarrow x_2y_2$ denotes more accurately that there is a monochromatic path from x_1 to y_2 passing through the edges x_1y_1 and x_2y_2 in G, that is, y_1 and x_2 are internal vertices in the path. Otherwise, we use $x_1y_1 \not\leftrightarrow x_2y_2$ (or $e_1 \not\leftrightarrow e_2$) to denote that such monochromatic path does not exist. Note that $x_1y_1 \leftrightarrow x_2y_2$ and $x_1y_1 \leftrightarrow y_2x_2$ are different.

Let v be a vertex with $d(v) = d$, denote $f_1, f_2, \ldots, f_d$ be the faces incident with v in a clockwise order, and $v_1, v_2, \ldots, v_d$ be the neighbors of

v, where v_i is incident with f_i, f_{i+1}, $i = 1, 2, \ldots, d$. Note that eventually f_1 and f_{d+1} denote the same face.

Theorem 1. Let G be a planar graph with $\Delta(G) \geq 5$. If G does not contain 5-cycles and 6-cycles, then $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$.

Proof. According to [6], if G is a planar graph with $\Delta(G) \geq 7$ and without 5-cycles, then $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$. According to [7] and [9], Conjecture A is true for all planar graphs. Henceforth, to prove Theorem 1, we only need to prove that every planar graph with $\Delta(G) = 6$ and without 5-cycles and 6-cycles has a 3-linear coloring. Let $G = (V, E, F)$ be a minimal counterexample to the theorem. First, we prove some lemmas for G.

Lemma 1. For any $uv \in E(G)$, $d_G(u) + d_G(v) \geq 8$.

The proof of Lemma 1 is similar to that of Lemma 1 in [6].

By Lemma 1, we have
(a) $\delta(G) \geq 2$, and
(b) any two 3^--vertices are not adjacent, and
(c) any 3-face is incident with three 4^+-vertices, or at least two 5^+-vertices.

Lemma 2. Every vertex is adjacent to at most two 2-vertices. Moreover, suppose that a vertex v is adjacent to two 2-vertices x, y, let x', y' be another neighbor of x, y, respectively. Then $x'v$, $y'v \notin E(G)$.

The proof of Lemma 2 can be found in [8].

In the proofs of the following Lemmas, the notation $(u, 1)$ denotes the edge incident with u and colored with 1.

Lemma 3. If a vertex u is adjacent to two 2-vertices v, w and incident with a 3-face $uxyu$. Then $\min\{d(x), d(y)\} \geq 4$.

Proof. Since u is adjacent to two 2-vertices v, w, then neither v nor w is incident with 3-faces by Lemma 2, so v, w, x, y are distinct vertices. Suppose that $\min\{d(x), d(y)\} \leq 3$. Without loss of generality, assume that $d(x) \geq d(y)$. By Lemma 2, $d(x) \geq d(y) \geq 3$ and so $d(y) = 3$. By Lemma 1, $d(x) \geq 5$ and $d(u) = 6$. Let v', w' be another neighbors of v, w, respectively. Since G is minimal, $G' = G - uv$ has a 3-linear coloring φ. Without loss of generality, assume $\varphi(vv') = 1$. If there is a color $c \in C^0_\varphi(u)$, or $c \in C^1_\varphi(u)\setminus\{1\}$, or $c = 1 \in C^1_\varphi(u)$ but $vv' \not\leftrightarrow (u,1)$, then color directly uv with c. So $C^0_\varphi(u) = \emptyset$, $C^1_\varphi(u) = \{1\}$ and $vv' \leftrightarrow (u,1)$. If $\varphi(uw) \neq 1$, then $ww' \not\leftrightarrow (u,1)$, and it follows that we can recolor uw with 1 and color uv with $\varphi(uw)$. So we have $\varphi(uw) = \varphi(ww') = 1$, $\varphi(ux) \neq 1$ and $\varphi(uy) \neq 1$. Now let's come back to discuss y and x. If $1 \notin C^2_\varphi(y)$, then we can recolor uy with 1, and color uv with $\varphi(uy)$. Otherwise, we have $\varphi(xy) = 1$ and then recolor ux with 1, xy with $\varphi(ux)$ and color uv with $\varphi(ux)$. Thus φ is extended to a 3-linear coloring of G, a contradiction. It completes the proof of Lemma 3.

Lemma 4. G has no subgraph isomorphic to the configuration in Figure 1 where $d(u) = 5, d(w) = d(v) = 3$.

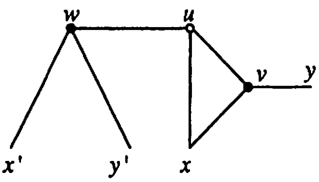

Figure 1

Proof. Suppose G has a configuration as depicted in Figure 1. By the minimality of G, $G' = G - uw$ has a 3-linear coloring φ. If there is a color c such that $c \notin C_\varphi(u,w)$, then color directly uw with c, so $C_\varphi(u,w) = \{1,2,3\}$.

Suppose $\varphi(wx') = \varphi(wy')$. Without loss of generality, let $\varphi(wx') = \varphi(wy') = 1$. Since $d_{G'}(u) = 4$, we have $C^0_\varphi(u) = \{1\}$. If $1 \notin C^2_\varphi(v)$, then recolor uv with 1 and color uw with $\varphi(uv)$. Otherwise, we have $\varphi(vx) = \varphi(vy) = 1$. Thus we can recolor ux with 1, vx with $\varphi(ux)$ and color uw with $\varphi(ux)$. It follows that G is 3-linear colorable, a contradiction.

Suppose $\varphi(wx') \neq \varphi(wy')$. Without loss of generality, let $\varphi(wx') = 1$, $\varphi(wy') = 2$, then $C^1_\varphi(u) = \{1,2\}$. If $wx' \not\leftrightarrow (u,1)$, then color directly uw with 1. Similarly, if $wy' \not\leftrightarrow (u,2)$, then color directly uw with 2. Otherwise, if $\varphi(uv) = 3$, since $|C^2_\varphi(v)| \leq 1$, we can assume $1 \in C^0_\varphi(v) \cup C^1_\varphi(v)$, and then we can recolor uv with 1 and color uw with 3. Otherwise, assume $\varphi(uv) = 1$. Since $wx' \leftrightarrow (u,1)$, we have $\varphi(vy) = 1$ or $\varphi(vx) = 1$. We recolor uv with 2, and color uw with 1. So φ is extended to a 3-linear coloring of G, a contradiction. We complete the proof of Lemma 4.

Lemma 5. G has no subgraph isomorphic to the configuration in Figure 2 where $d(u) = 6, d(w) = 2, d(v) = 3$.

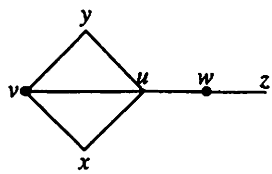

Figure 2

Proof. Suppose G has a configuration as depicted in Figure 2. By the minimality of G, $G' = G - uw$ has a 3-linear coloring φ. Without loss of generality, assume $\varphi(wz) = 1$. Similarly, we can assume that $C^0_\varphi(u) = \emptyset$, $C^1_\varphi(u) = \{1\}$ and $wz \leftrightarrow (u, 1)$. If $\varphi(uv) \neq 1$ and $1 \notin C^2_\varphi(v)$, then we can recolor uv with 1 and color uw with $\varphi(uv)$.

Suppose $\varphi(uv) \neq 1$ and $1 \in C^2_\varphi(v)$. Then $\varphi(vy) = \varphi(vx) = 1$. Since $C^1_\varphi(u) = \{1\}$, we have $\varphi(ux) \neq 1$ or $\varphi(uy) \neq 1$. Assume $\varphi(uy) \neq 1$. If $(u, 1) \leftrightarrow yv$, then $\varphi(ux) \neq 1$, and we can recolor ux with 1, vx with $\varphi(ux)$, and color uw with $\varphi(ux)$. Otherwise, we can recolor uy with 1, vy with $\varphi(uy)$, and color uw with $\varphi(uy)$.

Finally, we assume $\varphi(uv) = 1$. Then $\varphi(vx) = 1$ or $\varphi(vy) = 1$ (since $wz \leftrightarrow (u, 1)$). Without loss of generality, assume $\varphi(vy) = 1$ and $\varphi(vx) = 2$. If $\varphi(uy) = 2$ and $vx \leftrightarrow yu$, then $\varphi(ux) = 3$ and we can recolor uy with 1, vy and ux with 2, vx with 3, and color uw with 3. Otherwise, we can recolor uy with 1, vy with $\varphi(uy)$ and color uw with $\varphi(uy)$. Thus φ is extended to a 3-linear coloring of G, a contradiction. So we complete the proof of Lemma 5.

Lemma 6. G has no subgraph isomorphic to the configuration depicted in Figure 3 where $d(v) = 2, d(z) = 3$.

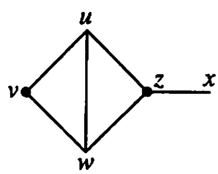

Figure 3

Proof. On the contrary, suppose G has a configuration as depicted in Figure 3. By the minimality of G, $G' = G - uv$ has a 3-linear coloring φ.

Without loss of generality, assume $\varphi(vw) = 1$. Similarly, we can assume that $C^0_\varphi(u) = \emptyset$, $C^1_\varphi(u) = \{1\}$ and $vw \leftrightarrow (u, 1)$.

Suppose $\varphi(uz) \neq 1$ and $\varphi(wz) \neq 1$, then we can recolor uz with 1 and color uv with $\varphi(uz)$.

Suppose $\varphi(uz) = 1$, then $\varphi(uw) \neq 1$ and $1 \in C^2_\varphi(z)$, assume $\varphi(uw) = 2$. If $\varphi(zx) = 1$, or $\varphi(wz) = 1$ and $zx \leftrightarrow wu$, then we can recolor vw, uz with 2, uw with 1, and color uv with 1. Otherwise, we can recolor wz with 2, uw with 1, and color uv with 2.

Suppose $\varphi(uz) \neq 1$ and $\varphi(wz) = 1$. Similarly $\varphi(uw) \neq 1$ and $1 \in C^2_\varphi(z)$ and we can assume $\varphi(uw) = 2$. Then we can recolor wz with 2, uw with 1, and color uv with 2.

Thus, we can obtain a 3-linear coloring of G, a contradiction. It completes the proof of Lemma 6.

Lemma 7. Since G contains no 5-cycles and 6-cycles and $\delta(G) \geq 2$, the following results hold:

(a) Any 4^+-vertex is incident with at most $\lfloor \frac{2d(v)}{3} \rfloor$ 3-faces.

(b) A 4^--face f is adjacent to a 4-face f' if and only if the two faces are incident with a common 2-vertex.

(c) If a face is adjacent to two nonadjacent 3-faces, then the face must be 7^+-face.

(d) If $d(v) \geq 5$, then v is incident with at most $d - 2$ 4^--faces.

Proof. (a), (b), (c) is similar to Lemma 4 in [6, 8]. Next we will show (d).

(d) If $f_4(v) = 0$, then (d) is obvious by (a). So we assume that $f_4(v) > 0$, let f_1 be a 4-face incident with v. If both f_2 and f_d are 4^--faces, then f_3 and f_{d-1} must be 7^+-faces by (b) and Lemma 2. Now suppose that f_d is a 7^+-face. If f_2 is a 4^--face, then one of f_3 and f_4 must be a 7^+-face. Thus we prove (d).

By Euler's formula $|V| - |E| + |F| = 2$, we have

$$\sum_{v \in V}(2d(v) - 6) + \sum_{f \in F}(d(f) - 6) = -12 < 0. \tag{1}$$

We define the initial charge $ch(x)$ of $x \in V(G) \cup F(G)$ to be $ch(v) = 2d(v) - 6$ if $v \in V(G)$ and $ch(f) = d(f) - 6$ if $f \in F(G)$. In the following, we will reassign a new charge denoted by $ch'(x)$ to each $x \in V(G) \cup F(G)$ according to the discharging rules. Since our rules only move charges around, and do not affect the sum, we have

$$\sum_{x\in V(G)\cup F(G)} ch'(x) = \sum_{x\in V(G)\cup F(G)} ch(x) = -12. \qquad (2)$$

In the following, we will show that $ch'(x) \geq 0$ for each $x \in V(G)\cup F(G)$, a contradiction to (2), completing the proof.

The discharging rules are defined as follows.

R1. The 6-vertices give 1 to each of their adjacent 2-vertices.

R2. For a 3-face f.

R2.1. f receives $\frac{3}{2}$ from each of its incident 5^+-vertices if f is incident with a 3^--vertex.

R2.2. f receives 1 from each of its incident vertices if f is incident with no 3^--vertex.

R3. For a 4-face f.

R3.1. f receives 1 from each of its incident 5^+-vertices if f is incident with two 3^--vertices .

R3.2. f receives $\frac{2}{3}$ from each of its incident 4^+-vertices if f is incident with only one 3^--vertex.

R3.3. f receives $\frac{1}{2}$ from each of its incident 4^+-vertices if f is incident with no 3^--vertex.

R4. Each 6-vertex v, if v is incident with a 7^+-face f, then v receives $\frac{d(f)-6}{n}$ from f.(n denotes the number of 6-vertices incident with f)

Let f be a face of G. If $d(f) = 3$, then $ch(f) = 3 - 6 = -3$, and it follows that $ch'(f) \geq ch(f) + min\{\frac{3}{2} \times 2, 1 \times 3\} = 0$ by Lemma 1 and R2.

If $d(f) = 4$, then $ch(f) = 4 - 6 = -2$, and $ch'(f) \geq ch(f) + \min\{1 \times 2, \frac{2}{3} \times 3, \frac{1}{2} \times 4\} = 0$ by Lemma 1 and R3.

If $d(f) \geq 7$, then $ch'(f) = ch(f) - \frac{ch(f)}{n} \times n = 0$ by R4.

Let v be a vertex of G. If $d(v) = 2$, then $ch'(v) = ch(v) + 2 = 0$ by Lemma 1 and R1.

If $d(v) = 3$, $ch'(v) = ch(v) = 0$.

If $d(v) = 4$, then v is incident with at most two 4^--faces, it follows that $ch'(v) \geq ch(v) - \max\{2 \times 1, \frac{2}{3} + 1, \frac{2}{3} \times 2\} = 0$ by R2 and R3.

If $d(v) = 5$, then $f_3(v) \leq 3$ by lemma 7. If $f_3(v) = 3$, then $n_3(v) \leq 1$ by Lemma 4. Hence $ch'(v) \geq ch(v) - \frac{3}{2} \times 2 - 1 = 0$ by R2. If $f_3(v) < 3$, then $ch'(v) \geq ch(v) - \max\{\frac{3}{2} \times 2 + \frac{2}{3}, \frac{3}{2} + 1 \times 2, 1 \times 3\} = \frac{1}{3} > 0$ by R2 and R3.

If $d(v) = 6$, then v is adjacent to at most two 2-vertices by Lemma 2, and $f_3(v) + f_4(v) \leq 4$ by Lemma 7.

If v is adjacent to two 2-vertices, then the 3-face incident with v must be a (6, 4^+, 4^+)-face by Lemma 3. Hence $ch'(v) \geq ch(v) - (2 + 4 \times 1) = 0$ by Lemma 7, R1, R2 and R3.

If v is adjacent to one 2-vertex.

Case 1. $f_3(v) = 4$, then $n_3(v) \leq 2$ by Lemma 5 and Lemma 6. It's easy to verify that v is incident with a (6, 2, 6)-face, a 7^+-face and a 8^+-face, and the 3-face adjacent to the (6, 2, 6)-face must be a (6, 6, 4^+)-face by Lemma 6. If $n_3(v) = 2$,then $ch'(v) \geq 6 - 1 - \frac{3}{2} - 1 - \frac{3}{2} \times 2 + \frac{1}{6} + \frac{2}{6} = 0$ by R1, R2 and R4. Otherwise $ch'(v) \geq 6 - 1 - \frac{3}{2} - 1 - \frac{3}{2} - 1 = 0$.

Case 2. $f_3(v) = 3$, then $f_4(v) \leq 1$. If $f_4(v) = 1$, then $f_{4^-}(v) > 3$, the 4-face must be adjacent to a 3-face, then the two faces are incident with a common 2-vertex by Lemma 7. Without loss of generality, we assume that f_1 is the (6, 2, 6)-face, f_6 is the 4-face, then f_6 must be a (6, 4^+, 6, 2)-face by Lemma 6. Similarly if f_2 is a 3-face, then f_2 must be a (6, 6, 4^+)-face, hence $ch'(v) \geq 6 - 1 - \frac{3}{2} - \frac{2}{3} - 1 - \frac{3}{2} = \frac{1}{3} > 0$. Otherwise, f_2 and f_5 must be 7^+-faces, then $ch'(v) \geq 6 - 1 - \frac{3}{2} - \frac{2}{3} - \frac{3}{2} \times 2 + \frac{2}{7} = \frac{5}{42} > 0$. If $f_4(v) = 0$, then $ch'(v) \geq 6 - 1 - \frac{3}{2} \times 3 = \frac{1}{2} > 0$ by R1 and R2.

Case 3. $f_3(v) \leq 2$, then $ch'(v) \geq 6 - 1 - \frac{3}{2} \times 2 - 1 \times 2 = 0$ by R1, R2 and R3.

If v is adjacent to no 2-vertex, then $ch'(v) \geq 6 - \frac{3}{2} \times 4 = 0$ by R2 and R3.

Hence we complete the proof of the Theorem 1. □

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References

[1] J.Akiyama, G.Exoo, F.Harary, Covering and packing in graphs III: cyclic and acyclic invariants *Math. Slovaca*, **30**(1980), 405-417.

[2] J.Akiyama, G.Exoo, F.Harary, Covering and packing in graphs IV: Linear arboricity *Networks*, **11**(1981), 69-72.

[3] H. Enomoto and B. Péroche, The linear arboricity of some regular graphs *J. Graph Theory*, **8**(1984), 309-324.

[4] F.Guldan, The linear arboricity of 10 regular graphs, *Math. Slovaca*, **36**(1986), 225-228.

[5] F.Harary, Covering and packing in graphs I, *Ann. N.Y. Acad. Sci.*, **175**(1970), 198-205.

[6] J.L. Wu, J.F. Hou and G.Z. Liu, The linear arboricity of planar graphs with no short cycles, *Theor. Comp. Sci.*, **381**(2007), 230-233.

[7] J.L. Wu, On the linear arboricity of planar graphs, *J. Graph theory*, **31**(1999), 129-134.

[8] J.L. Wu, J.F. Hou and X.Y. Sun, A note on the linear arboricity of planar graphs without 4-cycles, *ISORA '09, Lecture Notes in Operations Research* ,**10** (2009), 174-178.

[9] J.L. Wu and Y.W. Wu, The linear arboricity of planar graphs of maximum degree seven are four, *J. Graph Theory*, **58**(2008),210-220.