Some narcissistic power-sequence $\mathbb{Z}_{n+1}$ terraces with n an odd prime power

Ian Anderson[a], D.A. Preece[b,c]

[a] *Department of Mathematics, University of Glasgow, University Gardens, Glasgow G12 8QW, UK*
ia@maths.gla.ac.uk

[b] *School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS, UK*
D.A.Preece@qmul.ac.uk

[c] *Institute of Mathematics, Statistics and Actuarial Science, Cornwallis Building, University of Kent, Canterbury, Kent CT2 7NF, UK*

Abstract

A terrace for $\mathbb{Z}_m$ is an arrangement $(a_1, a_2, \dots, a_m)$ of the m elements of $\mathbb{Z}_m$ such that the sets of differences $a_{i+1} - a_i$ and $a_i - a_{i+1}$ ($i = 1, 2, \dots, m-1$) between them contain each element of $\mathbb{Z}_m \setminus \{0\}$ exactly twice. For m odd, many procedures are available for constructing power-sequence terraces for $\mathbb{Z}_m$; each such terrace may be partitioned into segments one of which contains merely the zero element of $\mathbb{Z}_m$ whereas each other segment is either (a) a sequence of successive powers of a non-zero element of $\mathbb{Z}_m$ or (b) such a sequence multiplied throughout by a constant. For n an odd prime power satisfying $n \equiv 1$ or $3 \pmod 8$, this idea has previously been extended by using power-sequences in $\mathbb{Z}_n$ to produce some $\mathbb{Z}_m$ terraces $(a_1, a_2, \dots, a_m)$ where $m = n + 1 = 2\mu$, with $a_{i+1} - a_i = -(a_{i+1+\mu} - a_{i+\mu})$ for all $i \in [1, \mu - 1]$. Each of these "*da capo* directed terraces" consists of a sequence of segments, one containing just the element 0 and another just containing the element n, the remaining segments each being of type (a) or (b) above with each of its distinct entries x from $\mathbb{Z}_n \setminus \{0\}$ evaluated so that $1 \le x \le n-1$. Now, for many odd prime powers n satisfying $n \equiv 1 \pmod 4$, we similarly produce narcissistic terraces for $\mathbb{Z}_{n+1}$; these have $a_{i+1} - a_i = a_{m-i+1} - a_{m-i}$ for all $i \in [1, \mu - 1]$.

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1 Introduction

Let $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ be an arrangement of the elements of $\mathbb{Z}_m$, and let $\mathbf{b} = (b_1, b_2, \ldots, b_{m-1})$ be the ordered sequence $b_i = a_{i+1} - a_i$ for $i = 1, 2, \ldots, m-1$. The arrangement $\mathbf{a}$ is a *terrace* for $\mathbb{Z}_m$, with $\mathbf{b}$ as the corresponding *2-sequencing* or *quasi-sequencing* for $\mathbb{Z}_m$, if the sequences $\mathbf{b}$ and $-\mathbf{b}$ between them contain exactly two occurrences of each element x from $\mathbb{Z}_m \setminus \{0\}$. Clearly, if $\mathbf{a}$ is a terrace for $\mathbb{Z}_m$ (in short, a $\mathbb{Z}_m$ terrace), then so is any of its *translates* $(a_1 + y, a_2 + y, \ldots, a_m + y)$, all entries being calculated modulo m, where $y \in \mathbb{Z}_m$.

A $\mathbb{Z}_m$ terrace is *directed* [9] if all the elements in its 2-sequencing $(b_1, b_2, \ldots, b_{m-1})$ are distinct. For m even, a $\mathbb{Z}_m$ directed terrace is a *da capo* directed terrace [7] if $b_i \equiv -b_{(m/2)+i} \pmod{m}$ for all i satisfying $1 \leq i \leq (m-2)/2$, and $b_{m/2} \equiv m/2 \pmod{m}$. For m odd or even, a $\mathbb{Z}_m$ terrace is *narcissistic* [2, 6] if its 2-sequencing has $b_i = b_{m-i}$ for all $i = 1, 2, \ldots, (m-1)$; for m even, we clearly must then again have $b_{m/2} \equiv m/2 \pmod{m}$.

Some expositions include the zero element of $\mathbb{Z}_m$ in $\mathbf{b}$, as an extra element at the start, but we find this practice inconvenient; we follow various precedents by not adopting it.

Terraces were originally defined by Bailey [9] for a general finite group G, but the general case does not concern us here. They have been used in the construction of solutions to the Lucas round-dance problem [10] and the generalised Oberwolfach problem [15], and of combinatorial designs used in statistical applications involving carry-over effects [9, 1] and neighbour effects, but the present paper is not concerned with applications.

Anderson and Preece [2, 3, 4, 5] gave general constructions for "power-sequence" terraces for $\mathbb{Z}_m$ where m is odd. Each of these terraces can be partitioned into segments one of which contains merely the zero element of $\mathbb{Z}_m$, whereas each other segment is either (a) a sequence of successive powers of an element of $\mathbb{Z}_m$, or (b) such a sequence multiplied throughout by a constant. Many of the sequences $x^0,\ x^1,\ \ldots\ ,\ x^{s-1}$ of distinct elements are "full-cycle" sequences such that $x^s = x^0$, but partial cycles are used too.

The techniques used in [2, 3, 4, 5] are not adaptable to produce terraces from power-sequences in $\mathbb{Z}_m$ where m is even. However, for n odd, sequences of powers of non-zero elements of $\mathbb{Z}_n \pmod{n}$ can be used (perhaps counter-intuitively) to construct terraces for $\mathbb{Z}_{n-2}$, $\mathbb{Z}_{n-1}$, $\mathbb{Z}_{n+1}$ and $\mathbb{Z}_{n+2}$ [7, §1]. In [7] we used this approach to produce *da capo* directed terraces for $\mathbb{Z}_{n+1}$ where n is an odd prime power satisfying $n \equiv 1$ or $3 \pmod{8}$. We now similarly produce narcissistic terraces for $\mathbb{Z}_{n+1}$ where n is an odd prime power satisfying $n \equiv 1 \pmod 4$.

Lemma 1.1 *Suppose that a narcissistic terrace exists for $\mathbb{Z}_{n+1}$ where n is an odd positive integer. Then $n \equiv 1 \pmod 4$.*

Proof: Suppose that a narcissistic terrace exists for $\mathbb{Z}_{n+1}$ where $n \equiv 3 \pmod 4$, and write $n+1 = 4q$ for some integer q. The middle entry in the 2-sequencing for a narcissistic terrace for $\mathbb{Z}_{n+1}$ must be $2q$, so we need consider only the terrace's translate **a** such that $a_{2q} = q$ and $a_{2q+1} = 3q$. Then, by the narcissistic property, we must have $a_{2q-j} + a_{2q+1+j} \equiv 4q \pmod{4q}$ for $j = 0, 1, \ldots, 2q-1$, and so $\sum_{i=1}^{n+1} a_i \equiv 0 \pmod{4q}$. But $\sum_{i=1}^{n+1} a_i \equiv 0+1+2+\ldots+(4q-1) \equiv (4q-1)4q/2 \not\equiv 0 \pmod{4q}$, which gives us a contradiction. □

To construct each of our narcissistic terraces, we use power-sequences in $\mathbb{Z}_n$ to give us all elements of $\mathbb{Z}_n \setminus \{0\}$, each such element x always being written so as to satisfy $0 < x < n$. Then, as in [7], we introduce two further elements 0 and n to produce a sequence where each element y of $\mathbb{Z}_{n+1}$ occurs exactly once, written with $0 \leq y < n+1$. These two extra elements are, of course, both congruent to 0 (mod n), so our final sequence has two zeros when viewed modulo n. To avoid confusion when the construction of a $\mathbb{Z}_{n+1}$ terrace is specified in terms of operations in $\mathbb{Z}_n$, we represent the two zeros as 0_0 and 0_n, the subscript in each denoting the value to be taken when the terrace-sequence is interpreted modulo $n+1$. As in [7], we write $\mathbb{Z}_n^{\oplus} = (\mathbb{Z}_n \setminus \{0\}) \cup \{0_0\} \cup \{0_n\}$ where each element x from $\mathbb{Z}_n \setminus \{0\}$ is written with $0 < x < n$. When we consider our terrace-sequences modulo n, we regard 0_n as being the negative of 0_0.

Each narcissistic $\mathbb{Z}_{n+1}$ terrace **a** that we obtain has $a_i \equiv -a_{n+2-i}$, modulo n (not modulo $n+1$), so the second half of the terrace is the negative (mod n) of the reverse of the first half; in short, we refer to the second half as the *negative image* of the first half. Thus the narcissistic power-sequence $\mathbb{Z}_{14}$ terrace

13 | : 8 4 2 1 7 10 : ‖ : 3 6 12 11 9 5 : | 0

can be written more briefly as

13 | : 8 4 2 1 7 10 : ‖ neg. image .

Here, the double fence ‖ indicates the middle of the terrace, and any other fence | separates segments. The use of a single colon : at the start and end of a segment indicates a segment containing a sequence based on a half-cycle of successive powers (mod n) of some element from $\mathbb{Z}_n \setminus \{0\}$; the above example employs successive decreasing powers of 2 (mod 13), for which a full cycle is

$$8\ 4\ 2\ 1\ 7\ 10\ 5\ 9\ 11\ 12\ 6\ 3\,.$$

Some of our constructions have segments containing a quarter-cycle of elements (mod n), indicated by a double colon :: at the start and end of each such segment. For example, taking $n = 29$ in our Theorem 2.2 yields the narcissistic power-sequence $\mathbb{Z}_{30}$ terrace

$$::\ 3\ 6\ 12\ 24\ 19\ 9\ 18\ ::\ |\ 29\ |\ ::\ 16\ 8\ 4\ 2\ 1\ 15\ 22\ ::\ \|\ \text{neg. image}\,.$$

In the more succint notation used in [8], this $\mathbb{Z}_{30}$ terrace can be written

$$::\ 3\ \xrightarrow{2}\ ::\ |\ 29\ |\ ::\ 16\ \xleftarrow{2}\ ::\ \|\ \text{neg. image}\,.$$

Here and in similar circumstances, the first element of a segment is followed by $\xrightarrow{2}$ or $\xleftarrow{2}$ according as successive elements in the segment are obtained by multiplying, modulo n, by 2 or 2^{-1} respectively.

In *all* our present constructions, as in our constructions [7] of *da capo* terraces for $\mathbb{Z}_{n+1}$, each successive element in a power-sequence segment is obtained from the previous element by multiplying by 2 or by 2^{-1} (mod n). Thus, if two successive elements in a segment are a_i and a_{i+1}, one or other of the differences $a_{i+1} - a_i$ or $a_i - a_{i+1}$ is congruent to d_i (mod n) where $0 < d_i < n/2$, and that same difference is also congruent to d_i (mod $n+1$). Consequently, the within-segment differences between successive elements can be said to be *undisturbed* by the change from construction modulo n to interpretation modulo $n + 1$. Once again, of course, the difference (mod $n + 1$) at the double fence must be $(n + 1)/2$, and the differences at the other fences must compensate for the "missing" differences lost by the breaking of cycles of powers of 2. In discussing differences, we again take a difference in whichever direction (right-minus-left or left-minus-right) yields a positive value less than or equal to $(n + 1)/2$. Sometimes, in compensating for a lost difference d, where $0 < d < n/2$, a construction provides a fence-difference that is indeed d modulo $n+1$ but is $d-1$ modulo n; we call such a compensating difference a *raised difference.* Such raised differences arise from the constructions in Theorems 2.12 to 2.14 below.

If a fence separates the element 0_n from an element x that lies in $(0, n)$, then the fence difference is undisturbed in the above sense if and only if $x > n/2$, *i.e.* if and only if $2x$ is odd when evaluated, modulo n, so as to lie in $(0, n)$.

2 $\mathbb{Z}_{n+1}$ terraces for primes n

As our terraces must have $n \equiv 1 \pmod 4$, we have either $n \equiv 1 \pmod 8$, in which case 2 is a square in $\mathbb{Z}_n$, or $n \equiv 5 \pmod 8$, in which case 2 is not a square in $\mathbb{Z}_n$. We start with two theorems where 2 is a primitive root of n, so that $n \equiv 5 \pmod 8$. The statements of these and all subsequent theorems contain sequences each of which is an arrangement of the elements of $\mathbb{Z}_n^{\oplus}$ and each of which becomes a narcissistic terrace for $\mathbb{Z}_{n+1}$ when evaluated as follows. *First, each element from $\mathbb{Z}_n \setminus \{0\}$ is evaluated, modulo n, in the interval $(0, n)$. Then the entire sequence is interpreted modulo $n+1$, with 0_0 interpreted as 0, and 0_n interpreted as n.*

Theorem 2.1 *Let n be a prime, $n \equiv 5 \pmod 8$, that has 2 as a primitive root. A narcissistic terrace for $\mathbb{Z}_{n+1}$ is obtainable from the sequence*

$$0_n \;\mid\; : \; -2^{-3} \quad -2^{-4} \quad \ldots \quad +2^{-2} \; : \;\| \text{ neg. image .}$$

Proof: The sequence is

$$n \;\mid\; (5n-1)/8 \;\; \ldots \;\; (3n+1)/4 \;\|\; (n-1)/4 \;\; \ldots \;\; (3n+1)/8 \;\mid\; 0 \,.$$

The fence differences are $(3n+1)/8$ (twice) and $(n+1)/2$ (at the double fence). The missing differences are $2^{-3} = (3n+1)/8$ (twice), so each is compensated for by a fence difference, and the difference $(n+1)/2$ is achieved. □

Example 2.1: For $n = 13$, Theorem 2.1 gives the narcissistic $\mathbb{Z}_{14}$ terrace

$$13 \;\mid\; : \; 8 \;\; 4 \;\; 2 \;\; 1 \;\; 7 \;\; 10 \; : \;\|\; : \; 3 \;\; 6 \;\; 12 \;\; 11 \;\; 9 \;\; 5 \; : \;\mid\; 0 \,.$$

Theorem 2.2 *Let n be a prime, $n \equiv 5 \pmod 8$, that has 2 as a primitive root. Write $n = 8\nu + 5$. A narcissistic terrace for $\mathbb{Z}_{n+1}$ is obtainable from the sequence*

$$:: \; 2^{2\nu-1} \quad 2^{2\nu} \quad \ldots \quad 2^{4\nu-1} \; :: \mid 0_n \mid$$

$$:: \; 2^{2\nu-2}\alpha \quad 2^{2\nu-3}\alpha \quad \ldots \quad 2^{-2}\alpha \; :: \;\| \text{ neg. image}$$

where $\alpha = +1$ if $n/2 < 2^{2\nu-2} < n$ and $\alpha = -1$ if $0 < 2^{2\nu-2} < n/2$.

Proof: Consider the case $\alpha = +1$. We have $2^{4\nu-1} = -2^{-3} = (5n-1)/8$. The sequence is

$$2^{2\nu-1} \;\; \ldots \;\; (5n-1)/8 \;\mid\; n \;\mid\; 2^{2\nu-2} \;\; \ldots \;\; (3n+1)/4 \;\|\; (n-1)/4 \;\; \ldots$$

where $n/2 < 2^{2\nu-2} < n$. So the fence differences $(3n+1)/8$ and $n - 2^{2\nu-2}$ compensate for the missing differences, and the required extra difference $(n+1)/2$ is achieved at the double fence.

The case $\alpha = -1$ is dealt with similarly. □

Example 2.2: For $(n, \alpha) = (13, -1)$, Theorem 2.2 gives the narcissistic $\mathbb{Z}_{14}$ terrace

$$:: \ 2 \ 4 \ 8 \ :: \ | \ 13 \ | \ :: \ 12 \ 6 \ 3 \ :: \ \| \ \text{neg. image} \ .$$

We now present two theorems where $\text{ord}_n(2) = (n-1)/2$, so that $n \equiv 1 \pmod 8$. We use $\langle 2 \rangle$ to denote the set of elements of $\mathbb{Z}_n$ that are generated by the element 2.

Theorem 2.3 *Let n be a prime, $n \equiv 1 \pmod 8$, such that $\text{ord}_n(2) = (n-1)/2$. A narcissistic terrace for $\mathbb{Z}_{n+1}$ is obtainable from the sequence*

$$: \ -2^1c \ \ -2^2c \ \ldots \ +2^0c \ : \ | \ 0_n \ | \ : \ +2^{-3} \ \ +2^{-4} \ \ldots \ -2^{-2} \ : \ \| \ \text{neg. image}$$

where c is any element satisfying $(n+1)/2 < c < n-1$ and $c \notin \langle 2 \rangle$.

Proof: As $n \equiv 1 \pmod 4$, the element -1 is a square in $\mathbb{Z}_n$. The squares in $\mathbb{Z}_n$ are precisely the powers of 2, and $2^{(n-1)/4} \equiv -1 \pmod n$. The sequence can be written as

$$-2c \ \ldots \ c \ | \ n \ | \ (7n+1)/8 \ \ldots \ (n-1)/4 \ \| \ (3n+1)/4 \ \ldots \ .$$

Again, the fence differences compensate for the missing differences and give $(n+1)/2$ as the middle difference. □

Example 2.3: For $(n, c) = (17, 11)$, Theorem 2.3 yields the narcissistic $\mathbb{Z}_{18}$ terrace

$$: \ 12 \ 7 \ 14 \ 11 \ : \ | \ 17 \ | \ : \ 15 \ 16 \ 8 \ 4 \ : \ \| \ : \ 13 \ 9 \ 1 \ 2 \ : \ | \ 0 \ | \ : \ 6 \ 3 \ 10 \ 5 \ : \ .$$

Theorem 2.4 *Let n be a prime, $n \equiv 17 \pmod{24}$, such that $\text{ord}_n(2) = (n-1)/2$. A narcissistic terrace for $\mathbb{Z}_{n+1}$ is obtainable from the sequence*

$$0_0 \ | \ : \ -3 \cdot 2^{-3} \ \ -3 \cdot 2^{-4} \ \ldots \ +3 \cdot 2^{-2} \ : \ |$$

$$: \ +2^{-1} \ \ +2^0 \ \ldots \ -2^{-2} \ : \ \| \ \text{neg. image} \ .$$

Proof: As $n \equiv 5 \pmod{12}$, the element 3 is not a square in $\mathbb{Z}_n$ and hence $3 \notin \langle 2 \rangle$. Again, $-1 = 2^{(n-1)/4}$. The sequence is

$$0 \ | \ (3n-3)/8 \ \ldots \ (n+3)/4 \ | \ (n+1)/2 \ \ldots \ (n-1)/4 \ \| \ (3n+1)/4 \ \ldots \ .$$

The fence differences are $(3n-3)/8$ (twice), $(n-1)/4$ (twice) and $(n+1)/2$. The missing differences are $-3 \cdot 2^{-3} = (3n-3)/8$ (twice), and $-2^{-2} = (n-1)/4$ (twice). □

Example 2.4: For $n = 17$, Theorem 2.4 yields the narcissistic $\mathbb{Z}_{18}$ terrace

$$0 \mid : 6\ 3\ 10\ 5 : \mid : 9\ 1\ 2\ 4 : \| : 13\ 15\ 16\ 8 : \mid : 12\ 7\ 14\ 11 : \mid 17 .$$

We now give a theorem where $\text{ord}_n(2) = (n-1)/3$. We use $\langle 2, 3\rangle$ to denote the set of elements of $\mathbb{Z}_n$ that are generated by the elements 2 and 3. We here need $n \equiv 1 \pmod 3$, so that, by Lemma 1.1, we have $n \equiv 1 \pmod{12}$. If we were to take $n \equiv 1 \pmod{24}$, then the elements 2 and 3 would be squares in $\mathbb{Z}_n$ and so $\langle 2, 3\rangle$ could not be $\mathbb{Z}_n \setminus \{0\}$. So we restrict ourselves to $n \equiv 13 \pmod{24}$.

Theorem 2.5 *Let n be a prime, $n \equiv 13 \pmod{24}$, such that $\text{ord}_n(2) = (n-1)/3$ and $\mathbb{Z}_n \setminus \{0\} = \langle 2, 3\rangle$. Narcissistic terraces for $\mathbb{Z}_{n+1}$ are obtainable from the sequences*
(i)

$$: +3{\cdot}2^{-2}\ \ +3{\cdot}2^{-3}\ \ \ldots\ \ -3{\cdot}2^{-1}\ : \mid : -3^2{\cdot}2^{-2}\ \ -3^2{\cdot}2^{-1}\ \ \ldots\ \ +3^2{\cdot}2^{-3}\ : \mid$$
$$0_0 \mid : +2^{-3}\ \ +2^{-4}\ \ \ldots\ \ -2^{-2}\ : \|\ \text{neg. image}$$

(ii)

$$: +3^2\cdot 2^{-2}\ \ +3^2\cdot 2^{-1}\ \ \ldots\ \ -3^2\cdot 2^{-3}\ : \mid 0_n \mid$$
$$: -3\cdot 2^{-3}\ \ -3\cdot 2^{-4}\ \ \ldots\ \ +3\cdot 2^{-2}\ : \mid$$
$$: +2^{-1}\ \ +2^{0}\ \ \ldots\ \ -2^{-2}\ : \|\ \text{neg. image}$$

(iii)

$$0_0 \mid : +3^2\cdot 2^{-3}\ \ +3^2\cdot 2^{-4}\ \ \ldots\ \ -3^2\cdot 2^{-2}\ : \mid$$
$$: -3\cdot 2^{-1}\ \ -3\cdot 2^{0}\ \ \ldots\ \ +3\cdot 2^{-2}\ : \mid$$
$$: +2^{-1}\ \ +2^{0}\ \ \ldots\ \ -2^{-2}\ : \|\ \text{neg. image}\ .$$

Proof: As $(n-1)/3$ is even, we have $2^{(n-1)/6} = -1$ and so $-1 \in \langle 2\rangle$.

(i) The sequence is

$$: (n+3)/4\ \ \ldots\ \ (n-3)/2\ : \mid : (n-9)/4\ \ \ldots\ \ (3n+9)/8\ : \mid 0 \mid$$
$$: (3n+1)/8\ \ \ldots\ \ (n-1)/4\ : \| : (3n+1)/4\ \ \ldots\ .$$

The missing differences are $3 \cdot 2^{-2} = (n+3)/4$ and $3^2 \cdot 2^{-3} = (3n+9)/8$ and $2^{-3} = (3n+1)/8$ (each twice); the fence differences are precisely these numbers, as well as $(n+1)/2$ once.

(ii) Here the sequence is

$$: (3n+9)/4\ \ \ldots\ \ (5n-9)/8\ : \mid n \mid : (7n-3)/8\ \ \ldots\ \ (n+3)/4\ : \mid$$
$$: (n+1)/2\ \ \ldots\ \ (n-1)/4\ : \| : (3n+1)/4\ \ \ldots$$

and the differences are easily checked.

(iii) Here the sequence is

$$0 \mid : (3n+9)/8 \ \ldots \ (n-9)/4 : \mid : (n-3)/2 \ \ldots \ (n+3)/4 : \mid$$
$$: (n+1)/2 \ \ldots \ (n-1)/4 : \parallel : (3n+1)/4 \ \ldots$$

and again the differences are easily checked. □

Examples 2.5: For $n = 109$, Theorem 2.5 gives the following narcissistic terraces for $\mathbb{Z}_{110}$:

(i)

$$: 28 \ 14 \ \ldots \ 53 : \mid : 25 \ 50 \ \ldots \ 42 : \mid 0 \mid : 41 \ 75 \ \ldots \ 27 : \parallel \text{neg. image}$$

(ii)

$$: 84 \ 59 \ \ldots \ 67 : \mid 109 \mid : 95 \ 102 \ \ldots \ 28 : \mid : 55 \ 1 \ \ldots \ 27 : \parallel \text{neg. image}$$

(iii)

$$0 \mid : 42 \ 21 \ \ldots \ 25 : \mid : 53 \ 106 \ \ldots \ 28 : \mid : 55 \ 1 \ \ldots \ 27 : \parallel \text{neg. image}$$

Next we have a theorem where $\mathrm{ord}_n(2) = (n-1)/4$. For $(n-1)/4$ to be even, so that $-1 \in \langle 2 \rangle$, we need $n \equiv 1 \pmod 8$. Then 2 is a square in $\mathbb{Z}_n$ and so, if we wish to have $\mathbb{Z}_n \setminus \{0\} = \langle 2, 3 \rangle$, the element 3 must be a non-square in $\mathbb{Z}_n$ and so $n \equiv 5 \pmod{12}$. Thus we consider $n \equiv 17 \pmod{24}$.

Theorem 2.6 *Let n be a prime, $n \equiv 17 \pmod{24}$, such that* $\mathrm{ord}_n(2) = (n-1)/4$ *and* $\mathbb{Z}_n \setminus \{0\} = \langle 2, 3 \rangle$. *Narcissistic terraces for* $\mathbb{Z}_{n+1}$ *are obtainable from the sequences*

(i)

$$: -3\cdot 2^{-2} \ \ -3\cdot 2^{-3} \ \ldots \ +3\cdot 2^{-1} : \mid : +3^2\cdot 2^{-2} \ \ +3^2\cdot 2^{-3} \ \ldots \ -3^2\cdot 2^{-1} : \mid$$
$$: -3^3\cdot 2^{-2} \ \ -3^3\cdot 2^{-1} \ \ldots \ +3^3\cdot 2^{-3} : \mid 0_n \mid$$
$$: +2^{-3} \ \ +2^{-4} \ \ldots \ -2^{-2} : \parallel \text{neg. image}$$

(ii)

$$: -3^2\cdot 2^{-2} \ \ -3^2\cdot 2^{-3} \ \ldots \ +3^2\cdot 2^{-1} : \mid$$
$$: +3^3\cdot 2^{-2} \ \ +3^3\cdot 2^{-1} \ \ldots \ -3^3\cdot 2^{-3} : \mid$$
$$0_0 \mid : -3\cdot 2^{-3} \ \ -3\cdot 2^{-4} \ \ldots \ +3\cdot 2^{-2} : \mid$$
$$: +2^{-1} \ \ +2^{0} \ \ldots \ -2^{-2} : \parallel \text{neg. image}$$

(iii)

$$: \ -3^3\cdot 2^{-2} \quad -3^3\cdot 2^{-1} \ \ldots \ +3^3\cdot 2^{-3} \ : \ | \ 0_n \ |$$
$$: \ +3^2\cdot 2^{-3} \quad +3^2\cdot 2^{-4} \ \ldots \ -3^2\cdot 2^{-2} \ : \ |$$
$$: \ -3\cdot 2^{-1} \quad -3\cdot 2^{0} \ \ldots \ +3\cdot 2^{-2} \ : \ |$$
$$: \ +2^{-1} \quad +2^{0} \ \ldots \ -2^{-2} \ : \ \| \text{ neg. image}$$

(iv)

$$0_0 \ | \ : \ -3^3\cdot 2^{-3} \quad -3^3\cdot 2^{-4} \ \ldots \ +3^3\cdot 2^{-2} \ : \ |$$
$$: \ +3^2\cdot 2^{-1} \quad +3^2\cdot 2^{0} \ \ldots \ -3^2\cdot 2^{-2} \ : \ |$$
$$: \ -3\cdot 2^{-1} \quad -3\cdot 2^{0} \ \ldots \ +3\cdot 2^{-2} \ : \ |$$
$$: \ +2^{-1} \quad +2^{0} \ \ldots \ -2^{-2} \ : \ \| \text{ neg. image .}$$

Proof: Similar to that for Theorem 2.5. □

Example 2.6: For $n = 113$, sequence (i) of Theorem 2.6 yields the narcissistic $\mathbb{Z}_{114}$ terrace

$$: \ 84 \ 42 \ \ldots \ 58 \ : \ | \ : \ 87 \ 100 \ \ldots \ 52 \ : \ | \ : \ 78 \ 43 \ \ldots \ 74 \ : \ |$$
$$113 \ | \ : \ 99 \ 106 \ \ldots \ 28 \ : \ \| \text{ neg. image .}$$

Theorem 2.7 *Let n be a prime, $n \equiv 17 \pmod{24}$, such that $\mathrm{ord}_n(2)$ is odd and equal to $(n-1)/2h$ where $h > 1$, and such that $\mathbb{Z}_n \setminus \{0\} = \langle 2, 3\rangle$. Suppose that the integers $3^0\cdot 2^{-2}, 3^1\cdot 2^{-2}, \ldots, 3^{h-1}\cdot 2^{-2}$ are all odd when evaluated, modulo n, to lie in the interval $(0, n)$. Narcissistic terraces for $\mathbb{Z}_{n+1}$ are obtainable from the sequences*

(i)

$$3^1\cdot 2^{-2} \ \overset{2}{\leftarrow} \ | \ 3^2\cdot 2^{-2} \ \overset{2}{\leftarrow} \ | \ \cdots \ | \ 3^{h-2}\cdot 2^{-2} \ \overset{2}{\leftarrow} \ |$$
$$3^{h-1}\cdot 2^{-2} \ \overset{2}{\rightarrow} \ | \ 0_n \ | \ 3^0\cdot 2^{-3} \ \overset{2}{\leftarrow} \ \| \text{ neg. image}$$

(ii)

$$0_n \ | \ 3^{h-1}\cdot 2^{-3} \ \overset{2}{\leftarrow} \ |$$
$$3^{h-2}\cdot 2^{-1} \ \overset{2}{\rightarrow} \ | \ 3^{h-3}\cdot 2^{-1} \ \overset{2}{\rightarrow} \ | \ \cdots \ | \ 3^0\cdot 2^{-1} \ \overset{2}{\rightarrow} \ \| \text{ neg. image}$$

(iii) , *with* $1 < i < h-1$,

$$3^{h-i}\cdot 2^{-2} \ \overset{2}{\leftarrow} \ | \ 3^{h-i+1}\cdot 2^{-2} \ \overset{2}{\leftarrow} \ | \ \cdots \ | \ 3^{h-2}\cdot 2^{-2} \ \overset{2}{\leftarrow} \ |$$
$$3^{h-1}\cdot 2^{-2} \ \overset{2}{\rightarrow} \ | \ 0_n \ | \ 3^{h-i-1}\cdot 2^{-3} \ \overset{2}{\leftarrow} \ |$$
$$3^{h-i-2}\cdot 2^{-1} \ \overset{2}{\rightarrow} \ | \ 3^{h-i-3}\cdot 2^{-1} \ \overset{2}{\rightarrow} \ | \ \cdots \ | \ 3^0\cdot 2^{-1} \ \overset{2}{\rightarrow} \ \| \text{ neg. image .}$$

Proof: The requirement that $\mathrm{ord}_n(2)$ is odd forces 2 to be a square in $\mathbb{Z}_n$, so $n \equiv 1 \pmod 8$. Then 3 must not be a square in $\mathbb{Z}_n$ so $n \equiv 5 \pmod{12}$. This explains the condition $n \equiv 17 \pmod{24}$. Also, the given conditions imply that $-1 \in 3^h\langle 2\rangle$.

(i) The fence differences around 0_n are undisturbed as $3^i 2^{-3} \in (n/2, n)$ since $3^i 2^{-2}$ is odd, and $2^{-3} = (7n+1)/8 > n/2$. All the segments before 0_n contain the pattern $x \;\ldots\; 2x \mid 3x \;\ldots$. If $x < n/3$, then both missing and fence differences x are undisturbed. If $n/3 < x < 2n/3$, the value taken by the element $3x$ is $3x - n$, which is even, so this possibility does not arise. If $2n/3 < x < n$, then we have $x \;\ldots\; 2x - n \mid 3x - 2n \;\ldots$, and the differences are $n - x < n/2$.

(ii) and (iii) have similar proofs. □

Note 2.7(a): Theorem 2.7 is the first of several where we have found the use of the arrow notations $\xrightarrow{2}$ and $\xleftarrow{2}$ to be helpful. In this set of theorems there is a simple rule for determining the direction of each arrow: *In each half of a terrace, the arrows in the two segments on either side of the zero* 0_0 *or* 0_n *should point towards the zero; the other arrows should point away from the zero.* The same rule applies throughout all our previous theorems, if these are rewritten using the arrow notation.

Note 2.7(b): The requirement in Theorem 2.7, and in many of the subsequent theorems, that each member of a sequence

$$3^0c,\ 3^1c,\ \ldots,\ 3^{\gamma-1}c$$

is odd when evaluated as described, is to ensure that the fence differences are undisturbed and hence compensate for the missing differences. (See the proof of Theorem 2.7.) This requirement is equivalent to the requirement

$$c \in (2\delta n/3^{\gamma-1}, (2\delta+1)n/3^{\gamma-1})$$

where δ is any integer $3^0\beta_0 + 3^1\beta_1 + \cdots + 3^{\gamma-2}\beta_{\gamma-2}$ such that each of the $\gamma - 1$ values β_i $(i = 0, 1, \ldots, \gamma-2)$ is separately either 0 or 1. Thus

$$\gamma = 2: \quad c \in (0, n/3) \cup (2n/3, n)\ ;$$
$$\gamma = 3: \quad c \in (0, n/9) \cup (2n/9, n/3) \cup (2n/3, 7n/9) \cup (8n/9, n);$$

and so on.

So, for successive values of γ, the successively smaller disconnected regions within which c must lie are, apart trivially from the end-points of the intervals, successive approximations to the *Cantor set* [13, 16], to which they converge as $\gamma \to \infty$ (see discussions of "triadic Cantor dust" in, for example, [14, §1.1]). We judge the above requirement on a value c to be more useful for checking correctness of c-values than for finding them in the first place.

Note 2.7(c): If a prime n satisfies the conditions in the first sentence of Theorem 2.7, and $h \le 4$, then consideration of Note 2.7(b) shows that the supposition in the second sentence of the statement of Theorem 2.7 is invariably true. For example, $3 \cdot 2^{-2} = (n+3)/4$ lies in $(2n/9,\ n/3)$.

Note 2.7(d): In the range $3 < n < 1000$, Theorem 2.7 covers $(n, h) = (89, 4)$, $(233, 4)$ and $(881, 8)$.

Example 2.7: With $(n, h, i) = (89, 4, 2)$, sequence (iii) in Theorem 2.7 yields the narcissistic $\mathbb{Z}_{90}$ terrace

$$69 \;\; 79 \;\; \ldots \;\; 49 \mid 29 \;\; 58 \;\; \ldots \;\; 59 \mid 89 \mid$$

$$56 \;\; 28 \;\; \ldots \;\; 23 \mid 45 \;\; 1 \;\; \ldots \;\; 67 \parallel \text{neg. image}\,.$$

The next two constructions apply where 2 and 3 generate the set of non-zero squares in $\mathbb{Z}_n$. For 2 and 3 to be squares we require $n \equiv 1 \pmod{24}$.

Theorem 2.8 *Let n be a prime, $n \equiv 1 \pmod{24}$, such that $\mathrm{ord}_n(2)$ is odd and equal to $(n-1)/4g$ where $g > 1$, and such that $\langle 2, 3\rangle$ is the set of non-zero squares in $\mathbb{Z}_n$. Suppose that the integers $3^0 \cdot 2^{-2}, 3^1 \cdot 2^{-2}, \ldots, 3^{g-1} \cdot 2^{-2}$ are all odd when evaluated, modulo n, to lie in the interval $(0, n)$. Suppose further that there exists a non-square c in $\mathbb{Z}_n$ such that the integers $3^0 c$, $3^1 c, \ldots, 3^{g-1} c$ are all odd when evaluated, modulo n, to lie in the interval $(0, n)$. A narcissistic terrace for $\mathbb{Z}_{n+1}$ is obtainable from the sequence*

$$3^0 c \;\overset{2}{\leftarrow} \mid 3^1 c \;\overset{2}{\leftarrow} \mid \cdots \mid 3^{g-2} c \;\overset{2}{\leftarrow} \mid 3^{g-1} c \;\overset{2}{\rightarrow} \mid 0_n \mid 3^{g-1} . 2^{-3} \;\overset{2}{\leftarrow} \mid$$

$$3^{g-2} \cdot 2^{-1} \;\overset{2}{\rightarrow} \mid 3^{g-3} \cdot 2^{-1} \;\overset{2}{\rightarrow} \mid \cdots \mid 3^0 \cdot 2^{-1} \;\overset{2}{\rightarrow} \parallel \text{neg. image}\,.$$

Proof: As -1 is a square in $\mathbb{Z}_n$, we have $-1 \in \langle 2, 3\rangle$ and so $-1 \in 3^g\langle 2\rangle$. Thus the negatives of the elements of $\bigcup_{i=0}^{g-1} 3^i\langle 2\rangle$ are precisely the elements of $\bigcup_{i=g}^{2g-1} 3^i\langle 2\rangle$. So the given sequence contains all the elements of $\mathbb{Z}_n \setminus \{0\}$.

The fence differences around 0_n are undisturbed as $3^{g-1}c$ and $3^{g-1} \cdot 2^{-2}$ are both odd. To the left of 0_n, the pattern $x \;\ldots\; 2x \mid 3x$ occurs, and to the right the pattern is $3y \mid 2y \;\ldots\; y$. The argument now follows that of the previous theorem. □

Note 2.8(a): The supposition in the second sentence of the statement of Theorem 2.8 is invariably true if $g \le 4$.

Note 2.8(b): In the range $3 < n < 1000$, Theorem 2.8 covers $(n, g) = (73, 2)$ and $(337, 4)$.

Example 2.8: For $n = 73$ we have $g = 2$ in Theorem 2.8, so the odd integer c must satisfy $c \in (0, n/3)$ or $(2n/3, n)$ as well as $c \notin \langle 2, 3\rangle$. Thus the admissible values of c are 5, 7, 11, 13, 15, 17, 21, and 51, 53, 59, 63. With $c = 5$ we obtain the narcissistic $\mathbb{Z}_{74}$ terrace

$$5\ 39\ \ldots\ 10\ |\ 15\ 30\ \ldots\ 44\ |\ 73\ |$$

$$46\ 23\ \ldots\ 19\ |\ 37\ 1\ \ldots\ 55\ \|\ \text{neg. image}\ .$$

Theorem 2.9 *Let n be a prime, $n \equiv 1 \pmod{24}$, such that $\mathrm{ord}_n(2)$ is even and equal to $(n-1)/2h$ where $h > 1$, and such that $\langle 2, 3\rangle$ is the set of non-zero squares in $\mathbb{Z}_n$. Suppose that the integers $+3^0 \cdot 2^{-2}$, $+3^1 \cdot 2^{-2}$, $\ldots$, $+3^{h-1} \cdot 2^{-2}$ are all odd when evaluated, modulo n, to lie in the interval $(0, n)$. Suppose further that there exists a non-square c in $\mathbb{Z}_n$, such that the integers 3^0c, 3^1c, $\ldots$, $3^{h-1}c$ are all odd when evaluated, modulo n, to lie in the interval $(0, n)$. A narcissistic terrace for $\mathbb{Z}_{n+1}$ is obtainable from the sequence*

$$:\ (-1)^1 \cdot 3^0 c\ \stackrel{2}{\leftarrow}\ :\ |\ :\ (-1)^2 \cdot 3^1 c\ \stackrel{2}{\leftarrow}\ :\ |\ \cdots\ |\ :\ (-1)^{h-1} \cdot 3^{h-2} c\ \stackrel{2}{\leftarrow}\ :\ |$$

$$:\ (-1)^h \cdot 3^{h-1} c\ \stackrel{2}{\rightarrow}\ :\ |\ \alpha\ |\ :\ (-3)^{h-1} \cdot 2^{-3}\ \stackrel{2}{\leftarrow}\ :\ |$$

$$:\ (-3)^{h-2} \cdot 2^{-1}\ \stackrel{2}{\rightarrow}\ :\ |\ :\ (-3)^{h-3} \cdot 2^{-1}\ \stackrel{2}{\rightarrow}\ :\ |\ \cdots\ |$$

$$:\ (-3)^0 \cdot 2^{-1}\ \stackrel{2}{\rightarrow}\ :\ \|\ \text{neg. image}$$

where α is again 0_n or 0_0 according as h is odd or even.

Proof: Similar to that of Theorem 2.8. Here the pattern is $-x\ \ldots\ 2x\ |\ 3x$ to the left of α. □

Note 2.9: In the range $3 < n < 1000$, Theorem 2.9 covers $(n, h) = (241, 5)$, $(433, 3)$ and $(457, 3)$. It fails for $(n, h) = (673, 7)$, as then we have $3^{h-1} \cdot 2^{-2} = 3^6 \cdot 2^{-2} = 14$, which is even; here we have a "very near miss", which we circumvent by Theorem 2.14 below.

Examples 2.9 For $(n, h) = (241, 5)$, Theorem 2.9 provides narcissistic $\mathbb{Z}_{242}$ terraces with $c = 7$, 19, 55, 73, 163, 167, 179, 185 or 215. For each of $(n, h) = (433, 3)$ and $(457, 3)$ the possible c-values include 5.

Theorem 2.10 *Let n be a prime, $n \equiv 5 \pmod 8$, such that* $\mathrm{ord}_n(2) = (n-1)/k$ *where k is odd, $k > 3$, and such that $\mathbb{Z}_n \setminus \{0\} = \langle 2, 3 \rangle$. Suppose that, for some integer i satisfying $1 < i < k-1$, the integers $-3^0 \cdot 2^{-2}$, $-3^1 \cdot 2^{-2}, \ldots, -3^{k-i-1} \cdot 2^{-2}$ are all odd when evaluated, modulo n, to lie in the interval $(0, n)$. Suppose further that there exists an integer c, satisfying $c \in 3^{-i}\langle 2 \rangle$, such that the integers $3^0 c$, $3^1 c$, $\ldots$, $3^{i-1} c$ are all odd when evaluated, modulo n, to lie in the interval $(0, n)$. A narcissistic terrace for $\mathbb{Z}_{n+1}$ is obtainable from the sequence*

$$: -3^0 c \overset{2}{\leftarrow} : \mid : +3^1 c \overset{2}{\leftarrow} : \mid \;\cdots\; \mid : (-1)^{i-1} \cdot 3^{i-2} c \overset{2}{\leftarrow} : \mid$$

$$: (-1)^i \cdot 3^{i-1} c \overset{2}{\rightarrow} : \mid \alpha \mid : (-3)^{k-i-1} \cdot 2^{-3} \overset{2}{\leftarrow} : \mid$$

$$: (-3)^{k-i-2} \cdot 2^{-1} \overset{2}{\rightarrow} : \mid : (-3)^{k-i-3} \cdot 2^{-1} \overset{2}{\rightarrow} : \mid \cdots \mid$$

$$: +3^0 \cdot 2^{-1} \overset{2}{\rightarrow} : \parallel \text{neg. image}$$

where $\alpha = 0_n$ or 0_0 according as i is odd or even.

Proof: As $\mathrm{ord}_n(2)$ is even, we have $-1 \in \langle 2 \rangle$. If i is odd, then the term to the left of α is $3^{i-1}c \cdot 2^{-1} = (3^{i-1}c + n)/2 > n/2$, so the difference is undisturbed if $\alpha = 0_n$. Also, if i is odd then, as k is odd, the term to the right of α is $-3^{k-i-1} \cdot 2^{-3}$, which is greater than $n/2$ since $-3^{k-i-1} \cdot 2^{-2}$ is odd. A similar anlaysis holds when i is even.

To the left of α the pattern is $x \;\ldots\; -2x \mid -3x$ and to the right it is $-3y \mid -2y \;\ldots\; y$; again the proof is as for Theorem 2.7. □

Note 2.10: In the range $3 < n < 1000$, Theorem 2.10 covers only one n-value satisfying $n \equiv 13 \pmod{24}$, namely $n = 397$, for which $k = 9$. The smallest n-value satisfying $n \equiv 5 \pmod{24}$ that is covered is $n = 1181$, for which $k = 5$. (The value $n = 1013$, for which $k = 11$, fails.)

Example 2.10: For $n = 397$, Theorem 2.10 yields one narcissistic terrace or more for each of the values $i = 2, 3, \ldots, 7$. For example, for $i = 3$ we can take $c =$ 15, 17, 95, 103, 123, 125, 267, 277, 363, 367.

Theorem 2.11 *Let n be a prime, $n \equiv 17 \pmod{24}$, such that* $\mathrm{ord}_n(2) = (n-1)/k$ *where* $\mathrm{ord}_n(2)$ *and k are both even, $k \geq 4$, and such that $\mathbb{Z}_n \setminus \{0\} = \langle 2, 3 \rangle$. Suppose that, for some integer i satisfying $1 < i < k-1$, the integers $+3^0 \cdot 2^{-2}$, $+3^1 \cdot 2^{-2}, \ldots, +3^{k-i-1} \cdot 2^{-2}$ are all odd when evaluated, modulo n, to lie in the interval $(0, n)$. Suppose further that there exists an integer c, satisfying $c \in 3^{-i}\langle 2 \rangle$, such that the integers $3^0 c$, $3^1 c$, $\ldots$, $3^{i-1} c$ are all odd when evaluated, modulo n, to lie in the interval $(0, n)$. A narcissistic terrace for $\mathbb{Z}_{n+1}$ is obtainable from the sequence given in* Theorem 2.10, *where α is again 0_n or 0_0 according as i is odd or even.*

Proof: As $n \equiv 17 \pmod{24}$, the element 2 is a square in $\mathbb{Z}_n$ and 3 is not. As $\mathrm{ord}_n(2)$ is even, $-1 \in \langle 2 \rangle$. The proof is similar to the previous one. □

Note 2.11: In the range $3 < n < 1000$, Theorem 2.11 with $k > 4$ covers only $(n, k) = (257, 16)$, $(641, 10)$ and $(953, 14)$. With $k = 4$ (which is what we have for $n = 113$, 281, 353, 593 and 617), the terraces (ii) of Theorem 2.6 are obtainable from Theorem 2.11 with $i = 2$ and $c = 3^2 \cdot 2^{-2}$, but other c-values can be used too.

Examples 2.11: For $(n, k) = (257, 16)$, the only possible solution from Theorem 2.11 has $(i, c) = (10, 79)$. For $(n.k) = (641, 10)$ various solutions exist, including $(i, c) = (6, 7)$, $(5, 67)$ and $(4, 11)$. Likewise for $(n, k) = (953, 14)$ various solutions exist, including $(i, c) = (8, 871)$, $(7, 707)$ and $(6, 73)$.

Within the range $3 < n < 1000$, our theorems so far have failed to cover $n = 577$, 601, 673, 937 and 997. Indeed, we have no construction for $n = 601$. However, using constructions that depend on raised differences, we can cover the other four of the n-values. We now present Theorems 2.12 and 2.13 which, between them, cover $n = 577$, 937 and 997. Like Theorem 2.10 in [7], these next two theorems employ a sequence of odd integers c_i with $c_{i+1} = 3c_i + 2$. (As illustrated by Theorem 5.4 of [8], the equation $c_{i+1} = 3c_i + 2$ would be replaced by $c_{i+1} = 3c_i + 4$ if we were using $\mathbb{Z}_n$ power-sequences to construct terraces for $\mathbb{Z}_{n+2}$ instead of $\mathbb{Z}_{n+1}$.)

Theorem 2.12 *Let n be a prime, $n \equiv 1 \pmod 4$, such that $\mathrm{ord}_n(2) = (n-1)/k$ where $\mathrm{ord}_n(2)$ is even and $k > 2$. Suppose that $\mathbb{Z}_n \setminus \{0\} = \langle 2 \rangle \cup c_1\langle 2 \rangle \cup c_2\langle 2 \rangle \cup \cdots \cup c_{k-1}\langle 2 \rangle$ where the integers c_i $(i = 1, 2, \ldots, k-1)$, satisfying $c_{i+1} = 3c_i + 2$ $(i = 1, 2, \ldots, k-2)$, with $1 < c_1$ and $c_{k-1} < n-2$, are all odd. Suppose further that $c_1 \equiv 3 \pmod 4$. A narcissistic terrace for $\mathbb{Z}_{n+1}$ is obtainable from the sequence*

$$: (-1)^0 \cdot 2^{-1}c_1 \overset{2}{\longleftarrow} : \mid : (-1)^1 \cdot 2^{-1}c_2 \overset{2}{\longleftarrow} : \mid \cdots \mid$$

$$: (-1)^{k-3} \cdot 2^{-1}c_{k-2} \overset{2}{\longleftarrow} : \mid$$

$$: (-1)^{k-2} \cdot 2^{-1}c_{k-1} \overset{2}{\longrightarrow} : \mid \alpha \mid : +2^{-3} \overset{2}{\longleftarrow} : \parallel \text{ neg. image}$$

where $\alpha = 0_0$ if $n \equiv 5 \pmod 8$ (and hence k is odd), or $\alpha = 0_n$ if $n \equiv 1 \pmod 8$ (and hence k is even). If $c_1 \equiv 1 \pmod 4$, instead of $c_1 \equiv 3$, a narcissistic terrace for $\mathbb{Z}_{n+1}$ is obtained from the above sequence by replacing each term to the left of α by its negative.

Proof: If θ is a primitive root of p, and $2 = \theta^\alpha$, then $k = \gcd(n-1, \alpha)$. When $n \equiv 1 \pmod 8$, 2 is a square modulo n and hence k is even; and when $n \equiv 5 \pmod 8$, 2 is not a square and hence k is odd.

If k is odd, the term to the left of α is $2^{-2}c_{k-1} = (c_{k-1}+n)/4 < n/2$ if $c_{k-1} \equiv 3 \pmod 4$, so the choice $\alpha = 0_0$ is appropriate; if $c_{k-1} \equiv 1 \pmod 4$, the term is $-2^{-2}c_{k-1} = (n - c_{k-1})/4 < n/2$. The term to the right of α is $2^{-3} = (3n+1)/8 < n/2$. A similar analysis holds when k is even.

The general pattern is $-2^{-1}c_i \;\; \ldots \;\; c_i \;\; | \;\; 2^{-1}c_{i+1}$ or its negative. The missing difference here is $(n - c_i)/2$ and the (raised) fence difference is $c_i + (n + 1 - (c_{i+1} + n)/2) = (n - c_i)/2$. □

Note 2.12: In the range $3 < n < 1000$, the coverage of Theorem 2.12 is as follows:

$k=3$, $n \equiv 5 \pmod 8$	
n	Specimen (c_1, c_2)
109	(31, 95)
157	(17, 53)
229	(7, 23)
277	(11, 35)
733	(13, 41)
997	(7, 23)

$k=4$, $n \equiv 1 \pmod 8$	
n	Specimen (c_1, c_2, c_3)
113	(3, 11, 35)
281	(13, 41, 125)
353	(13, 41, 125)
577	(25, 77, 233)
593	(57, 173, 521)
617	(3, 11, 35)

The smallest n-value covered with $n \equiv 5 \pmod 8$ and $k = 5$ is $n = 1181$, for which we can take $(c_1, c_2, c_3, c_4) = (7, 23, 71, 215)$.

Example 2.12(a): For $n = 109$, with $(c_1, c_2) = (31, 95)$, and thus with $c_1 \equiv 3 \pmod 4$, Theorem 2.12 yields the following narcissistic $\mathbb{Z}_{110}$ terrace:

$$: 70 \;\; 35 \;\; \ldots \;\; 47 \;\; 78 \; : \; | \; : 7 \;\; 14 \;\; \ldots \;\; 80 \;\; 51 \; : \; | \; 0 \; |$$

$$: 41 \;\; 75 \;\; \ldots \;\; 54 \;\; 27 \; : \; \| \; \text{neg. image} .$$

Example 2.12(b): For $n = 113$, with $(c_1, c_2, c_3) = (3, 11, 35)$, and thus with $c_1 \equiv 3 \pmod 4$, Theorem 2.12 yields the following narcissistic $\mathbb{Z}_{114}$ terrace:

$$: 58 \;\; 29 \;\; \ldots \;\; 107 \;\; 110 \; : \; | \; : 51 \;\; 82 \;\; \ldots \;\; 22 \;\; 11 \; : \; | \; : 74 \;\; 35 \;\; \ldots \;\; 38 \;\; 76 \; : \; |$$

$$113 \; | \; : 99 \;\; 106 \;\; \ldots \;\; 56 \;\; 28 \; : \; \| \; \text{neg. image} .$$

Example 2.12(c): For $n = 281$, with $(c_1, c_2, c_3) = (13, 41, 125)$, and thus with $c_1 \equiv 1 \pmod 4$, Theorem 2.12 yields the following narcissistic $\mathbb{Z}_{282}$ terrace:

$$: 134\ \ 67\ \ \ldots\ \ 26\ \ 13\ :\ |\ :\ 161\ \ 221\ \ \ldots\ \ 199\ \ 240\ :\ |$$

$$: 78\ \ 156\ \ \ldots\ \ 121\ \ 242\ :\ |$$

$$281\ \ |\ :246\ \ 123\ \ \ldots\ \ 140\ \ 70\ :\ \|\ \text{neg. image}\ .$$

In our next theorem we must take $n \equiv 1 \pmod 8$, as taking $n \equiv 5 \pmod 8$ would make 2 a non-square in $\mathbb{Z}_n$ so that $\mathrm{ord}_n(2)$ would be even.

Theorem 2.13 *Let n be an odd prime, $n \equiv 1 \pmod 8$, such that $\mathrm{ord}_n(2) = (n-1)/2h$ where $\mathrm{ord}_n(2)$ is odd and $h > 2$. Suppose that*

$$\mathbb{Z}_n \setminus \{0\} \ = \ \langle 2\rangle \ \cup \ \bigcup_{i=1}^{h-1} c_i\langle 2\rangle \ \cup \ -\langle 2\rangle \ \cup \ -\bigcup_{i=1}^{h-1} c_i\langle 2\rangle$$

where the integers c_i $(i = 1, 2, \ldots, h-1)$ satisfy $c_1 \equiv 1 \pmod 4$ and $c_{i+1} = 3c_i + 2$ $(i = 1, 2, \ldots, h-2)$, with $1 < c_1$ and $c_{h-1} < n-2$. A narcissistic terrace for $\mathbb{Z}_{n+1}$ is obtainable from the sequence

$$2^{-1}c_1 \ \overset{2}{\leftarrow} \ |\ 2^{-1}c_2 \ \overset{2}{\leftarrow} \ |\ \cdots\ |\ 2^{-1}c_{h-2} \ \overset{2}{\leftarrow} \ |$$

$$2^{-1}c_{h-1} \ \overset{2}{\rightarrow} \ |\ 0_n \ |\ 2^{-3} \ \overset{2}{\leftarrow} \ \|\ \text{neg. image}\ .$$

Proof: Similar to the proof of Theorem 2.12. □

Note 2.13: In the range $3 < n < 1000$, Theorem 2.13 yields $\mathbb{Z}_{n+1}$ terraces as follows:

n	h	Specimen $(c_1, c_2, \ldots, c_{h-1})$
73	4	–
89	3	–
233	4	$(13, 41, 125)$
937	4	$(37, 113, 341)$

Example 2.13: For $n = 937$, with the c_i-values given in the table above, Theorem 2.13 yields the narcissistic $\mathbb{Z}_{938}$ terrace

$$487\ \ 712\ \ \ldots\ \ 74\ \ 37\ \ |\ 525\ \ 731\ \ \ldots\ \ 226\ \ 113\ \ |\ 639\ \ 341\ \ \ldots\ \ 394\ \ 788\ \ |$$

$$937\ \ |\ 820\ \ 410\ \ \ldots\ \ 469\ \ 703\ \ \|\ \text{neg. image}\ .$$

Table 1
The coverage of theorems from §2, for $3 < n < 1000$

n	Theorem	n	Theorem	n	Theorem
5	2.1, 2.2	281	2.6, 2.11, 2.12	641	2.11
13	2.1, 2.2	293	2.1, 2.2	653	2.1, 2.2
17	2.3, 2.4	313	2.3	661	2.1, 2.2
29	2.1, 2.2	317	2.1, 2.2	673	2.14
37	2.1, 2.2	337	2.8	677	2.1, 2.2
41	2.3, 2.4	349	2.1, 2.2	701	2.1, 2.2
53	2.1, 2.2	353	2.6, 2.11, 2.12	709	2.1, 2.2
61	2.1, 2.2	373	2.1, 2.2	733	2.5, 2.12
73	2.8	389	2.1, 2.2	757	2.1, 2.2
89	2.7	397	2.10	761	2.3, 2.4
97	2.3	401	2.3, 2.4	769	2.3
101	2.1, 2.2	409	2.3	773	2.1, 2.2
109	2.5, 2.12	421	2.1, 2.2	797	2.1, 2.2
113	2.6, 2.11, 2.12	433	2.9	809	2.3, 2.4
137	2.3, 2.4	449	2.3, 2.4	821	2.1, 2.2
149	2.1, 2.2	457	2.9	829	2.1, 2.2
157	2.5, 2.12	461	2.1, 2.2	853	2.1, 2.2
173	2.1, 2.2	509	2.1, 2.2	857	2.3, 2.4
181	2.1, 2.2	521	2.3, 2.4	877	2.1, 2.2
193	2.3	541	2.1, 2.2	881	2.7
197	2.1, 2.2	557	2.1, 2.2	929	2.3, 2.4
229	2.5, 2.12	569	2.3, 2.4	937	2.13
233	2.7, 2.13	577	2.12	941	2.1, 2.2
241	2.9	593	2.6, 2.11, 2.12	953	2.11
257	2.11	601	–	977	2.3, 2.4
269	2.1, 2.2	613	2.1, 2.2	997	2.12
277	2.5, 2.12	617	2.6, 2.11, 2.12		

Our last theorem in this section provides an *ad hoc* construction for $n = 673$ (a value for which we failed [7] to find a *da capo* directed terrace). The construction is achieved with a single raised difference. Generalisation of this construction to cover other n-values would be too cumbersome to be justified here.

Theorem 2.14 *Take the prime $n = 673$, and write $c = 521$. A narcissistic terrace for $\mathbb{Z}_{n+1}$ is obtainable from the sequence*

$$: (-3)^0 c \;\overset{2}{\leftarrow} : \mid : (-3)^1 c \;\overset{2}{\leftarrow} : \mid \;\cdots\; \mid : (-3)^6 c \;\overset{2}{\leftarrow} : \mid$$

$$: -3^4 \cdot 2^4 \;\overset{2}{\leftarrow} : \mid : +3^5 \cdot 2^4 \;\overset{2}{\leftarrow} : \mid : -3^6 \cdot 2^4 \;\overset{2}{\rightarrow} : \mid 0_n \mid$$

$$: +3^3 \cdot 2^{-3} \;\overset{2}{\leftarrow} : \mid : -3^2 \cdot 2^{-1} \;\overset{2}{\rightarrow} : \mid$$

$$: +3^1 \cdot 2^{-1} \;\overset{2}{\rightarrow} : \mid : -3^0 \cdot 2^{-1} \;\overset{2}{\rightarrow} : \parallel \text{ neg. image .}$$

Proof: For $n = 673$ we have $\text{ord}_n(2) = 48$. However, $|\langle 2,3\rangle| = 336$ and $c \notin \langle 2,3\rangle$, so that $\mathbb{Z}_n \setminus \{0\} = \langle 2,3\rangle \cup c\langle 2,3\rangle$. The values $3^0 \cdot 2^{-2}$, $3^1 \cdot 2^{-2}$, $3^2 \cdot 2^{-2}$, $3^3 \cdot 2^{-2}$ are 505, 169, 507, 175 respectively, and thus are all odd. The values $3^4 \cdot 2^4$, $3^5 \cdot 2^4$, $3^6 \cdot 2^4$ are 37, 111, 333 respectively, and thus are all odd. The values 3^0c, 3^1c, ..., 3^6c are also all odd. Accordingly, the proof proceeds similarly to the proofs of Theorems 2.9, 2.10 and 2.11, save that we must additionally check that the raised difference at the anomalous seventh fence compensates for the difference missing from the seventh segment; this check is easily made. Numerically the terrace is as follows:

$$: 521 \;\ldots\; 304 : \mid : 456 \;\ldots\; 434 : \mid : 651 \;\ldots\; 44 : \mid : 66 \;\ldots\; 541 : \mid$$

$$: 475 \;\ldots\; 396 : \mid : 594 \;\ldots\; 158 : \mid : 237 \;\ldots\; 199 : \mid : 636 \;\ldots\; 74 : \mid$$

$$: 111 \;\ldots\; 451 : \mid : 340 \;\ldots\; 503 : \mid \quad 673 \quad \mid : 424 \;\ldots\; 498 : \mid$$

$$: 332 \;\ldots\; 507 : \mid : 338 \;\ldots\; 504 : \mid : 336 \;\ldots\; 505 : \parallel \text{ neg. image.}$$

□

For the range $3 < n < 1000$, Table 1 shows which primes n satisfying $n \equiv 1 \pmod 4$ are covered by Theorems 2.1 to 2.14 inclusive.

3 $\mathbb{Z}_{n+1}$ terraces for $n = p^2$ (p an odd prime)

We start this Section with two theorems where 2 is a primitive root of both p and p^2; for these theorems we have $p \equiv 3$ or $5 \pmod 8$. No prime p is known such that 2 is a primitive root of p but not of p^2 (see [12]).

Theorem 3.1 *Let $n = p^2$ where p is a prime such that p and p^2 both have 2 as a primitive root. Let d be any even integer, $0 < d < p$. A narcissistic terrace for $\mathbb{Z}_{n+1}$ is obtainable from the sequence*

$$: \ 2^0 dp \quad 2^1 dp \quad \ldots \quad 2^{(p-3)/2} dp \ : \ | \ 0_n \ |$$

$$: \ 2^{-3} \quad 2^{-4} \quad \ldots \quad -2^{-2} \ : \ \| \ \text{neg. image} \, .$$

Proof: Here $2^{(p-3)/2} dp = n - (d/2)p > n/2$, so the first fence difference compensates for the missing difference from the first segment. Likewise, as $n \equiv 1 \pmod 8$, we have $2^{-3} = (7n+1)/8 > n/2$; the second fence difference compensates for the missing difference from the third segment. □

Note 3.1: If we take $d = 2$ in Theorem 3.1, the first segment of the terrace becomes $2p \ \ 4p \ \ \ldots \ \ (p-1)p$.

Example 3.1: With $(n, p, d) = (25, 5, 4)$, Theorem 3.1 yields the narcissistic $\mathbb{Z}_{26}$ terrace

$$: \ 20 \ \ 15 \ : \ | \ 25 \ | \ : \ 22 \ \ 11 \ \ 18 \ \ 9 \ \ 17 \ \ 21 \ \ 23 \ \ 24 \ \ 12 \ \ 6 \ : \ \| \ \text{neg. image} \, .$$

Our second theorem in this Section yields terraces where some segments contain partial cycles of "irregular length". As in [8], we mean by this that, if s is the number of elements in a full cycle, and τ is the number of terms in any one of the corresponding partial cycles, then, in general, $s \neq 2^\eta \tau$ for any integer η. We again indicate a partial cycle of irregular length by a scream ! at the start and the end of a segment containing a partial cycle.

Theorem 3.2 *Let $n = p^2$ where p is a prime, $p > 3$, such that both p and p^2 have 2 as a primitive root. Write $s = (p-5)/2$. Define π_i $(i = 0, 1, \ldots, (p-3)/2)$ to be the value taken by $2^{s+i(p-1)}$ when evaluated, modulo n, to lie in the interval $(0, n)$. Suppose that, for some particular value j of i, we have $\pi_j > (n-1)/4$. Then $\pi_j - (n-1)/4$ is a multiple $x_j p$ of p. A narcissistic terrace for $\mathbb{Z}_{n+1}$ is obtainable from the sequence*

$$! \ -2^{-1} \quad -2^0 \quad -2^1 \quad \ldots \quad -2^{s+j(p-1)-1} \ ! \ | \ \alpha \ |$$

$$: \ 2^{(p-3)/2} x_j p \quad 2^{(p-5)/2} x_j p \quad \ldots \quad 2^0 x_j p \ : \ |$$

$$! \ +2^{s+j(p-1)} \quad +2^{s+j(p-1)+1} \quad \ldots \quad -2^{-2} \ ! \ \| \ \text{neg. image}$$

where α is 0_0 or 0_n according as π_j is odd or even respectively.

Proof: As each $\pi_i \equiv -4^{-1} \pmod p$, the values π_i are equally spaced throughout the interval $[0, n]$, so there must exist values of i for which

$\pi_i > (n-1)/4$. Also, x_i is even if and only if π_i is even whenever $\pi_i > (n-1)/4$. The first missing difference is $-2^{-1}\pi_j$, so to avoid raised differences we must take $\alpha = n$ if π_j is even, and $\alpha = 0$ if π_j is odd. On the right of α, the missing difference is $-2^{-1}x_j\,p$, so we need $\alpha = n$ if x_j and π_j are even, as on the left. The third missing difference is $(n-1)/4$, and the third fence difference is $\pi_j - x_j p = (n-1)/4$. □

Example 3.2(a): For $n = 25$, Theorem 3.2 has $p = 5$, whence $s = 0$. Thus $\pi_1 = 16$, whence $x_1 p = 10$. Accordingly we have the following narcissistic $\mathbb{Z}_{26}$ terrace, the quarter-cycle segments being specific to $n = 25$:

$$:: \; 12 \; 24 \; 23 \; 21 \; 17 \; :: \; | \; 25 \; | \; : \; 20 \; 10 \; : \; | \; :: \; 16 \; 7 \; 14 \; 3 \; 6 \; :: \; \| \; \text{neg. image}\,.$$

Examples 3.2(b): For $n = 121$, Theorem 3.2 has $p = 11$, whence $s = 3$. Thus $(\pi_0, \pi_1, \pi_2, \pi_3, \pi_4) = (8, 85, 41, 118, 74)$. We discard $\pi_0 = 8$, as it is less than $(n-1)/4$. But $\pi_1 = 85$, whence $x_1 p = 55$; these values yield the narcissistic $\mathbb{Z}_{122}$ terrace

$$! \underbrace{60 \;\; 120 \;\; \ldots \;\; 9 \;\; 18}_{\text{14 terms}} ! \; | \; 0 \; | \; : \; 33 \;\; 77 \;\; 99 \;\; 110 \;\; 55 \; : \; |$$

$$! \underbrace{85 \;\; 49 \;\; \ldots \;\; 15 \;\; 30}_{\text{41 terms}} ! \; \| \; \text{neg. image}\,.$$

Likewise $\pi_3 = 118$ and $x_3 p = 88$; these values yield the terrace

$$! \underbrace{60 \;\; 120 \;\; \ldots \;\; 31 \;\; 62}_{\text{34 terms}} ! \; | \; 121 \; | \; : \; 77 \;\; 99 \;\; 110 \;\; 55 \;\; 88 \; : \; |$$

$$! \underbrace{118 \;\; 115 \;\; \ldots \;\; 15 \;\; 30}_{\text{21 terms}} ! \; \| \; \text{neg. image}\,.$$

Our next two theorems are analogues of the previous two. Instead of having 2 a primitive root of both p and p^2, we now have $\mathrm{ord}_p(2) = (p-1)/2$ and $\mathrm{ord}_n(2) = p(p-1)/2$. The theorems thus do not cover the Wieferich prime $p = 3511$, which [11] is the only prime known to have $\mathrm{ord}_n(2) = \mathrm{ord}_p(2) = (p-1)/2$.

Theorem 3.3 *Let $n = p^2$ where p is a prime, $p \equiv 7 \pmod 8$, such that $\mathrm{ord}_p(2) = (p-1)/2$ and $\mathrm{ord}_n(2) = p(p-1)/2$. Let d be any odd integer, $0 < d < p$. A narcissistic terrace for $\mathbb{Z}_{n+1}$ is obtainable from the sequence*

$$2^0dp \;\; 2^1dp \;\; \ldots \;\; 2^{-1}dp \;\; | \;\; 0_n \;\; | \;\; 2^{-3} \;\; 2^{-4} \;\; \ldots \;\; 2^{-2} \;\; \| \;\; \text{neg. image}\,.$$

Proof: For 2 to be a square in $\mathbb{Z}_n$, we require $p \equiv 1$ or $7 \pmod 8$. For -1 not to be in $\langle 2 \rangle$ we require $\mathrm{ord}_p(2)$ to be odd, *i.e.* $(p-1)/2$ to be odd, so we need $p \equiv 7 \pmod 8$. The rest of the proof is similar to the proof of Theorem 3.1. □

Theorem 3.4 *Let $n = p^2$ where p is a prime, $p \equiv 7 \pmod 8$, such that $\mathrm{ord}_p(2) = (p-1)/2$ and $\mathrm{ord}_n(2) = p(p-1)/2$. Write $s = (p-5)/2$. Define ξ_i $(i = 0, 1, \ldots, p-2)$ to be the value taken by $-2^{s+(i(p-1)/2)}$ when evaluated, modulo n, to lie in the interval $(0, n)$. Suppose that, for some particular value j of i, we have $\xi_j > (n-1)/4$. Then $\xi_j - (n-1)/4$ is a multiple $x_j p$ of p. A narcissistic terrace for $\mathbb{Z}_{n+1}$ is obtainable from the sequence*

$$! -2^{-1} \quad -2^0 \quad -2^1 \quad \ldots \quad -2^{s+(j(p-1)/2)-1} \; ! \mid \alpha \mid$$

$$2^{-1}x_j p \quad 2^{-2}x_j p \quad \ldots \quad 2^0 x_j p \mid$$

$$! -2^{s+j(p-1)/2} \quad -2^{s+(j(p-1)/2)+1} \quad \ldots \quad -2^{-2} \; ! \; \| \text{ neg. image}$$

where α is 0_0 or 0_n according as ξ_j is even or odd respectively.

Proof: Similar to the proof of Theorem 3.2. □

Example 3.4: For $(n, p, s) = (49, 7, 1)$, Theorem 3.4 has

$$(\xi_0, \xi_1, \ldots, \xi_5) = (47, 33, 29, 5, 40, 26),$$

from which we must discard $\xi_3 = 5$, as it is less than $(n-1)/4$. But we can, for example, use $\xi_1 = 33$ to obtain the narcissistic $\mathbb{Z}_{50}$ terrace

$$! \underbrace{24 \;\; 48 \;\; 47 \;\; 45 \;\; 41}_{\text{5 terms}} ! \mid 49 \mid 35 \;\; 42 \;\; 21 \mid ! \underbrace{33 \;\; 17 \;\; \ldots \;\; 6 \;\; 12}_{\text{16 terms}} ! \; \| \text{ neg. image}.$$

Finally in this Section, we provide a theorem that covers, for example, $p = 17$, which has been missed so far. We now need terraces with more segments than have been present hitherto.

Theorem 3.5 *Let $n = p^2$ where p is a prime, $p \equiv 17 \pmod{24}$, such that $\mathrm{ord}_p(2) = (p-1)/2$ and $\mathrm{ord}_n(2) = p(p-1)/2$. A narcissistic terrace for $\mathbb{Z}_{n+1}$ is obtainable from the sequence*

$$: -2^0 p \quad -2^{-1}p \quad \ldots \quad +2^{+1}p \; : \mid : \; +2^0 \cdot 3p \quad +2^1 \cdot 3p \quad \ldots \quad -2^{-1} \cdot 3p \; : \mid$$

$$0_0 \mid : \; -3 \cdot 2^{-3} \quad -3 \cdot 2^{-4} \quad \ldots \quad +3 \cdot 2^{-2} \; : \mid$$

$$: \; +2^{-1} \quad +2^0 \quad \ldots \quad -2^{-2} \; : \; \| \text{ neg. image}.$$

Proof: As $p \equiv 17 \pmod{24}$, the element 3 is not a square in $\mathbb{Z}_n$ and hence $3 \notin \langle 2 \rangle$. Also $-1 \in \langle 2 \rangle$. When we note that $-3 \cdot 2^{-3} = 3(n-1)/8$, the rest of the proof follows easily. □

Example 3.5: With $(n,p) = (289, 17)$ Theorem 3.5 yields the narcissistic $\mathbb{Z}_{290}$ terrace

$$: \ 272 \ 136 \ 68 \ 34 \ : \mid \ : \ 51 \ 102 \ 204 \ 119 \ : \mid 0 \mid$$

$$: \underbrace{108 \ 54 \ \ldots \ 146 \ 73}_{68 \text{ terms}} : \mid : \underbrace{145 \ 1 \ \ldots \ 36 \ 72}_{68 \text{ terms}} : \parallel \text{ neg. image} .$$

4 $\mathbb{Z}_{n+1}$ terraces for $n = 3^{2t}$

Theorem 4.1 *Let $n = 3^{2t}$ where t is a positive integer. Narcissistic terraces for $\mathbb{Z}_{n+1}$ are obtainable from the sequences*
(i)

$$0_0 \mid : \ 3^{2t-1} \cdot 2^{-2} \ : \mid$$

$$: \ (-3)^{2t-2} \cdot 2^{-1} \ \xrightarrow{2} : \mid : \ (-3)^{2t-3} \cdot 2^{-1} \ \xrightarrow{2} : \mid \ \cdots \ \mid$$

$$: \ (-3)^0 \cdot 2^{-1} \ \xrightarrow{2} : \parallel \text{ neg. image}$$

(ii)

$$: \ -3^{2t-1} \ : \mid 0_n \mid : \ (-3)^{2t-2} \cdot 2^{-3} \ \xleftarrow{2} : \mid$$

$$: \ (-3)^{2t-3} \cdot 2^{-1} \ \xrightarrow{2} : \mid : \ (-3)^{2t-4} \cdot 2^{-1} \ \xrightarrow{2} : \mid \ \cdots \ \mid$$

$$: \ (-3)^0 \cdot 2^{-1} \ \xrightarrow{2} : \parallel \text{ neg. image} .$$

Proof: As n is an even power of 3, we have $n \equiv 1 \pmod 8$. If i is odd, $i < 2t-1$, then $3^i 2^{-2}$ takes the value $(n + 3^i)/4 < (n + n/3)/4 = n/3$, whereas if i is even, $-3^i 2^{-2}$ takes the value $(n - 3^i)/4 < n/3$. Thus in (i) the general form of the sequence after the first two short segments is $\ldots \ (n-3y) \mid (n-2y) \ \ldots \ y \mid$ where $0 < y < n/3$. So the missing difference y is compensated for by the fence difference y. A similar argument holds for (ii). □

Example 4.1(a): With $t = 1$, Theorem 4.1 yields the following type (ii) narcissistic $\mathbb{Z}_{10}$ terrace, which is also obtainable from Theorem 3.1:

$$: \ 6 \ : \mid 9 \mid : \ 8 \ 4 \ 2 \ : \parallel : \ 7 \ 5 \ 1 \ : \mid 0 \mid : \ 3 \ : .$$

Example 4.1(b): With $t = 2$, Theorem 4.1 yields the following type (i) narcissistic terrace for $\mathbb{Z}_{82}$:

$$0 \mid : 27 : \mid : 45 \ 9 \ 18 : \mid : 39 \ 78 \ \ldots \ 51 \ 21 : \mid$$

$$: 41 \ 1 \ \ldots \ 10 \ 20 : \| \text{ neg. image .}$$

Theorem 4.2 *Let $n = 3^{2t}$ where the integer t satisfies $t > 1$. Narcissistic terraces for $\mathbb{Z}_{n+1}$ are obtainable from the sequences*
(i)

$$: +2^{-1}c_1 \overset{2}{\leftarrow} : \mid : +2^{-1}c_2 \overset{2}{\leftarrow} : \mid \cdots \mid : +2^{-1}c_{2t-2} \overset{2}{\leftarrow} : \mid$$

$$: -3^{2t-1} : \mid 0_n \mid : 2^{-3} \overset{2}{\leftarrow} : \| \text{ neg. image}$$

(ii)

$$: -2^{-1}c_2 \overset{2}{\leftarrow} : \mid : -2^{-1}c_3 \overset{2}{\leftarrow} : \mid \cdots \mid : -2^{-1}c_{2t-2} \overset{2}{\leftarrow} : \mid$$

$$: +3^{2t-1} : \mid 0_0 \mid : -3 \cdot 2^{-3} \overset{2}{\leftarrow} : \mid : 2^{-1} \overset{2}{\rightarrow} : \| \text{ neg. image}$$

where
(a) $c_{2t-2} = 2 \cdot 3^{2t-2}$ *or* $5 \cdot 3^{2t-2}$, *and*
(b) *the values* c_{i-1} $(i = 2t-2, 2t-3, \ldots, 2)$ *are then obtained successively from*
$c_{i-1} = 2 \cdot 3^{2t-1} - c_i/3$ *or* $3^{2t} - c_i/3$ *if* c_i *is even,*
or from
$c_{i-1} = 2 \cdot 3^{2t-1} - c_i/3$ *or* $3^{2t-1} - c_i/3$ *if* c_i *is odd,*
each successive value c_i $(i = 2t-2, 2t-3, \ldots, 1)$ *being evaluated, modulo* n, *to lie in the interval* $(0, n)$.

Proof: Modulo n, we have $5 \cdot 3^{2t-2} \equiv -4 \cdot 3^{2t-2} \in 3^{2t-2}\langle 2 \rangle$, so $c_{2t-2} \in 3^{2t-2}\langle 2 \rangle$. It is straightforward to show inductively that $c_i \in 3^i\langle 2 \rangle$, $\forall i = 1, 2, \ldots, 2t-2$.

Consider (i). Here $-3^{2t-1} = 2n/3 > n/2$ and $2^{-3} = (7n+1)/8 > n/2$, so the differences at 0_n match the missing differences in the adjacent segments. Before 0_n the pattern is $2^{-1}c_{i-1} \ \ldots \ -c_{i-1} \mid 2^{-1}c_i$. Consider for example the case where c_i is odd and $c_{i-1} = 2 \cdot 3^{2t-1} - c_i/3$. We then have $-c_{i-1} = 3^{2t-1} + c_i/3$. The missing difference is

$$2^{-1}c_{i-1} = (n+c_{i-1})/2 = (n+2n/3-c_i/3)/2 = (5n-c_i)/6 = -(n+c_i)/6 ,$$

and the fence difference is

$$2^{-1}c_i - (3^{2t-1} + c_i/3) = (n+c_i)/2 - (n+c_i)/3 = (n+c_i)/6 .$$

So the unraised fence difference compensates for the missing difference.

The proofs are similar in all the other cases for (i) and (ii). □

Examples 4.2: With $t = 2$, Theorem 4.2 yields narcissistic $\mathbb{Z}_{82}$ terraces that include

(i) with $c_2 = 18$, $c_1 = 75$:

$$: 78\ 39\ \ldots\ 12\ 6 : | : 9\ 45\ 63 : | : 54 : |\ 81\ |$$

$$: 71\ 76\ \ldots\ 40\ 20 : \|\ \text{neg. image}$$

and

(ii) with $c_2 = 18$:

$$: 72\ 36\ 18 : | : 27 : |\ 0\ |$$

$$: 30\ 15\ \ldots\ 42\ 21 : | : 41\ 1\ \ldots\ 10\ 20 : \|\ \text{neg. image}\ .$$

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