ON f-DERIVATIONS OF BCC-ALGEBRAS

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ABSTRACT. In this paper, the notion of left-right and right- left f-derivation of a BCC-algebra is introduced, and some related properties are investigated. Also, we consider regular f-derivation and d-invariant on f-ideals in BCC-algebras.

1. INTRODUCTION

In the theory of rings and near rings, the properties of derivations are important. Several authors [5], [6], [8], [9], [11],[12] have studied BCI-algebras, BCK-algebras and BCC-algebras and lattices. In [10], Jun and Xin applied the notion of derivations in rings and near-rings theory to BCI-algebras, and also introduced a regular derivation in BCI-algebras and in [7], [16],[17], Ferrari, Szasz, Xin, Li and Lu applied the notion of derivations to lattices. They investigated some of its properties, defined a d-invariant ideal and gave some conditions for an ideal to be d-invariant. In [1], Abujabal and Al-Shehri continued studying derivations in BCI-algebras. In [6], Dudek and Zhang introduced the notion of f-derivations of BCI-algebras and they gave a characterizations of a p-semisimple BCI-algebra using regular f-derivations. Later, symmetric bi-derivations, f-derivations, permuting tri-derivations, symmetric f bi-derivations on a lattice and some properties related with these derivations were discussed by [3], [4], [13], [15], respectively. In [14], Prabpayak and Leerawat applied the notion of a regular derivation in BCI-algebras to BCC-algebras and also investigated some of its related properties.

In this paper the notion of left-right and right-left f-derivation of a BCC-algebra is introduced, and some related properties are investigated.

2. PRELIMINARIES

By an algebra $G = (G, ., 0)$ we mean a non-empty set G together with a binary operation multiplication and a constant 0. In the sequel, a multiplication will be denoted by juxtaposition.

Key words and phrases. f-derivation, BCC-algebra, BCI-algebra, regular, d-invariant.

Definition 2.1. An algebra $G = (G, ., 0)$ is called BCC-algebra if for all x, y, $z \in G$ it satisfies the following axioms,

(1) $((xy)(zy))(xz) = 0$

(2) $0x = 0$

(3) $x0 = x$

(4) $xy = yx = 0$ implies $x = y$.

A non-empty subset S of a BCC-algebra G is called BCC-subalgebra of G if $xy \in S$ for all x, $y \in S$.

By (1) we get: $(xy)x = 0$ and $xx = 0$ for all x, $y \in G$.

In any BCC-algebra G, one can define a partial order "$\leq$" by putting $x \leq y$ if and only if $xy = 0$.

A non-empty subset A of a BCC-algebra G is called a BCC- ideal, if

(5) $0 \in A$

(6) $(xy)z \in A$ and $y \in A$ imply $xz \in A$.

Putting $z = 0$ in (6) we obtain : $xy \in A$ and $y \in A$ implies $x \in A$. In a BCC-algebra G, for elements x, y of G we denote $x \wedge y = y(yx)$.

Definition 2.2. [14] Let G be a BCC-algebra. A map $d : G \to G$ is said to be a left-right derivation (briefly, (l, r)- derivation) of G if it satisfies the identity $d(xy) = d(x)y \wedge xd(y)$ for all x, $y \in G$.

If d satisfies the identity $d(xy) = x\, d(y) \wedge d(x)y$ for all $x, y \in G$, then d is said to be a right-left derivation (briefly, (r, l)- derivation) of G. Moreover, if d is both (l, r) and (r, l)- derivation, then it is said to that d is a derivation.

3. THE f-DERIVATIONS OF BCC-ALGEBRAS

In what follows, let f be an endomorphism of G unless otherwise specified.

Definition 3.1. Let G be a BCC-algebra and f be an endomorphism of G. A map $d : G \to G$ is said to be left-right f-derivation (briefly, (l,r)- f-derivation) of G, if it satisfies the identity $d(xy) = (d(x)f(y)) \wedge (f(x)d(y))$ for all x, $y \in G$.

If d satisfies the identity $d(xy) = (f(x)\, d(y)) \wedge (d(x)\, f(y))$ for all $x, y \in G$, then d is said to be right-left f-derivation (briefly, (r, l)- f- derivation) of G. Moreover, if d is both (l,r) and (r, l)-f-derivation, then it is said to that d is an f-derivation.

Example 3.1. *Let $G = \{ 0, 1, 2, 3 \}$ be a BCC-algebra with Cayley table as follows.*

·	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	1	0

By [14] *we know that a map $d : G \to G$ for all x in G defined by*

$$d(x) = \begin{cases} 0, & x=0,\ 1,\ 3 \\ 2, & x=2 \end{cases}$$

is a derivation of G. Define an endomorphism f of G by $f(x) = 0$ for all $x \in G$. Then d is not an f-derivation of G since $d(2\ 1) = d(2) = 2$, but $(d(2)\ f(1)) \wedge (f(2)\ d(1)) = (2\ 0) \wedge (0\ 0) = 2 \wedge 0 = 0$, and thus $d(2\ 1) \neq d(2)\ f(1)) \wedge (f(2)\ d(1))$.

Remark 3.1. *Every derivation of G could be made an f-derivation of G by identity endomorphism of G.*

Example 3.2. *Let G be a BCC-algebra as in Example 3.1. Define a map $d : G \to G$ for all x in G by*

$$d(x) = \begin{cases} 0, & x=0,\ 1 \\ 2, & x=2,\ 3 \end{cases}$$

Then d is not a derivation of G since $d(3\ 1) = d(3) = 2$, but $(d(3)\ 1) \wedge (3\ d(1)) = (2\ 1) \wedge (3\ 0) = 2 \wedge 3 = 3$, and thus $d(3\ 1) \neq (d(3)\ 1) \wedge (3\ d(1))$. Define an endomorphism f of G for all x in G by

$$f(x) = \begin{cases} 0, & x=0,\ 1 \\ 2, & x=2,\ 3 \end{cases}$$

Then it is easily checked that d is an f-derivation of G.

Example 3.3. *Let G be a BCC-algebra as in Example 3.1. Define a map $d : G \to G$ for all x in G by*

$$d(x) = \begin{cases} 0, & x=0,\ 1,\ 2 \\ 3, & x=3 \end{cases}$$

Then d is not a derivation of G since $d(3\ 2) = d(1) = 0$ but $(3\ d(2)) \wedge (d(3)\ 2) = (3\ 0) \wedge (3\ 2) = 3 \wedge 1 = 1\ (1\ 3) = 1\ 0 = 1$, and thus $d(3\ 2) \neq (3\ d(2)) \wedge (d(3)\ 2)$.

Define an endomorphism f of G for all x in G by

$$f(x) = \begin{cases} 0, & x=0,\ 1 \\ 3, & x=2,\ 3 \end{cases}$$

Then it is easily checked that d is an f-derivation of G.

Remark 3.2. *From Example* 3.2 *and* 3.3, *we have seen that there is an f- derivation of G which is not a derivation of G.*

Definition 3.2. An f-derivation of a BCC-algebra G is said to be regular if $d(0) = 0$.

Theorem 3.3. *If d is a (r, l)-f-derivation of a BCC-algebra G, $d(0) = 0$.*

Proof: Since d is (r, l)-f-derivation of G,

$$\begin{aligned} d(0) &= d(0\ x) = (f(0)\ d(x)) \wedge (d(0)\ f(x)) \\ &= (0\ d(x)) \wedge (d(0)\ f(x)) = 0 \wedge (d(0)\ f(x)) \\ &= (d(0)\ f(x))\ ((d(0)f(x))\ (0)) \\ &= (d(0)\ f(x))\ (d(0)f(x)) = 0. \end{aligned}$$

Corollary 3.4. *An f-derivation d of a BCC-algebra G is regular.*

Proposition 3.5. *Let d be a self map of BCC-algebra G. Then the following hold.*

(1) If d is an (l, r)-f-derivation of G, then $d(x) = d(x) \wedge f(x)$ for all $x \in G$.

(2) If d is an (r, l)-f-derivation of G, then $d(x) = f(x) \wedge d(x)$ for all $x \in G$.

Proof: 1) Let d be an (l, r)-f-derivation of G. Then,
$d(x) = d(x\ 0) = (d(x)\ f(0)) \wedge (f(x)\ d(0)) = (d(x)\ 0) \wedge (f(x)\ 0) = d(x) \wedge f(x)$.
2) Let d be an (r, l)-f-derivation of G. Then,
$d(x) = d(x\ 0) = (f(x)\ d(0)) \wedge (d(x)\ f(0)) = (f(x)\ 0) \wedge (d(x)\ 0) = f(x) \wedge d(x)$.

Definition 3.6. Let G be a BCC-algebra and d be a f-derivation of G. $d^{-1}(0) = \{\ x \in G \mid d(x) = 0\ \}$.

Proposition 3.7. *Let G be a BCC-algebra with a partial order $\leq$, and let d be an f-derivation of G. Then the following hold:*

(1) $d(x) \leq f(x)$ for all $x \in G$.
(2) $d(x\ y) \leq d(x)\ f(y)$ for all $x, y \in G$.
(3) $d(x\ y) \leq f(x)\ d(y)$ for all $x, y \in G$.
(4) If $d \circ f = f \circ d$, then $d(f(x)\ d(x)) = 0$ for all $x \in G$.
(5) If $d \circ f = f \circ d$, then $d(d(x)) \leq f(f(x))$ for all $x \in G$.
(6) $d^{-1}(0)$ is a subalgebra of G.

Proof:
(1) Let x be an element of G.
By Proposition 3.5, $d(x) = d(x) \wedge f(x) = f(x)\ (f(x)d(x))$. Then
$d(x)f(x) = (f(x)\ (f(x)d(x)))\ f(x) = 0$. Thus $d(x) \leq f(x)$ for all $x \in G$.
(2) For any $x, y \in G$, we have

$d(x\ y) = (f(x)\ d(y)) \wedge (d(x)\ f(y))$
$\quad = (d(x)\ f(y))\ ((d(x)\ f(y))\ (f(x)\ d(y)))$.
Then $d(x\ y)\ (d(x)\ f(y)) = 0$.
Thus $d(x\ y) \leq d(x)\ f(y)$ for all $x, y \in G$.
(3) For any $x, y \in G$, we have
$d(x\ y) = (d(x)\ f(y)) \wedge (f(x)\ d(y))$
$\quad = (f(x)\ d(y))\ ((f(x)\ d(y))\ (d(x)\ f(y)))$.
Then $d(x\ y)\ (f(x)\ d(y)) = 0$.
Thus $d(x\ y) \leq f(x)\ d(y)$ for all $x, y \in G$.
(4) Let $d \circ f = f \circ d$, for any $x \in G$, we have
$d(f(x)\ d(x)) = (f(f(x))\ d(d(x))) \wedge (d(f(x))\ f(d(x)))$
$\quad = (f(f(x))\ d(d(x))) \wedge (f(d(x))\ f(d(x)))$
$\quad = (f(f(x))\ d(x)) \wedge 0$
$\quad = 0$
(5) If $d \circ f = f \circ d$, then for any $x \in G$, we have by (4)
$d(d(x)) = d(d(x) \wedge f(x))$
$\quad = d(f(x)(f(x)d(x)))$
$\quad = (d(f(x))\ f(f(x)\ d(x))) \wedge (f(f(x))\ d(f(x)d(x)))$
$\quad = (d(f(x))\ f(f(x)\ d(x))) \wedge (f(f(x))\ 0)$
$\quad = (d(f(x))\ (f(f(x)\ d(x))) \wedge (f(f(x)))$
$\quad = (f(f(x)))\ ((f(f(x)))\ (d(f(x))\ (f(f(x)\ d(x)))))$
Then $d(d(x))\ (f(f(x)) = 0$. Thus $d(d(x)) \leq f(f(x))$ for all $x \in G$.
(6) Since d is regular, $0 \in d^{-1}(0)$, that is, $d^{-1}(0) \neq \emptyset$.
Let $x, y \in d^{-1}(0)$, then $d(x) = 0 = d(y)$, and so
$d(x\ y) = (f(x)\ d(y)) \wedge (d(x)\ f(y))$
$\quad = (f(x)\ 0) \wedge (0\ f(y))$
$\quad = f(x) \wedge 0 = 0$, that is $x\ y \in d^{-1}(0)$.
Hence $d^{-1}(0)$ is a BCC-subalgebra of G.

Remark 3.3. *In general $d^{-1}(0)$ is not a BCC-ideal of G, as seen in Example 3.3. $d^{-1}(0) = \{ 0, 1, 2 \}$ is not a BCC-ideal of G since $(3\ 2)1 = 0 \in d^{-1}(0)$ and $2 \in d^{-1}(0)$, but $3\ 1 = 3 \notin d^{-1}(0)$.*

Definition 3.8. A BCC-ideal A of a BCC-algebra G is said to be f-BCC-ideal if $f(A) \subseteq A$.

Definition 3.9. Let d be a self map of a BCC-algebra G. An f-ideal A of G is said to be d-invariant if $d(A) \subseteq A$.

Example 3.4. *Let G be a BCC-algebra as in Example 3.1. Then $A = \{ 0, 1 \}$ is a BCC-ideal of G.*

Let d be an f-derivation of G as in Example 3.2. $f(A) = \{ 0 \} \subseteq A$. Thus A is an f-BCC-ideal. Also $d(A) = \{ 0 \} \subseteq A$. Hence A is d-invariant.

Theorem 3.10. *Let d be a f-derivation of BCC-algebra G. Then every f-ideal A of G is d-invariant.*

Proof: Let A be an f-ideal of BCC-algebra G. Let $y \in d\ (A)$. Then $y = d\ (x)$ for some $x \in A$. It follows that by Proposition 3.7.(1) we have $y\ f(x) = d(x)\ f(x) = 0 \in A$, since A is f-ideal we have $y \in A$. Thus $d(A) \subseteq A$. Hence A is d-invariant.

References

[1] Abujabal, H. A. S. A. and Al-Shehri, N. O. 2006. *Some results on derivations of BCI-algebras.* Coden Jnsmac 46:13-19.

[2] Abujabal, H. A. S. A. and Al-Shehri, N. O. . 2007. *On left derivations of BCI-algebras.* Soochow Journal of Mathematics 33 (3): 435-444.

[3] Çeven, Y., 2009 *Symmetric bi-derivations of lattices*, Quaestiones Mathematicae 32, 241-245.

[4] Çeven, Y. and Öztürk, M.A. 2008. *On f-Derivations of lattices*,Bull. Korean Math. Soc. 45, No.4, pp. 701-707.

[5] Dudek, W. A. 1992. *On constructions of BCC-algebras.*Selected Papers on BCK- and BCC-algebras 1:93-96.

[6] Dudek, W. A. and Zhang, X. 1998. *On ideals and congruences in BCC-algebras.* Czechoslovak Math. Journal. 48 (123): 21-29.

[7] Ferrari, L., 2001.*On Derivations of Lattices*, Pure Math. Appl. 12 , no.4, 365-382.

[8] Iseki, K. and S. Tanaka. 1976. *Ideal theory of BCK-algebras.* Math. Japonica. 21: 351-336.

[9] Iseki, K. and Tanaka, S. 1978. *An introduction to the theory of BCK-algebras.* Math. Japonica. 23: 1-26.

[10] Jun, Y. B. and Xin, L. 2004. *On derivations of BCI-algebras.* Information Sciences 159: 167-176.

[11] Meng, D. J. 1987. *BCI-algebras and abelian gorups.* Math. Japonica. 47 (1): 693-696.

[12] Meng, J. and Xin, X. L. 1992. *Commutative BCI-algebras.* Math. Japonica. 37 (3): 569-572.

[13] Öztürk, M.A., Yazarli, H. and Kyung, H. K. 2009 *Permuting Tri-Derivations in Lattices*, Quaestiones Mathematicae 3,415-425.

[14] Prabpayak C. and Leerawat U. 2009. *On derivations of BCC-algebras.*Kasetsart J. (Nat. Sci.) 43: 398-401.

[15] Özbal, S. A. and Firat A., 2010 *Symmetric f bi-derivations on lattices*, Submitted to A Canadian Journal of Combinatorics.

[16] Szasz, G. 1975. *Derivations of Lattices*, Acta Sci. Math. (Szeged) 37, 149-154.

[17] Xin, X. L. , Li, T.Y. and Lu ,J. H. 2008 *On derivations of Lattices*, Information Sciences 178, 307-316.

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