Using affine planes to partition full designs with block size three

Ken Gray*, Anne Penfold Street*
Mathematics, The University of Queensland, Brisbane 4072, Australia
R G Stanton†,
Computer Science, University of Manitoba, Winnipeg R3T 2N2, Canada

Abstract

We provide the specifics of how affine planes of orders three, four and five can be used to partition the full design comprising all triples on 9, 16 and 25 elements respectively. Key results of the approach for order five are generalised to reveal when there is potential for using suitable affine planes of order n to partition the complete sets of n^2 triples into sets of mutually disjoint triples covering either all n^2, or else precisely $n^2 - 1$, elements.

Key words and phrases: balanced incomplete block designs, full designs, affine planes, partitions, triples.

1. Preliminaries

A *block design* is a pair, $D = (V, \mathcal{B})$, where V is a set of v elements (or a *v-set*) and $\mathcal{B}$ is a set of β k-subsets (called *blocks*) chosen from V so that every element of V occurs in exactly r blocks. If every t-subset of V belongs to exactly λ blocks, the design is said to be *t-balanced*, and is called a *t-design*, with parameters t-(v, k, λ). Since we deal only with designs having $t = 2$, we refer to them as *balanced* and write their parameters as (v, k, λ). A design is *simple* if it contains no repeated blocks. The simple design comprising all k-subsets of V is called the *full* design on v elements and written as $\binom{V_v}{k}$, or simply $\binom{v}{k}$ when the set V involved is apparent.

An affine plane of order n is an $(n^2, n, 1)$ balanced design, known to exist when n is a prime or prime power: no affine planes are known to exist for other values of n. Throughout this paper we assume $n \geq 3$. For $n \leq 8$, affine planes are known to be unique up to isomorphism, whereas non-isomorphic affine planes are known to exist for some larger values; for example, precisely seven non-isomorphic affine planes of order nine exist. Any affine plane contains $(n^2 + n)$ blocks of size n, and these blocks can

*Research supported by ARC; corresponding author: aps@maths.uq.edu.au
†Died 21-04-2010

be uniquely arranged into $(n+1)$ *resolution classes.* We label these resolution classes $R_1, R_2, \ldots, R_n, R_{n+1}$. It follows that every element occurs in precisely $(n+1)$ blocks. Blocks in different resolution classes intersect in precisely one element.

Any affine plane of order n can be derived from $(n^2+n+1, n+1, 1)$ designs, known as *projective planes of order* n and denoted here by P_n. This is done by deleting any block of P_n and then also deleting all occurrences of elements of that block in the other blocks of P_n. Conversely, all projective planes can be constructed from affine planes. Each resolution class of the resultant affine plane comprises the blocks from which the same element of P_n has been deleted. (For these and other results relating to affine planes, see, for example, [2] and [11]).

Example 1 *The collection,* D_4 *say, of the* 20 *blocks in Table 1 constitutes a* 2-$(16,4,1)$ *design, that is an affine plane of order four, on the* 16*-element set* $V = \{0,1,2,\ldots,9,\mathtt{A},\mathtt{B},\mathtt{C},\mathtt{D},\mathtt{E},\infty\}$. *This particular design, arranged into resolution classes, has been obtained by developing, modulo* 15*, the starter blocks* 1248 *and* 05A∞ *(short).* (See [4] for one way such starter blocks can be obtained and [11] for an explanation of short starter blocks).

	R_1		R_2		R_3		R_4		R_5
1:	1248	5:	2359	9:	346A	13:	457B	17:	568C
2:	679D	6:	78AE	10:	89B0	14:	9AC1	18:	ABD2
3:	BCE3	7:	CD04	11:	DE15	15:	E026	19:	0137
4:	05A∞	8:	16B∞	12:	27C∞	16:	38D∞	20:	49E∞

Table 1: blocks of a 2-$(16,4,1)$ design, D_4, by resolution class

2. Motivation for studying partitions

For certain values of n, the triples contained in the blocks of a suitable affine plane of order n can be used to partition the full design comprising all triples on n^2 elements, with most components of this partition being sets of mutually exclusive triples. Similarly, larger blocks than triples may occur in a partition.

But why are we doing this?

Many important combinatorial problems are based on partitions. We refer first to the resolution classes of Example 1, which form a partition of the affine plane 2-$(16,4,1)$. Further examples follow.

Example 2 *Taking all* $\binom{7}{3}$ *triples from the set* $\{1,2,3,4,5,6,7\}$ *gives us a* $(7,3,5)$ *design, that is, a full design. We can partition this design into three separate subdesigns, as shown in Table 2.*

The $(7,3,3)$ design $S = S_1 \cup S_2 \cup S_3$ is *irreducible* [8], that is, it contains no $(7,3,1)$ design and thus no $(7,3,2)$ design either. Each of S_4 and S_5 is a $(7,3,1)$ design. We cannot partition S into subdesigns, but we can partition it into minimal defining sets, in two different ways. To see this, we apply the function $f(x) = 3x + 1$ to the set $\{1,2,3,4,5,6,7\}$, inducing the permutation $(146527)(3)$. Here we have two non-isomorphic minimal defining sets of the $(7,3,3)$ design S.

The first of these can be considered as S_1, S_2, or S_3, thus S can be partitioned into these three minimal defining sets. Similarly, minimal defining sets T_1, T_2, and T_3 also partition S, where blocks of T_1, T_2, and T_3 are labelled by $*, +$, and $-$ respectively, in Table 2; see [3].

S_1	S_2	S_3	S_4	S_5
123*	135*	147+	124	134
234+	246+	251*	235	245
345*	357-	362+	346	356
456*	461-	473+	457	467
567-	572-	514*	561	571
671-	613-	625+	672	612
712*	724+	736-	713	723

Table 2: a full design partitioned into three subdesigns in two different ways

Further examples of such partitions include the 55 disjoint projective planes of order 3 found by Chouinard [1] as a partition of the collection of all 4-subsets of a 13-set into these planes, and the 91 disjoint AG(2,4)s found by Mathon [6] as a partition of the collection of all 4-sets of a 16-set.

Example 3 *A* cycle decomposition *of a graph* G *is a collection of cycles whose edges partition the set of edges of* G*; see for instance [5].* If G has eleven vertices, which we label `0,1,2,3,4,5,6,7,8,9,X`, and if G is the complete graph K_{11}, then another example of a partition is that of the set of edges of G into 11 5-cycles, as follows: `01427, 12538, 23649, 3475X, 45860, 56971, 67X82, 78093, 891X4, 9X205, X0316.`

Example 4 *Two copies of all $\binom{7}{3}$ triples from the set $\{1,2,3,4,5,6,7\}$ can be partitioned into seven $2-(6,3,2)$ designs as shown in Table 3. This partition has an automorphism group of order* 5, *generated by* (12357). For details of this construction, see [10].

D_1	234	235	246	257	267	347	356	367	456	457
D_2	136	137	145	147	156	345	346	357	467	567
D_3	124	125	146	157	167	247	256	267	456	457
D_4	125	126	135	137	157	236	237	257	356	567
D_5	126	127	134	136	147	234	237	246	367	467
D_6	123	127	134	145	157	235	245	247	347	357
D_7	123	124	135	146	156	236	245	256	345	346

Table 3: partition of $2 \times \binom{7}{3}$ into seven 2-(6,3,2) designs

3. Definitions and results fundamental for this paper

We now show how, for certain values of n, those triples *not* contained in the blocks of a suitable affine plane of order n can be used to partition the full design comprising all triples on n^2 elements, with most components of this partition being sets of mutually exclusive triples. The first stage of our approach is to construct a 'pick-out' function. This is based on taking each triple $\{i,j,k\}$, or simply ijk, *not* contained in a block of the affine plane, and then identifying the three, necessarily distinct, blocks of the affine plane that contain the pairs $\{i,j\}$, $\{i,k\}$ and $\{j,k\}$. The following fundamental preliminary definitions and results pertain to the particular values of n considered in this paper, but can also apply to other, larger, values of n not specifically dealt with here (see Section 7 for their possible parameters). From now on, a specific affine plane of order n, on a set $V = \{0,1,2,\ldots,(\mathrm{n}-1)^2\}$ or some other set as needed, is denoted by D_n, and the set of all triples contained in the blocks of D_n is denoted by D'_n; thus the set of all triples on V not contained in the blocks of D_n is $\binom{n^2}{3}\backslash D'_n$.

Lemma 5 *If D'_n is the set of all triples contained in the blocks of an affine plane D_n of order n, then:*

(i) $|D'_n| = \binom{n}{3} \times (n^2+n)$;

(ii) $|\binom{n^2}{3}\backslash D'_n| = \binom{n^2}{3} - \binom{n}{3} \times (n^2+n) = \frac{n^3(n-1)^2(n+1)}{6}$;

(iii) for any three given resolution classes of D_n, there are $n^2(n-1)$ triples $ijk \in \binom{n^2}{3} \backslash D'_n$ such that the pairs $\{i,j\}, \{i,k\}$ and $\{j,k\}$ each appear in (precisely) one of those three resolution classes.

Proof:

- Results (i) and (ii) are both obtained by elementary counting.
- We first illustrate (iii) by an example based on the affine plane D_4, with the 20 blocks in Table 1. We aim to show that, for any three given resolution classes, R_1, R_2, R_3 say, there are precisely $n^2(n-1) = 48$ triples ijk of $\binom{16}{3} \backslash D'_4$ such that each pair $\{i,j\}, \{i,k\}, \{j,k\}$ appears in (precisely) one of those three classes.

  Take any two blocks of R_1 and R_2, 1248 and 2359 say, which must have an element in common, in this case element 2. Now the three blocks of D_4 containing pairs $\{1,3\}, \{1,5\}$, and $\{1,9\}$ cannot be in class R_1 or R_2, and no more than one of the three can occur in any resolution class. Hence one of the three blocks is in R_3, one in R_4 and one in R_5. Suppose the block containing pair $\{1,3\}$ is in R_3. Then triple 123, elicited from 1248 and 2359, has its three pairs in R_1, R_2 and R_3; we would obtain one suitable triple no matter which pair were chosen. Similarly we consider the blocks containing the three pairs $\{4,3\}, \{4,5\}$, and $\{4,9\}$ and then the three pairs $\{8,3\}, \{8,5\}$, and $\{8,9\}$, in each case eliciting another required triple from each, giving three triples in all. Each of the $n^2 = 16$ choices of one block from each of R_1 and R_2 similarly gives rise to three more triples, thus accounting for all 48 triples as required. The argument applies no matter which set of three resolution classes is under discussion.

  In general, starting with an affine plane of order n, we can choose one block from each of any two resolution classes in n^2 ways, and there are $n-1$ choices of the element from the first block which is to pair with each of the relevant elements of the second, thus giving in total $n^2(n-1)$ triples whose pairs cover any three chosen resolution classes.

  The number of triples in $\binom{n^2}{3} \backslash D'_n$, that is, $n^3(n-1)^2(n+1)/6$, divided by the number of choices of three resolution classes, $\binom{n+1}{3}$, gives $n^2(n-1)$ as required. This provides a rough check.

The following definitions allow blocks and resolution classes to have multiple labels.

Definition 6 *For a given affine plane D_n of order n on V and a given triple $ijk \in \binom{n^2}{3} \backslash D'_n$, define:*

(i) b_{ij} to be the (unique) block of D_n containing pair $\{i,j\} \in V$;

(ii) R_{ij} to be the resolution class holding block b_{ij};

(iii) $b_{ij}(k)$ to be the (unique) block of R_{ij} containing any other element $k \in V$, provided triple $ijk \notin D'_n$; such a block must exist, and contains no element of b_{ij};

(iv) $f(ijk)$ to be the set $\{b_{ij}(k) \cap b_{ik}(j), b_{ij}(k) \cap b_{jk}(i), b_{ik}(j) \cap b_{jk}(i)\}$.

Theorem 7 *For a given affine plane D_n of order n on V, a given triple $ijk \in \binom{n^2}{3} \backslash D'_n$, and the function f as defined in Definition 6 (iv):*

(i) $|\{b_{ij}(k) \cap b_{ik}(j)\}| = |\{b_{ij}(k) \cap b_{jk}(i)\}| = |\{b_{ik}(j) \cap b_{jk}(i)\}| = 1$;

(ii) if $b_{ij}(k), b_{ik}(j)$ and $b_{jk}(i)$ have a common element, $|f(ijk)| = 1$;

(iii) (a) if $b_{ij}(k), b_{ik}(j)$ and $b_{jk}(i)$ have no common element, $|f(ijk)| = 3$,

(b) when $|f(ijk)| = 3$ for all triples $ijk \in \binom{n^2}{3} \backslash D'_n$, we have the induced mapping $f : \binom{n^2}{3} \backslash D'_n \rightarrow \binom{n^2}{3} \backslash D'_n$, with no triple of $\binom{n^2}{3} \backslash D'_n$ mapping to itself.

Proof:

(i) Since the pairs of elements of ijk belong to different resolution classes, and any two blocks from different resolution classes of an affine plane have precisely one element in common, each of the intersections yields a single element.

(ii) This is self-evident.

(iii) (a), (b) The only non-trivial results to check are that, under function f, no triple maps onto itself and no triple is mapped to a block of D'_n. No triple can map on to itself, since this would require two blocks in the same resolution class to have an element in common, which is impossible. Now $f(ijk) = \{b_{ij}(k) \cap b_{ik}(j) = x, b_{ij}(k) \cap b_{jk}(i) = y, b_{ik}(j) \cap b_{jk}(i) = z\}$, where elements x, y, z are all different. So $\{x, y\} \in b_{ij}(k) \in R_{ij}, \{x, z\} \in b_{ik}(j) \in R_{ik}$ and $\{y, z\} \in b_{jk}(i) \in R_{jk}$, where resolution classes R_{ij}, R_{ik} and R_{jk} are all different. Since each pair in xyz is in a different resolution class, $xyz \notin D'(n)$.

Conceivably, affine planes exist such that function f, as in Definition 6 (iv), maps some triples onto triples but other triples onto single elements. Note also, from Theorem 8 which follows, that there exists an affine plane such that function f maps two triples to the same triple. In such a case, no inverse function for f can exist.

Theorem 8 *There exists an affine plane for which function f of Definition 6 (iv) maps at least two triples of $\binom{n^2}{3}\backslash D'_n$ onto the same triple of $\binom{n^2}{3}\backslash D'_n$.*

Proof: We take the affine plane obtained from $Hughes(9)$ [7] by deleting the elements of the first listed block $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Starting with triple $\{55, 57, 62\}$, our approach gives the chain of length three: $\{55, 57, 62\}; \{75, 25, 78\}; \{21, 60, 20\}$. However, we see that

$$f(\{21, 60, 20\}) = \{55, 57, 62\} = f(\{34, 35, 42\});$$

that is, two triples can give rise to the same triple under function f, hence sequences can contain repeated blocks. Table 4 gives the three blocks containing the three pairs from each of two different triples. Here the subscript refers to the element deleted from $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

block in R_0	$b_{34,35}(42)$	=	*20* 27 *42* **55** 56 **57** 58 59 60	=	$b_{21,61}(20)$
block in R_1	$b_{34,42}(35)$	=	12 28 *35* **55** *61* **62** 63 64 65	=	$b_{21,20}(61)$
block in R_2	$b_{35,42}(34)$	=	13 *21* *34* **57** **62** 70 71 72 73	=	$b_{20,61}(21)$

Table 4: blocks, derived from $Hughes(9)$ showing f is not always $1-1$

From now on we are concerned only with those functions f (as defined in Definition 6(iv)) which have the additional property that the function maps **every** triple to a triple, that is, that no triple maps to a single element.

Definition 9 *Suppose that* $\binom{n^2}{3}\backslash D'_n$ *is the set of all triples not contained in the blocks of an affine plane* D_n *of order* n *on* V *and that function* f *is as defined in Definition 6(iv). Suppose further that function* f *maps every triple to a triple, not to a single element. Then, for* $\mathbf{b} \in \binom{n^2}{3}\backslash D'_n$, *we define:*

(i) a sequence of blocks starting with triple $\mathbf{b}$, *denoted by* $S(f(\mathbf{b}))$, *to be the set of all triples* $f^{\alpha}(\mathbf{b})$ *(that is,* $S(f(\mathbf{b}))$ *is the set of all triples obtained under successive applications of* f, *starting with* $\mathbf{b}$*;*

(ii) a chain of blocks that includes triple $\mathbf{b}$, *denoted by* $C_{\mathbf{b}}$, *to be a series* $S(f(\mathbf{b}))$ *that both uniquely contains triple* $\mathbf{b}$ *and consists of blocks that are mutually disjoint;*

(iii) the length *of chain* $C_{\mathbf{b}}$, *denoted by* $|C_{\mathbf{b}}|$, *to be the number of distinct triples comprising* $C_{\mathbf{b}}$. *When all chains of blocks in* $\binom{n^2}{3}\backslash D'_n$ *are of the same length, we may simply use* $|C|$ *(see Theorem 11).*

Theorem 10 *Suppose that sequences are produced, as described in Definitions 6 and 9 and Theorem 7, by applying function* f *to the triples of* $\binom{n^2}{3}\backslash D'_n$, *where* D'_n *is the set of all triples not contained in the blocks of an affine plane* D_n *of order* n. *Then all the pairs contained in the set of triples forming a sequence must occur in blocks of precisely three of the resolution classes, with one pair from each triple occurring in a block of each class.*

Proof: This follows from the definition of function f in Definition 6(iv).

We complete this section by giving additional properties of our pick-out function f of Definition 6 (iv), and of the sequences it induces, when $\binom{n^2}{3}\backslash D'_n$ is transitive on the blocks.

Theorem 11 *Suppose that sequences are produced under the assumptions of Theorem 10. Suppose further that* $\binom{n^2}{3}\backslash D'_n$ *is transitive on the blocks. Then the following hold:*

(i) If some sequence is a chain, then all the sequences are chains;

(ii) All the sequences are of the same length (written as $|C|$, *when the sequences are chains);*

(*iii*) *When the sequences are chains, and every block of* $\binom{n^2}{3}\backslash D'_n$ *appears in precisely one chain, then there are* $n^2(n-1)/|C|$ *chains involving any three given resolution classes.*

Proof:

(i), (ii) Since the blocks are transitive, there exists a mapping ρ taking a given triple $\mathbf{b}$ in a (unique) chain to any arbitrary triple $\mathbf{b}^* \in \binom{n^2}{3}\backslash D'_n$, where these two triples may or may not ever occur in the same sequence. Then applying ρ to all the triples of the chain containing $\mathbf{b}$ produces a (unique) chain containing triple $\mathbf{b}^*$. Clearly, mapping ρ preserves mutual disjointness of the triples and chain length.

(iii) By Lemma 5 (iii), there are $n^2(n-1)$ triples ijk of $\binom{n^2}{3}\backslash D'_n$ such that the pairs $\{i,j\}$, $\{i,k\}$ and $\{j,k\}$ each appear in (precisely) one of any three given resolution classes. By parts (i) and (ii), under function f these triples each appear uniquely in a chain of $|C|$ triples involving those three resolution classes. The result follows.

4. Using the affine plane of order three to partition $\binom{9}{3}$

We now exemplify our approach by using a representative affine plane of order three, D_3, to partition $\binom{9}{3}$.

Theorem 12 *Let* D_3 *be an affine plane of order three, known to be transitive on the blocks. Then:*

(*i*) *the 72 blocks of* $\binom{9}{3}\backslash D'_3$ *(here identical to* $\binom{9}{3}\backslash D_3$*) can be partitioned into 24* 1-factors, *that is, into sets of three triples such that each of the nine elements occurs in precisely one triple;*

(*ii*) *the partition from (i) above gives rise to a partition of* $\binom{9}{3}$ *into 28 1-factors.*

	R_1		R_2		R_3		R_4
1:	123	4:	147	7:	267	10:	168
2:	456	5:	258	8:	159	11:	249
3:	789	6:	369	9:	348	12:	357

Table 5: blocks of a representative 2-(9,3,1) design, $\mathcal{D}_3$, by resolution class

Proof: Without loss of generality, we take D_3 to be the representative affine plane of order three with blocks given in Table 5.

(i) We follow the process outlined in the previous definitions and theorems. For each triple ijk not contained in any block of D_3', 124 say, we locate the three blocks of D_3' that each contain a pair of elements in ijk, in this case finding 123, 147 and 249.

- For 123, we locate 456, the block containing element 4 and in the same resolution class as 123, R_1.
- For 147, we locate 258, the block containing element 2 and in the same resolution class as 123, R_2.
- For 249, we locate 168, the block containing element 1 and in the same resolution class as 123, R_4.

The pairwise intersections of the three found blocks give triple 568; that is, $f(124) = 568$. Repeating the process, we obtain $f(568) = 379$ which returns to $f(379) = 124$. Thus we have obtained a chain of length three, which is the 1-factor that comprises the latter three blocks. Since D_3' is transitive on the blocks, generating chains by each block of $\binom{9}{3}\backslash D_3'$ in turn will give rise to $72/3 = 24$ disjoint 1-factors that are a partition of $\binom{9}{3}\backslash D_3'$.

(ii) Simply supplement the partition of $\binom{9}{3}\backslash D_3'$ into 24 1-factors given in part (i) with the four 1-factors that are the resolution classes of D_3'.

5. Using the affine plane of order four to partition $\binom{16}{3}$

When an affine plane of order four is involved, blocks $b_{ij}(k), b_{ik}(j)$ and $b_{jk}(i)$ of Definition 6 have an element in common. For example, if we start the process using the affine plane of Table 1 and block 123, we obtain $\mathtt{b}_{12}(3) = \mathtt{BCE3}, \mathtt{b}_{13} = \mathtt{2ABD}$ and $\mathtt{b}_{23}(1) = \mathtt{16B}\infty$, which have element B in common. Hence $|f(ijk)| = 1$ (see Theorem 7 (ii)). This means that function f does not map all triples to triples.

We now show in this section how, under a *modified version* of our approach, the properties of any affine plane of order four induce a partition of $\binom{16}{3}$ not into chains, but into 48 sets of ten triples, each with ten of the 16 elements occurring in precisely three triples, together with the design comprising all 80 triples contained in the affine plane. This affine plane is transitive on the blocks; that is, for any two blocks of the design, there is an automorphism of the design that maps one block to the other. Hence, without loss of generality, the proofs here of Lemma 13 and Theorem 14 use examples based on the particular affine plane D_4 of Example 1.

Lemma 13 *For an affine plane of order four, D_4, on V:*

(i) every block of D_4 belongs to 12 sets of five blocks of D_4 containing precisely ten of the elements of V, with each of these ten elements occurring precisely twice;

(ii) there are precisely 48 sets of five blocks (and 10 elements) as in (i);

(iii) there are 48 sets of size 6, the complements in V of the sets of size 10 as in (ii).

Proof: We take design D_4 of Example 1 as the representative affine plane of order four.

(i) It is sufficient here to show that a typical block 1248 of D_4 belongs to 12 such sets. Since every element occurs twice, the set of five blocks must contain another block with element 1, block 16B∞, say. The set can then be completed with {346A, 38D∞, 2BDA}, {89B0, E026, 49E∞} or {27C∞, 457B, 568C}. Three other choices of a block containing element 1 are possible, and each results in three sets, giving 12 different sets in all;

(ii) Each of the 20 blocks belongs to 12 such sets of five blocks, so the number of such sets is $\frac{20\times 12}{5} = 48$;

(iii) The size and number of these sets is self-evident. Note that these 48 6-sets for affine plane D_4 can be developed, modulo 15, from starter blocks C_1 = 0579CE, C_2 = 039ADE, C_3 = 379AB∞ and C_4 = 0369C∞ (short), and form a $(16, 6, 6)$ design.

Theorem 14 *For an affine plane of order four, D_4, on V, let D'_4 be the set of all 80 triples on V contained in a block of D_4. Then:*

(i) each of the 480 blocks of $\binom{V}{3}\backslash D'_4$ can be uniquely associated with one of the 6-sets of Lemma 13 (iii);

(ii) *each given* 6*-set of Lemma 13 (iii) can be associated with ten blocks of* $\binom{V}{3}\backslash D_4'$*;*

(iii) *the 480 blocks of* $\binom{V}{3}\backslash D_4'$ *can be partitioned into 48 isomorphic sets of ten blocks as in (ii).*

Proof: From the 20 blocks of design D_4 of Example 1, we have $D_4' =$ {124, 128, 148, 248, 135, 239, 259, 359, ..., 49E, 49∞, 4E∞, 9E∞}. Since D_4 is obtained by developing, modulo 15, the starter blocks 1248 and 05A∞, the 80 blocks of D_4' are obtained by developing, modulo 15, starter blocks 124, 128, 148, 248 and 05∞, and short starter block 05A.

(i) Start with a particular block of $\binom{V}{3}\backslash D_4'$, 012 say. We locate the occurrences of its three pairs 01, 02 and 12 in (necessarily) precisely three blocks of D_4, namely in 0137, 026E and 1248. The six additional elements so found, in this case {3, 4, 6, 7, 8, E}, can be partitioned, uniquely, into two triples that occur in blocks of D_4, here 346A and 78EA. These two triples always have an element in common and so we have associated triple 012 with the 7-set {3, 4, 6, 7, 8, A, E}. The elements of V that are in neither this 7-set nor the original triple, here {5, 9, B, C, D, ∞}, form the 6-set associated with the triple, 012; this particular 6-set is a block in cycle C_3 of Lemma 13.

(ii), (iii) Given alongside each starter 6-set from Lemma 13 (ii) is a set of 10 triples associated with that 6-set.

C_1 = 0369C∞; C_1' = {12D, 451, 784, AB7, DEA, 24B, 57E, 8A2, BD5, E18};

C_2 = 0579CE; C_2' = {348, 12B, AB6, 24A, 461, BD∞, AD3, 36∞, D28, 18∞};

C_3 = 039ADE; C_3' = {458, 56B, 12∞, 247, 57C, 681, 14B, 8C2, 7B∞, 6C∞};

C_4 = 379AB∞; C_4' = {DE0, CD5, 56E, 12E, 682, 024, D14, C06, 581, 48C}.

To confirm that, when C_1', C_2', C_3', C_4', are developed (modulo 15), each triple obtained can come from only one of the four starter sets of ten blocks above, and none is in D_4', it is sufficient to note that:

- the first five of the ten triples associated with C_1 are from the same cycle of triples (modulo 15), the last five blocks are all from another cycle, and together C_1', $C_1' + 1$, $C_1' + 2$ contain all 30 blocks of these two cycles of triples;

- none of the two representative triples {12D, 24B}, the 30 triples of C'_2, C'_3, C'_4 and the two starter triples of D'_4 can be obtained from another under addition, modulo 15; that is, they all belong to different cycles of triples.

Hence developing (modulo 15) C'_1, C'_2, C'_3 and C'_4 produces $3+3\times 15 = 48$ sets of ten blocks, thus accounting for all $\binom{16}{3} - 80 = 480$ blocks of $\binom{V}{3}\backslash D'_4$ and partitioning this set into 48 sets of 10 blocks.

It can be easily checked that all four derived starter sets of 10 blocks, and hence all 48 developed sets of ten blocks, are isomorphic. Note: the remaining 80 blocks of $\binom{16}{3}$, namely those of D'_4, cannot be partitioned into sets of ten blocks isomorphic to those already obtained.

Thus the first stage of our approach has proven useful for generating chains of triples of D'_4, that is, when $n = 4$. However, in this case the additional condition in part (iii) of Theorem 7, namely that the elements common to the three pairs of blocks are not identical, is not satisfied, thus necessitating the more complicated algorithm. The application of the previous theorems to an affine plane of order $n = 5$, on the other hand, proves to be straightforward.

6. Using the affine plane of order five to partition $\binom{25}{3}$

This section provides the details of the partition, under our approach, of the sets of triples on 25 elements that are **not** contained in the blocks of an affine plane of order five.

Theorem 15 *Let D_5 be an affine plane of order five and let D'_5 be the set of triples contained in the blocks of D_5. Then the 2000 blocks of $\binom{V}{3}\backslash D'_5$ can be partitioned into 250 sets of eight mutually disjoint triples (with each set containing 24 of the 25 elements).*

Proof: Without loss of generality, we take V to be {0,1,2,..., 9, A,B,C,...N,∞} and D_5 to be the affine plane of order five obtained by developing, modulo 24, the starter blocks 0149B and 06CI∞ (short).

Starting with triple $012 \notin D'_5$, we locate $b_{01(2)}$ = 28EK∞, $b_{02(1)}$ = LM168 and $b_{12(0)}$ = DEHMO, thus obtaining, on taking intersections, $f(012)$ = 8EM. Applying function f successively, we obtain $f(8EM)$ = 37I and $f(37I)$ =

9FA, finally returning to $f(\texttt{9FA}) = \texttt{012}$. Thus, in all, we have a chain comprising four mutually disjoint triples of $\binom{V}{3}\backslash D_5'$. (Note that the 'middle' triple 37I comprises the three elements not found in any of 0149B, 125AC, 02FGJ, DEMH, 68LM and 8EK∞.)

To complete the proof, we provide lists of triples in Table 6 and Table 7 which follow. These lists confirm that, as expected, such chains of four mutually disjoint blocks are obtained no matter which triple from $\binom{25}{3}\backslash D_5'$ is used to start. Further, the listing in Table 7 shows that, for every such chain, there exists another (unique) complementary chain such that the resultant pair of chains comprises eight mutually disjoint triples. The two sets of three resolution classes associated with a complementary pair of chains are themselves disjoint.

To allow easy checking, the 2300 triples of $\binom{25}{3}$ are treated in these tables as the development, modulo 24, of (ordered) starter blocks 012, 013, ..., 01N, 024, 025, ..., 07F, 07G, 01∞, 02∞, ..., 0B∞, 0C∞ (short).

Table 6 gives triples of D_5' contained in the two starter blocks 0149B and 06CI∞ for D_5 and the starter block (denoted start.), from the list above, of the cycle of triples to which each belongs. (Starter block 0C∞ is short, generating only 12 blocks, and so only one representative triple contained in 0C∞ is given for each cycle.) Thus these triples of D_5' give rise to the $12 \times 24 + 12 = 30 \times \binom{5}{2} = 300$ triples of D_5'.

	triples from different cycles contained in 0149B									
triple	014	019	01B	09B	19B	49B	149	14B	049	04B
start.	014	019	01B	02F	02G	02J	038	03A	049	04B
	(representative) triples contained in 06CI∞									
triple	06C	06∞	0C∞							
start.	06C	06∞	0C∞							

Table 6: triples from different cycles contained in the starter blocks of D_5

Table 7 gives representative pairs of chains (denoted rep.) comprising, in all, eight mutually disjoint triples of $\binom{V}{3}\backslash D_5'$, again with the respective starter blocks of the cycles involved. The 250 sets of eight mutually disjoint triples are obtained by developing, modulo 24, these representative sets. In Table 7, (n) indicates that the development of particular representatives is short and of length n. A dash (-) indicates that the block is in the same cycle as a previous block in that same set of eight triples. Hence development of pairs of chains 7.1 & 7.2, 8.1 & 8.2 and 9.1 & 9.2 contributes six, three and one sets of eight mutually disjoint triples respectively. Thus, the total number of triples accounted for in Table 7 is $10\times8\times24+8\times6+8\times3+8\times1 =$ 2000 which, since there are 300 triples in D_5', completes the proof.

Notes:

(i) in Table 7, labels identify each chain, with * denoting that the representative chain can be obtained by applying the automorphism $a(x) = 5x$ of D_5' to the representative chain directly above it: lack of such a corresponding chain indicates that the set of eight mutually disjoint triples is fixed under $a(x)$. This accounts for the labelling of the chains.

(ii) the chains of triples generated in Theorem 12 and Theorem 15 from affine places of orders three and five, respectively, cannot be the orbits of a single permutation applied to all the triples not in the blocks of the affine plane. If they were, every element would always map to the same element, which is clearly not so. For the affine plane of order three, the triples 012, 02∞ and 256, which have element 2 in common, map, respectively, to triples 347, 746 and 03∞, which have no common element. For the affine plane of order five, we see from Table 7 that triples 5HI (1.2) and CH5 (2.1) map, respectively, to disjoint triples 69A and 70E.

7. Results for affine planes of other orders

We now consider what happens when we apply our approach to affine planes of other orders.

7.1 Application to small affine planes obtained from $PG(2, n)$

Affine planes of orders six and ten are known not to exist (see [2]). Affine planes obtained from $PG(2, n)$ which include all affine planes of order less than nine and one of the seven planes of order nine (see [2]), are of special interest in that their blocks are *doubly transitive* (see [9]). We have already dealt with the application of our approach to affine planes of orders three, four and five, and nothing useful results when the order is two, since it is trivial that the three blocks on which function f of Definition 6 (iv) relies must have an element in common.

The other three affine planes we mentioned of order less than ten satisfy both the conditions for which sequences are defined and for which the results of Theorem 11 hold. Applying our function to them gives the following.

For order seven, we can obtain a chain of disjoint triples of length six. Taking $PG(2,7)$ to be the projective plane on $V = \{0,1,2,\ldots,56\}$ developed from starter block $\{0,5,6,8,18,37,41,48\}$ (modulo 57), we derive affine plane D_7 by deleting the elements of this starter block from each block of $PG(2,7)$ Starting with block $\{1,2,3\}$ of D_7, we obtain the chain: $\{1,2,3\}; \{12,27,56\}; \{17,40,50\}; \{55,49,28\}; \{20,11,22\}; \{53,34,13\}$. Since the chain is of length six and contains 18 elements, no group of chains of this length can contain precisely 48 or 49 of the $7^2 = 49$ elements, with each element occurring precisely once.

For order eight, direct application of our approach is impossible since, as with orders two and four, three blocks on which function f relies have a common element. For example, take D_8 to be the design derived by deleting the elements of starter block $V = \{1,2,4,8,16,32,37,55,64\}$ (modulo 72) for projective plane $PG(2,8)$ on $V = \{0,1,2,\ldots,72\}$ from each of the blocks of P_8. The blocks of D_8 in which pairs of $\{4,5,6\}$ occur are the following: $\{4,5,7,11,19,35,40,58,67\}i$, $\{4,6,10,18,34,39,57,66\}$ and $\{5,6,8,12,20,41,59,68\}$, giving $b_{45}(6) = \{6,9,13,21,37,42,60,69\}$, $b_{46}(5) = \{2,5,9,17,33,38,56,65\}$, and $b_{56}(4) = \{46,47,49,53,61,4,9,27\}$, all of which contain element 9.

For order nine, we take the (unique) affine plane derived from $PG(2,9)$ by deleting the elements of the first block $\{0,1,2,3,4,5,6,7,8,9\}$ listed in [7]. Starting with triple $\{10,11,19\}$, our approach gives the small chain of length three: $\{10,11,19\}; \{34,35,42\}; \{27,20,12\}$.

7.2 Application to affine planes of orders higher than nine

Our results for orders three to nine, and some preliminary investigation of order eleven, motivate the following general questions covering all affine planes currently known to exist.

First, for what values of n, where n is a prime or prime power, **might** applying our approach to D_n, an affine plane of order n, induce analogous partitions of the triples of $\binom{n^2}{3}\backslash D'_n$?

Secondly, For what affine planes of viable orders **does** our approach indeed induce analogous partitions?

More specifically: For what orders of n, and for what specific affine planes or families of affine planes, can the triples of $\binom{n^2}{3}\backslash D'_n$ be partitioned, using our approach, into collections of either $n^2/3$ or $(n^2-1)/3$ mutually disjoint triples [thus covering n^2 or $(n^2 - 1)$ elements, respectively], such that each collection of triples comprises chains induced by function f, and of constant length $|C|$, with the triples of each such set either:

Case 1: all coming from chains from a single set of three resolution classes, or

Case 2: coming from chains across $(n+1)/3$ mutually disjoint sets of three resolution classes (thus partitioning the $(n+1)$ resolution classes of D_n), with each resolution class contributing precisely the same number of chains to each collection?

For convenience, we call such a partition, of either case, an *affine-induced partition*. We first determine the orders of those affine planes known to exist that allow our approach to induce such a partition.

Lemma 16 *Suppose that D_n is an affine plane of order n, where n is either prime or a prime power, and that every triple not in a block of D_n belongs uniquely to a chain, where the chains have constant length $|C|$. Then $|C| \leq n^2/3$, and $|C|$ is either a power of the same prime as n, but less than n^2, or else $|C|$ divides $(n-1)$.*

Proof: From Lemma 16, $|C|$ divides $n^2(n-1)$. Since n is prime or a prime power, $|C|$ must either be a power of n or n or else it divides $n-1$. Since the triple cannot cover more than n^2 elements, $|C| \leq n^2/3$, which rules out $|C| = n^2$.

Note: Section 4 showed that, when $n = 3$, the (constant) chain length $|C|$ divides n, whereas Section 6 showed that, when $n = 5$, it divides $(n-1)$.

Theorem 17 *The parameters of an affine plane of order n, where n is a prime or a prime power, allow the possibility of an affine-induced partition if and only if n is equal to:*

- $6m+1$. *Case 2; $n^2 - 1$ elements covered; $|C|$ divides $4m(3m+1)$;*
- 3^m. *Case 1; n^2 elements covered; $|C| = 3^q$, where $q \leq m-1$;*
- 2^m. *For $3|C|$ dividing n: Case 1; $n^2 - 1$ elements covered; $|C|$ divides n. For m odd: Case 2; $n^2 - 1$ elements covered; $|C|$ divides, but is not equal to, $n-1$;*

- $n = 6m+5$. *Case 1; n^2-1 elements covered; $|C| = n-1$. Case 2; n^2-1 elements covered; $|C|$ divides, but is not equal to, $n-1$.*

Proof: In each case we need three to divide n^2 or n^2-1 and the chain length $|C|$ to be a factor of n^2, with $|C| \leq n^2$, or else to divide $n-1$. Since each of $6m, 6m+2$ and $6m+4$ is always divisible by 2, and $6m+3$ is divisible by 3, all primes or prime powers are either of the form $6m+1$ or 3^m, the two cases we deal with first, or else of the form 2^m or $6m+5$.

Affine plane of orders $n = 6m+1$ or $n = 3^m$ have $6m+2$ and 3^m+1 resolution classes respectively, and thus cannot be partitioned into sets of three resolution classes as needed. Hence in these cases an affine-induced partition exists only if all the chains come from a single set of three resolution classes.

When $n = 6m+1$, $n^2 = 12m(3m+1)+1$ which, divisibilty by three implies the chains can cover n^2-1, but not n^2, elements. We cannot have chains of a length which is a factor of n^2, since such a factor cannot divide n^2-1. If the chain length $|C|$ divides $n-1$, then chains clearly have the potential to cover the n^2-1 elements, provided $3|C|$ divides $n^2-1 = 12m(3m+1)$. This final condition is met if and only if $|C|$ divides $4m(3m+1)$.

When $n = 3^m$, n^2-1 is not divisible by three, and hence n^2-1 elements cannot be covered by triples. So we must look to cover n^2 elements with chains of length $|C|$ comprising 3^{2m-1} triples in all, which can be done if and only if the chains are of length 3^q, where $q \leq m-1$.

We now look at the situations where Case 2 appears possible. When $n = 2^m$, it is impossible to cover $n^2 = 2^{2m}$ elements with triples. We try to cover n^2-1 elements. Case 1 is satisfied by chains of length $|C|$, provided $3|C|$ divides 2^m-1. If Case 2 applies,then there are 2^m+1 resolution classes, which can be partitioned into sets of three if and only if $m$is odd. Then taking $c \geq 2$ chains of length $|C|$ from each set of three resolution classes, we have $\frac{2^m+1}{3}.c.|C|.3 = \frac{2^m+1}{3}.c.|C|3 = 2^{2m}-1$, which reduces to $c = \frac{2^m-1}{|C|} \geq 2$ and hence to $|C|$ dividing, but not equalling, $n-1$.

When $n = 6m+5$, $n^2 = 36m^2+60m+25$ is not divisible by three, but n^2-1 allows $4(3m+2)(m+1)$ triples. There are $2m+2$ sets of three resolution classes, so taking c chains of length $|C|$ from each set gives $(2m+2)c.|C| = 4(3m+2)(m+1)$ and hence $c.|C| = 2(3m+2) = n-1$. So Case 1 is possible with chain length $n-1$ and Case 2 is possible for chain lengths dividing, but not equalling $n-1$.

This leaves open the second question, "Of those affine planes or families of affine planes of orders allowing an affine-induced partition, which do indeed induce such partitions?" Finally, we point out that it might be useful to investigate the number and nature of the sequences generated by the blocks of affine planes of the same order when determining whether or not they are isomorphic.

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Table 7: representative sets of eight mutually disjoint triples in D_5

rep.	1.1	DEF	L3B	GK7	MN4	1.2	5HI	69A	018	2J∞
start		012	06E	04F	016		01C	01L	018	07∞
rep.	1*.1	HM3	9F7	84B	EKJ	1*.2	1DI	6L2	05G	AN∞
start.		05A	028	03K	01J		05C	04J	05G	0B∞
rep.	2.1	CH5	70E	FIG	A9∞	2.2	3MK	28N	DL4	JB1
start.		05H	70E	013	01∞		027	039	07G	06G
rep.	2*.1	CD1	BOM	3I8	2L∞	2*.2	FE4	AJG	H9K	N75
start.		01D	02D	05F	05∞		01E	03I	03G	02I
rep.	3.1	6GI	K4H	MDF	79∞	3.2	05B	2J3	C8E	1LN
start.		02E	03B	029	02∞		05B	01H	02K	024
rep.	3*.1	68I	4KD	EH3	BL∞	3*.2	017	ANF	CGM	59J
start.		02C	07F	03D	0A∞		017	05D	04A	04E
rep.	4.1	LH9	05E	AN1	4J∞	4.1	M67	3IC	DGK	F82
start.		04G	05E	02B	09∞		01G	06F	037	06D
rep.	4*.1	9DL	01M	2J5	KN∞	4*.2	E6B	FIC	H84	3GA
start.		04C	01M	03H	03∞		03J	036	04D	06H
rep.	5.1	01A	7HD	IGL	359	5.2	N2B	EFJ	2C8	M6∞
start.		01A	04I	025	026		03C	015	048	08∞
rep.	6.1	03F	CEM	156	JN∞	6.2	892	BD4	AGL	HK7
start.		03F	02A	01K	04∞		01I	02H	05I	03E
rep.	7.1	035	69B	CFH	ILN	7.2	12G	78M	DE4	JKA
start.		02L	-	-	-		01F	-	-	-
rep.	8.1	04H	6AN	CGI	M5B	8.2	37K	9D2	FJL	18E
start.		04H	-	-	-		-	-	-	-
rep.	9.1	08G	MCE	K4C	IA2	9.2	19H	NOF	L5D	JB3
start.		08G	-	-	-	(8)	-	-	-	-