# A new sufficient condition for graphs of $f$-class 1*

**Xia Zhang[a]†, Guizhen Liu[b], Jiansheng Cai[c] and Jianfeng Hou[d]**
[a]Department of Mathematics, Shandong Normal University
Jinan 250014, China
[b]School of Mathematics, Shandong University
Jinan 250100, China
[c]School of Mathematics and Information Sciences, Weifang University
Weifang 261061, China
[d]Center for Discrete Mathematics, Fuzhou University
Fuzhou 350002, China

**Abstract** An $f$-coloring of a graph $G$ is an edge-coloring of $G$ such that each color appears at each vertex $v \in V(G)$ at most $f(v)$ times. The minimum number of colors needed to $f$-color $G$ is called the $f$-chromatic index of $G$. A simple graph $G$ is of $f$-class 1 if the $f$-chromatic index of $G$ equals $\Delta_f(G)$, where $\Delta_f(G) = \max_{v\in V(G)}\{\lceil d(v)/f(v)\rceil\}$. In this article, we find a new sufficient condition for a simple graph to be of $f$-class 1, which is strictly better than a condition presented by Zhang and Liu in 2008 and is sharp. Combining the previous conclusions with this new condition, we improve a result of Zhang and Liu in 2007.

**Keywords:** Edge-coloring; $f$-Coloring; Classification of graph; $f$-Chromatic index

## 1 INTRODUCTION

All graphs considered in this paper are finite and undirected. They allow multiple edges but no loops. If a graph has neither loops nor multiple

*Supported by NSFC(10901097), NSFC(10871119), RFDP(200804220001), Tianyuan Youth Foundation of Mathematics(10926099) and NSF of Shandong(ZR2009AM009) of China.

†Corresponding author. Email: pandarhz@sina.com

edges, we will call it a *simple graph*. The reader is referred to [1] for the undefined terms.

An *edge-coloring* of $G$ is an assignment of colors to all the edges of $G$. Let $Q \subseteq E(G)$ and $Q \neq \emptyset$. A *partial edge-coloring* of $G$ is an edge-coloring of a subgraph $G[Q]$ of $G$. When $Q = E(G)$, a partial edge-coloring of $G$ is exactly an edge-coloring of $G$. Let $G$ be a graph and let $f$ be a function which assigns a positive integer $f(v)$ to each vertex $v \in V(G)$. An *$f$-coloring* of $G$ is an edge-coloring of $G$ such that each vertex $v \in V(G)$ has at most $f(v)$ edges colored with the same color. The minimum number of colors needed to $f$-color $G$ is called the *$f$-chromatic index* of $G$ and denoted by $\chi'_f(G)$. If $f(v) = 1$ for all $v \in V(G)$, the $f$-coloring is reduced to the *proper edge-coloring*, and the $f$-chromatic index of $G$ is reduced to the *chromatic index* of $G$ and denoted by $\chi'(G)$.

We define

$$\Delta_f(G) = \max_{v \in V(G)} \{\lceil \frac{d(v)}{f(v)} \rceil\},$$

where $\lceil x \rceil$ is the smallest integer not smaller than $x$. It is trivial to show that $\chi'_f(G) \geq \Delta_f(G)$.

Hakimi and Kariv [2] generalized proper edge-colorings to $f$-colorings and obtained many interesting results, some of which will be used in the rest of this article as follows.

**Lemma 1.1** *[2] Let $G$ be a bipartite graph. Then $\chi'_f(G) = \Delta_f(G)$.*

**Lemma 1.2** *[2] Let $G$ be a graph. If $f(v)$ is positive and even for all $v \in V(G)$, then $\chi'_f(G) = \Delta_f(G)$.*

**Lemma 1.3** *[2] Let $G$ be a simple graph. Then*

$$\Delta_f(G) \leq \chi'_f(G) \leq \max_{v \in V(G)} \{\lceil \frac{d(v)+1}{f(v)} \rceil\} \leq \Delta_f(G) + 1.$$

We say that a simple graph $G$ is of *$f$-class* 1 if $\chi'_f(G) = \Delta_f(G)$, and of *$f$-class* 2 otherwise. The problem of deciding whether a simple graph $G$ is of $f$-class 1 or $f$-class 2 is called the *classification problem* on $f$-colorings. If $f(v) = 1$ for all $v \in V(G)$, a well-known theorem of Vizing [5], i.e. $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$, can be deduced.

Zhang and Liu [6, 7, 8, 9, 10], Zhang et al. [11] studied the classification problem of complete graphs, simple regular graphs and some other special classes of simple graphs on $f$-colorings. Liu et al. [4] studied some properties of $f$-critical graphs (i.e. connected simple graphs $G$ of $f$-class 2 satisfying that $\chi'_f(G - e) < \chi'_f(G)$ for any edge $e \in E(G)$). When $f(v) = 1$ for all $v \in V(G)$, many known results in the classification problem on proper edge-colorings can be deduced.

We denote the neighbor set of vertex set $S$ in $G$ by $N_G(S)$ for $S \subseteq V(G)$. Let

$$V_0^*(G) = \{v \in V(G) : \frac{d(v)}{f(v)} = \Delta_f(G)\}.$$

Then $N_G(V_0^*(G)) = \{v \in V(G) : uv \in E(G), u \in V_0^*(G)\}$. The *f-core* of a graph $G$ is the subgraph of $G$ induced by the vertices of $V_0^*(G)$ and is denoted by $G_{\Delta_f}$. The number $d(v)/f(v)$ is called the *f-ratio* of vertex $v$ in $G$. A graph $G$ is *$\Delta_f(G)$-peelable*, if all the vertices of $G$ can be iteratively peeled off using the following peeling operation: Removal of a vertex $v$, which has at most one remaining neighbor of $f$-ratio $\Delta_f(G)$.

So far, the best results based on $V_0^*(G)$ are the following.

**Theorem 1.1** *[8] Let $G$ be a simple graph. Suppose that $V_0^*(G) \neq \emptyset$. If $f(v)$ is positive and even for all $v \in V_0^*(G) \cup N_G(V_0^*(G))$, then $G$ is of $f$-class* 1.

**Theorem 1.2** *[9] Let $G$ be a simple graph. If $G$ is $\Delta_f(G)$-peelable, then $G$ is of $f$-class* 1.

As shown in [9], a simple graph in which $V_0^*(G) = \emptyset$ or $G_{\Delta_f}$ is a forest is $\Delta_f(G)$-peelable. $\Delta_f(G)$-peelable simple graphs are the most extensive class of graphs of $f$-class 1 obtained by coloring the edges in a particular order so far.

We call a graph $G$ *RP-removable*, if all the vertices of $G$ can be iteratively removed using the following vertex removal operations:

(1) removal of a vertex $v$ with degree at most $(f(v) - 1)\Delta_f(G) + 1$;

(2) removal of a vertex $v$, which has at most one remaining neighbor of $f$-ratio $\Delta_f(G)$.

In Section 2, we prove that every $RP$-removable simple graph is of $f$-class 1. In particular, we show that this sufficient condition is strictly better than the one in Theorem 1.2 and is sharp. Also, we give a class of graphs with $\chi'_f(G) = \Delta_f(G)$. In Section 3, we discuss non-$RP$-removable simple graphs. We find a more general class of simple graphs which are of $f$-class 1 by combining the previous conclusions with the vertex $RP$-removability condition. As a result, we improve Theorem 1.1. In Section 4, we present an open problem for further research.

# 2 MAIN RESULTS

Set $\nu(G) = |V(G)|$ and $E_u = \{uv : v \in V(G) \text{ and } uv \in E(G)\}$. Given an $f$-coloring of a subgraph $G'$ of $G$ and an uncolored edge $e$ in $G$, we say that $e$ *can be f-colored* if $G' + e$ can be $f$-colored. Let $C$ be a set of colors. Suppose that $G$ has been given a partial edge-coloring $\tilde{c}$ with colors in $C$. An edge colored with color $\alpha \in C$ is called an *$\alpha$-edge*. Denote by $|\alpha(v)|$ the number of $\alpha$-edges incident with the vertex $v$ in $G$. Define $M(v) = \{\alpha : |\alpha(v)| < f(v), \alpha \in C\}$ for $\tilde{c}$. It is easy to see that $M(v) \neq \emptyset$ if $f(v)|C| > \sum_{\alpha \in C} |\alpha(v)|$ in $\tilde{c}$. When $|C| \geq \Delta_f(G)$, we have $f(v)|C| \geq f(v)\Delta_f(G) \geq d_G(v) \geq \sum_{\alpha \in C} |\alpha(v)|$ for any $\tilde{c}$. Thus, $M(v) \neq \emptyset$ for a vertex $v \in V(G)$ if $|C| > \Delta_f(G)$, $|C| = \Delta_f(G)$ and $d_G(v)/f(v) < \Delta_f(G)$, or $|C| = \Delta_f(G)$ and $v$ is incident with at least one uncolored edge in $\tilde{c}$. We adopt the convention that any subgraph $G'$ of $G$ has $f_{G'}(v) = f_G(v)$ for all $v \in V(G')$.

We now describe a tool commonly used for $f$-colorings.

Define $m(v,\alpha) = f(v) - |\alpha(v)|$ for each $v \in V(G)$ and each $\alpha \in C$. Clearly, $M(v) = \{\alpha : m(v,\alpha) \geq 1, \alpha \in C\}$. For two distinct colors $a, b \in C$, a walk $W = v_0e_1v_1e_2v_2 \ldots e_hv_h$ is called an *ab-alternating walk* if $W$ satisfies the following conditions:

(a) the edges of $W$ are colored alternately with $a$ and $b$, and the first edge of $W$ is colored with $b$;

(b) $m(v_0, a) \geq 1$ if $v_0 \neq v_h$,
$m(v_0, a) \geq 2$ if $v_0 = v_h$ and $|W|$ is odd;

(c) $m(v_h, b) \geq 1$ if $v_0 \neq v_h$ and $|W|$ is even,
$m(v_h, a) \geq 1$ if $v_0 \neq v_h$ and $|W|$ is odd.

In particular, any closed walk $W$ of even length whose edges are colored with $a$ and $b$ alternately is an $ab$-alternating walk. The operation, interchanging the colors $a$ and $b$ of the edges in an $ab$-alternating walk $W$, is called *switching* $W$. After $W$ was switched, $m(v_i, a)$ and $m(v_i, b)$ remain as they were if $i \neq 0, h$, while $m(v_0, b) \geq 1$ if $W$ is not a closed walk of even length. A *maximal* alternating walk is one whose length cannot be increased.

In this section, we give two useful lemmas. One is the following.

**Lemma 2.1** *Let $G$ be a graph, $u \in V(G)$ and $k$ be a positive integer with $k \geq \Delta_f(G)$. If $G - u$ can be $f$-colored with $k$ colors and $d_G(u) \leq (f(u) - 1)k + 1$, then $G$ can be $f$-colored with $k$ colors.*

**Proof.** Clearly, for graph $G$, an $f$-coloring with $m$ colors is also an $f$-coloring with $n$ colors if $n \geq m$. If $\Delta_f(G)$=1, which implies that $\chi'_f(G) = 1$, then $G$ can be $f$-colored with $k \geq 1$ colors. Next, we assume that $\Delta_f(G) \geq 2$.

Suppose that $G - u$ is $f$-colored with the colors in $C = \{c_1, c_2, \ldots, c_k\}$. Next, we present an $f$-coloring of $G$ with the colors in $C$.

For any edge $e = uv \in E_u$, we consider two cases.

**Case 1.** If there exists some color $\alpha \in C$ such that $|\alpha(u)| < f(u)$ and $|\alpha(v)| < f(v)$, then color $e$ with $\alpha$.

**Case 2.** Otherwise, we choose a color $\beta \in C$ such that $|\beta(v)| < f(v)$. Color $e$ with color $\beta$. Since before coloring $e$ we had $|\beta(u)| = f(u)$, after coloring $e$ we have $|\beta(u)| = f(u)+1$. We claim that there must exist a color, without loss of generality, say $\gamma$, in $C$ such that $|\gamma(u)| \leq f(u) - 2$. (Otherwise, the number of the colored edges incident with $u$ is at least $(f(u) - 1)(k - 1) + (f(u) + 1) \geq (f(u) - 1)k + 2$. It contradicts $d_G(u) \leq (f(u) - 1)k + 1$.) Find a maximal $\gamma\beta$-alternating walk $P = ue_1v_1e_2v_2 \ldots e_hv_h$. In particular, when the walk $P$ returns to $u$ with a $\gamma$-edge, we continue to extend $P$ by choosing a $\beta$-edge incident with $u$. Since $|\beta(u)| - |\gamma(u)| \geq 3$, there exists such a $\beta$-edge which has not been included in $P$ so far. If $v_h = u$, then $P$ must end at $u$ with a $\beta$-edge. After switching $P$, we have $|\beta(u)| = f(u) - 1$ and $|\gamma(u)| \leq f(u)$. If $v_h \neq u$, then after switching $P$ we have $|\beta(u)| = f(u)$ and $|\gamma(u)| \leq f(u) - 1$.

In either case, $e$ can be $f$-colored with a color in $C$. Repeat this operation until all edges in $E_u$ are $f$-colored with the colors in $C$. ∎

Let $t$ be a function which assigns a positive integer $t(v)$ to each vertex $v \in V(G)$. A graph $G$ is *t-removable*, if all vertices of $G$ can be iteratively removed using the following vertex removal operation: removal of a vertex $v$ with degree at most $t(v)$.

**Theorem 2.1** *Let $G$ be a graph. If $G$ is $(f-1)\Delta_f(G)+1$-removable, then $\chi'_f(G) = \Delta_f(G)$.*

**Proof.** Suppose that the vertices of $G$ can be $(f-1)\Delta_f(G)+1$-removed in the order $v_1, v_2, \ldots, v_{\nu(G)}$. Let $G_i = G - \{v_1, v_2, \ldots, v_i\}(1 \le i < \nu(G))$ and let $v_s$ be the vertex such that $E(G_s) = \emptyset$ and $E(G_{s-1}) \neq \emptyset$. Clearly, $G_{s-1}$ is a star. By Lemma 1.1, we have $\chi'_f(G_{s-1}) = \Delta_f(G_{s-1})$. Obviously, $\Delta_f(G_{s-1}) \le \Delta_f(G)$. Thus, $G_{s-1}$ can be $f$-colored with $\Delta_f(G)$ colors. By hypothesis, $d_{G_{s-2}}(v_{s-1}) \le (f(v_{s-1})-1)\Delta_f(G)+1$, so $G_{s-2}$ can be $f$-colored with $\Delta_f(G)$ colors by Lemma 2.1. Iteratively applying Lemma 2.1 to $G_i$ $(i = s-2, s-3, \ldots, 1)$, then obtain that $G$ can be $f$-colored with $\Delta_f(G)$ colors. ∎

Theorem 2.1 describes a class of graphs with $f$-chromatic indices equal to $\Delta_f(G)$. For a simple graph, we have the following corollary.

**Corollary 2.1** *Let $G$ be a simple graph. If $G$ is $(f-1)\Delta_f(G)+1$-removable, then $G$ is of $f$-class 1.*

Next, we confine ourselves to simple graphs. The following lemma is a standard result on $f$-colorings of simple graphs.

**Lemma 2.2** *[8, 9] Let $C$ denote the set of colors available to color the edges of a simple graph $G$. Suppose that $e_0 = wv_0$ is an uncolored edge in $G$, and graph $G - e_0$ is $f$-colored with the colors in $C$. If every neighbor $v$ of either $w$ or $v_0$ has $M(v) \neq \emptyset$, then we can $f$-color $e_0$ with a color in $C$.*

Now, we give the other useful lemma.

**Lemma 2.3** *Let $G$ be a simple graph, $u \in V(G)$ and $k$ be a positive integer with $k \ge \Delta_f(G)$. If $u$ can be $\Delta_f(G)$-peeled and $G-u$ can be $f$-colored with $k$ colors, then $G$ can be $f$-colored with $k$ colors.*

**Proof.** We denote the neighbor set of vertex $u$ in $G$ by $N_G(u)$. Suppose that $G-u$ is $f$-colored with the colors in $C = \{c_1, c_2, \ldots, c_k\}$. Note that $k \ge \Delta_f(G)$. We consider two cases.

**Case 1.** $u$ has no neighbor of $f$-ratio $\Delta_f(G)$ in $G$.

In this case, $d_G(v)/f(v) < \Delta_f(G)$ for any $v \in N_G(u)$. Thus we always have $M(v) \neq \emptyset$ for every $v \in N_G(u)$ when coloring the edges in $E_u$ one by one. So we can $f$-color every edge in $E_u$ with the colors in $C$ by Lemma 2.2.

**Case 2.** $u$ has exactly one neighbor of $f$-ratio $\Delta_f(G)$ in $G$.

Let $u' \in N_G(u)$ and $d_G(u')/f(u') = \Delta_f(G)$. Clearly, $u$ has no neighbor of

$f$-ratio $\Delta_f(G)$ in $G - uu'$. As shown in Case 1, $G - uu'$ can be $f$-colored with the colors in $C$. After $f$-coloring $G - uu'$ with the colors in $C$, we still have $M(v) \neq \emptyset$ for each $v \in N_G(u)$ and $v \neq u'$. And now, $M(u') \neq \emptyset$ since $uu'$ is uncolored so far. Then $uu'$ can be $f$-colored with a color in $C$ by Lemma 2.2.

In either case, we can obtain an $f$-coloring of $G$ with $k$ colors. ∎

We are now ready to prove the main result of this article.

**Theorem 2.2** *Let $G$ be a simple graph. If $G$ is RP-removable, then $G$ is of $f$-class* 1.

**Proof.** First, we suppose that the vertices of $G$ can be $RP$-removed in the order $v_1, v_2, \ldots, v_{\nu(G)}$. Furthermore, let $G_i = G - \{v_1, v_2, \ldots, v_i\}(1 \leq i < \nu(G))$ and let $v_s$ be the vertex such that $E(G_s) = \emptyset$ and $E(G_{s-1}) \neq \emptyset$. As proved in Theorem 2.1, $G_{s-1}$ can be $f$-colored with $\Delta_f(G)$ colors. Consider vertex $v_{s-1}$. If $v_{s-1}$ is $RP$-removed under operation (1), then $d_{G_{s-2}}(v_{s-1}) \leq (f(v_{s-1}) - 1)\Delta_f(G) + 1$. By Lemma 2.1, $G_{s-2}$ can be $f$-colored with $\Delta_f(G)$ colors. If $v_{s-1}$ is $RP$-removed under operation (2), then $G_{s-2}$ also can be $f$-colored with $\Delta_f(G)$ colors by Lemma 2.3. In the same way, we can iteratively prove that $G_i$ can be $f$-colored with $\Delta_f(G)$ colors for $i = s-3, s-4, \ldots, 1$ and $G$ can be $f$-colored with $\Delta_f(G)$ colors. ∎

Clearly, Theorem 1.2 and Corollary 2.1 are just two special cases of Theorem 2.2. Here, we show that Theorem 2.2 provides a strictly better sufficient condition for a simple graph to be of $f$-class 1 than Theorem 1.2 and Corollary 2.1. See the graph $G$ in Fig. 2.1. We have $\Delta_f(G) = 3$ and $V_0^*(G) = V(G) \setminus \{v_1\}$. $G_{\Delta_f}$ is indicated by thick lines. It is easy to see that $G$ is $RP$-removable in the order $v_1, v_2, \ldots, v_9$, in which $v_1, v_3$ are $RP$-removed under operation (1) and the others under operation (2). But it is neither $\Delta_f(G)$-peelable nor $(f-1)\Delta_f(G) + 1$-removable. (In $G$, no vertex can be $\Delta_f(G)$-peeled since each vertex has at least two neighbors of $f$-ratio $\Delta_f(G)$, and no vertex can be $(f-1)\Delta_f(G) + 1$-removed except $v_1$.)

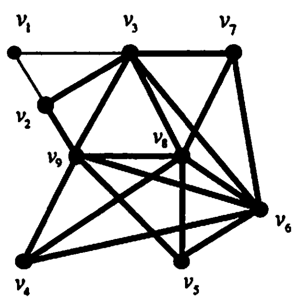


Fig. 2.1. An $RP$-removable simple graph $G$ with $f(v_i) = 1$ $(i = 2, 4, 5, 7)$ and $f(v_j) = 2$ $(j = 1, 3, 6, 8, 9)$.

We call a graph $G$ *$RP^1$-removable*, if all the vertices of $G$ can be iteratively removed using the following operation (1′) and the $RP$-removal operation (2); a graph $G$ *$RP^2$-removable*, if all the vertices of $G$ can be iteratively removed using the following operation (2′) and the $RP$-removal operation (1):

(1′) removal of a vertex $v$ with degree at most $(f(v) - 1)\Delta_f(G) + 2$;

(2′) removal of a vertex $v$, which has at most two remaining neighbor of $f$-ratio $\Delta_f(G)$.

To see that the sufficient condition in Theorem 2.2 is sharp, we now show that either class of $RP^1$-removable simple graphs and $RP^2$-removable simple graphs contain simple graphs of $f$-class 2. Consider a simple graph $G'$ which is obtained from $G$ in Fig. 2.1 by changing the function $f$ into $f(v_i) = 1$ $(i = 1, 2, 4, 5, 7)$ and $f(v_j) = 2$ $(j = 3, 6, 8, 9)$. We have $\Delta_f(G') = 3$ and $V_0^*(G') = V(G')\setminus\{v_1\}$. Clearly, $d_{G'}(v_1) = 2 \leq (f(v_1)-1)\Delta_f(G')+2$ and $v_1$ has exactly two remaining neighbors of $f$-ratio $\Delta_f(G')$. So, $v_1$ can be $RP^1$-removed under operation (1′) or $RP^2$-removed under operation (2′). It is easy to see that $G'$ is $RP^1$-removable as well as $RP^2$-removable in the same order $v_1, v_2, \ldots, v_9$. However, we say that $G'$ is of $f$-class 2, for otherwise, there will exist a color such that the number of edges colored with the color, i.e. $(2 \times 4 + 1 \times 5)/2$, is not an integer.

When $f(v) = 1$ for all $v \in V(G)$, an $RP$-removable graphs $G$ is reduced to a *$\Delta(G)$-peelable* graph, i.e. a graph all the vertices of which can be iteratively peeled off using the following peeling operation: Removal of a vertex $v$, which has at most one remaining neighbor of degree $\Delta(G)$.

**Corollary 2.2** *[3] Let $G$ be a simple graph. If $G$ is $\Delta(G)$-peelable, then $G$ is of class* 1.

# 3 FURTHER DISCUSSION

Let $G$ be a graph. An *$RP$-remaining graph* of $G$ is a subgraph obtained by removal of some vertices using the $RP$-removal operation in $G$. In particular, every graph is an $RP$-remaining graph of itself. We call an $RP$-remaining graph $H$ of $G$ *minimal* if no vertex can be $RP$-removed in $H$. A graph $G$ has no minimal $RP$-remaining graph if and only if $G$ is an $RP$-removable graph. According to the definition of $RP$-removability, it is easy to see that, if there are several choices available, we may perform the $RP$-removal operations for those vertices in any order. So, for a non-$RP$-removable simple graph $G$, the minimal $RP$-remaining graph of $G$ is exclusive and is denoted by $G[U]$, where $U$ is the set of "non-$RP$-removable" vertices in $G$. Note that a non-$RP$-removable graph $G$ has $G[U] = G$ if no vertex can be $RP$-removed in $G$.

We consider a non-$RP$-removable simple graph $G$. Clearly, every vertex $v \in U$ is adjacent to at least two vertices of $f$-ratio $\Delta_f(G)$ in $G[U]$ and $\Delta_f(G[U]) = \Delta_f(G)$.

**Proposition 3.1** *Let $G$ be a simple graph. If $G$ is not $RP$-removable, then* $\chi'_f(G) = \chi'_f(G[U])$.

**Proof.** Suppose that the vertices of $V(G) \setminus U$ can be $RP$-removed in the order $v_1, v_2, \ldots, v_s$, where $s = |V(G) \setminus U|$. Set $G_i = G - \{v_1, v_2, \ldots, v_i\}(1 \leq i \leq s)$. Furthermore, suppose that $G[U](= G_s)$ is $f$-colored with $\chi'_f(G[U])$ colors. Note that $\chi'_f(G[U]) \geq \Delta_f(G[U]) = \Delta_f(G)$.

If $v_s$ is $RP$-removed under operation (1), then $G_{s-1}$ can be $f$-colored with $\chi'_f(G[U])$ colors by Lemma 2.1. If $v_s$ is $RP$-removed under operation (2), then $G_{s-1}$ also can be $f$-colored with $\chi'_f(G[U])$ colors by Lemma 2.3. We can iteratively prove that $G_i$ can be $f$-colored with $\chi'_f(G[U])$ colors for $i = s-2, s-3, \ldots, 1$ and $G$ can be $f$-colored with $\chi'_f(G[U])$ colors. Since $\chi'_f(G) \geq \chi'_f(G[U])$, we have $\chi'_f(G) = \chi'_f(G[U])$. ∎

We have determined the $f$-chromatic indices of $RP$-removable simple graphs in Theorem 2.2. For a non-$RP$-removable simple graph $G$, we can obtain the $f$-chromatic index of $G$ if we easily determine the $f$-chromatic index of the minimal $RP$-remaining graph $G[U]$ according to Proposition 3.1.

It is easy to verify the following result by combining Proposition 3.1 with Lemma 1.1 and Lemma 1.2.

**Theorem 3.1** *Let $G$ be a simple graph. Then $G$ is of $f$-class 1 if one of the following conditions is true:*

(1) *$G$ is $RP$-removable;*

(2) *$G[U]$ is a bipartite graph;*

(3) *each vertex $v \in U$ has $f(v)$ positive and even.*

For a graph $G$, every vertex $v \in V(G) \setminus \{V_0^*(G) \cup N_G(V_0^*(G))\}$ can be $RP$-removed since $v$ has no neighbor of $f$-ratio $\Delta_f(G)$ in $G$. Consider a simple graph $G$, which satisfies that $V_0^*(G) \neq \emptyset$ and $f(v)$ is positive and even for all $v \in V_0^*(G) \cup N_G(V_0^*(G))$. If $G$ is $RP$-removable, then Condition (1) of Theorem 3.1 is satisfied. Otherwise, Condition (3) of Theorem 3.1 is satisfied since $U \subseteq V_0^*(G) \cup N_G(V_0^*(G))$. Hence we can deduce Theorem 1.1 from Theorem 3.1.

Fig. 2.1 and Fig. 3.1 show that the sufficient condition obtained by combining Condition (1) with Condition (3) in Theorem 3.1 is strictly better than the sufficient condition in Theorem 1.1 for a simple graph to be of $f$-class 1. In particular, for $G$ in Fig. 3.1, we have $\Delta_f(G) = 3$, $V_0^*(G) = V(G) \setminus \{v_2, v_5\}$ and $V_0^*(G) \cup N_G(V_0^*(G)) = V(G)$. $G_{\Delta_f}$ is indicated by thick lines. It is easy to see that no vertex can be $RP$-removed except that $v_1, v_2, \ldots, v_5$ can be $RP$-removed in the same order. Then $G$ is non-$RP$-removable and $U = \{v_6, v_7, \ldots, v_{12}\}$. Also, each vertex $v \in U$ has $f(v) = 2$, so $G$ is of $f$-class 1. Clearly, $G$ does not satisfy the condition in Theorem 1.1.

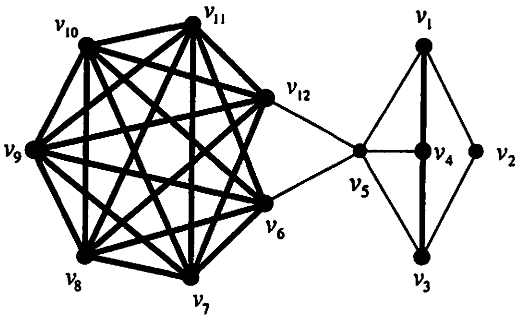


Fig. 3.1. A non-$RP$-removable simple graph $G$ with $f(v_i) = 1$ ($i = 1, \ldots, 4$) and $f(v_j) = 2$ ($j = 5, \ldots, 12$).

# 4 OPEN PROBLEMS

Consider the graph $G$ in Fig. 4.1. Clearly, $\Delta_f(G) = 3$ and $G$ is $RP$-removable in the order $v_1, v_2, v_3, v_4$. However, $\chi'_f(G) \neq 3$ (for otherwise, there will exist a color such that the number of edges colored with the color, i.e. $(1 \times 3)/2$, is not an integer).

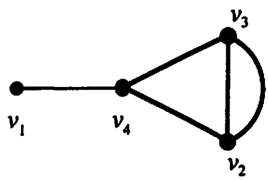


Fig. 4.1. An $RP$-removable graph $G$ with $f(v_i) = 1$ ($i = 1, \ldots, 4$).

Fig. 4.1 shows that, without condition "$G$ is a simple graph", the condition "$G$ is an $RP$-removable graph" does not insure that $\chi'_f(G) = \Delta_f(G)$. Thus we present the following problem:

**Problem 4.1** *Find sufficient conditions for an RP-removable graph G to have* $\chi'_f(G) = \Delta_f(G)$.

On the other hand, one could consider the classification problem on $f$-colorings for non-$RP$-removable simple graphs.

# References


[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, MacMillan, London (1976)

[2] S. L. Hakimi and O. Kariv, A generalization of edge-coloring in graphs, *Journal of Graph Theory*, 10 139-154 (1986)

[3] S. L. Hakimi and E. F. Schmeichel, Improved bounds for the chromatic index of graphs and multigraphs. *J. Graph Theory*, 32 311-326 (1999)

[4] G. Liu, J. Hou and J. Cai, Some results about $f$-critical graphs. *Networks*, 50(3) 197-202 (2007)

[5] V. G. Vizing, On an estimate of the chromatic class of a $p$-graph. *Diskret. Analiz*, 3 25-30 (1964)

[6] X. Zhang and G. Liu, The classification of complete graphs $K_n$ on $f$-coloring. *Journal of Applied Mathematics & Computing*, 19(1-2) 127-133 (2005)

[7] X. Zhang and G. Liu, Some sufficient conditions for a graph to be $C_f$ 1. *Applied Mathematics Letters*, 19 38-44 (2006)

[8] X. Zhang and G. Liu, A class of graphs of $f$-class 1. *7th International Conference on Computational Science (ICCS 2007)*, Part III, LNCS 4489: 362-369 (2007)

[9] X. Zhang and G. Liu, Some graphs of class 1 for $f$-colorings. *Applied Mathematics Letters*, 21 23-29 (2008)

[10] X. Zhang and G. Liu, $f$-Colorings of some graphs of $f$-class 1. *Acta Mathematica Sinica*, 24(5) 743-748 (2008)

[11] X. Zhang, J. Wang and G. Liu, The classification of regular graphs on $f$-colorings. *Ars Combinatoria*, 86 273-280 (2008)